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Network congestion analysis of gravity generated models

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HIGHLIGHTS

- Network congestion is examined under different topology and traffic types.
- Gravity networks suffer less congestion than random, scale-free or Jackson–Rogers ones.
- Gravity traffic pattern causes more severe congestion to all topology types.

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ABSTRACT

The network topology has lately proved to be critical to the appearance of traffic congestion, with scale-free networks being the less affected at high volumes of traffic. Here, the congestion dynamics are investigated for a class of networks that has experienced a resurgence of interest, the networks based on the gravity model. In addition, supplementary to the standard paradigm of uniform traffic volumes between randomly interacting node pairs, more realistic gravity traffic patterns are used to simulate the flows in the network. Results indicate that depending on the traffic pattern, the networks have different tolerance to congestion. Experiment simulation shows that the topologies created on the basis of the gravity model suffer less from congestion than the random, the scale-free or the Jackson–Rogers ones under both random and gravity traffic patterns. The congestion level is found to be approximately correlated with the network clustering coefficient in the case of random traffic, whereas in the case of gravity traffic such a correlation is not a trivial one. Other basic network properties such as the average shortest path and the diameter are seen to correlate fairly well with the congestion level. Further investigation on the adjustment of the gravity model parameters indicates particular sensitivity to network congestion. This work may have practical implications for designing traffic networks with both reasonable budget and good performance.

1. Introduction

Networks have been thoroughly investigated in the last decade with impressive research results on the universality of various topological characteristics [1] and on the mechanisms of topology generation [1–3]. Unexpected similarities and substantial common, but non-trivial, structural features (power-law degree distributions, high clustering coefficient, small geodesic path lengths, etc.) among real-world networks compose the most essential findings indicating a departure from the random network models having being proposed five decades earlier [4]. The contemporary knowledge on the networks' structure can now be used for the understanding of challenging underlying processes that take place on networks. For example, searching, diffusion, spreading, synchronization, traffic flow interactions and other dynamical phenomena have recently been put in the foreground.

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Traffic dynamics in complex networks have been extensively studied lately [5–22] due to their wide applications in telecommunications and transportation and as a result of the novel discoveries on network topology. Recent works focus on the efficiency improvement of traffic systems and bring out a relationship between controllable parameters of network topology and traffic-flow performance. Several of them deal with routing strategies on complex networks [13–16,19,20], some others with the capacity distribution [5,11,21], while others with the cascading failures [8,9] and the congestion on complex networks [6,7,10,12,17,18,22].

Most of the networks examined in the previous studies are random (Erdős–Rényi or *ER* algorithm [4]) or scale-free (Barabási–Albert or *BA* algorithm [1]) and the traffic flows are assumed to be homogeneous between randomly selected source and destination nodes, with barely a few exceptions [17,20]. However, in real networks, especially on spatial ones, the topology can deviate from those derived from the *ER* or *BA* models [23,24] and traffic is more likely to be generated/received unevenly at some nodes than at others, according to their characteristics [25–27]. Therefore, in this paper, gravity topologies are introduced as they have been found to share statistical properties with real-world networks while allowing for optimal expected traffic exchange. Their ability to combine intrinsic attributes and extrinsic features in a simple spatial weighted network model has widely established them in telecommunications and transportation [24,26–33]. Here, their behavior under congestion is compared to the behavior of random, scale-free and Jackson–Rogers (*JR*) topologies. The *JR* topologies are incorporated in the analysis since they can successfully reproduce all of the basic features of real-world networks [3], including a high cliquishness which may be a congestion determinant. In addition, network congestion is studied taking into consideration traffic flows obeying the more realistic gravity-based flow patterns, as recently observed in the related literature [34,35].

The main purpose of this paper is to examine the relationship between the traffic flows and congestion factors in different topologies. Traffic congestion can be improved either by developing better routing strategies or by network restructuring [36] which is more in the focus of this paper. In this analysis a series of implications are derived from what kind of congestion and cost level to expect for a given set of topological and traffic parameters. Thus knowledge is obtained on which way to (re)design a traffic network, e.g. transportation, telecommunications or other network, in order to alleviate congestion effects.

The paper is organized as follows. In Section 2, the network models and the traffic flow types are introduced. In Section 3, the simulation results are presented and discussed. Finally, the conclusion is given in Section 4.

2. Network types and traffic flows

2.1. Network representation

In formal terms, networks can be represented as graphs $G(N, K)$, which are mathematical entities from Graph Theory, defined by two sets, N and K. The first set, N, is a finite set of N elements called nodes or vertices, and K is a finite set of K elements containing unordered pairs of different nodes called links or edges. The graph G can be described by the $N \times N$ adjacency matrix, A, whose entry a_{ij} is equal to 1 if there is a direct edge between nodes i and j, and 0 otherwise. In the case of a weighted network, an additional set of values attached to the edges is characterizing the links. The matrix containing the edge weights could describe the traffic flows, the capacity, the cost, the length, etc. Here, the graph nodes are also considered to have a precise position on a planar map $\{x_i, y_i\}_{i=1,\ldots,N}$ and a fitness value $\{f_i\}_{i=1,\ldots,N}$.

The basic statistical properties of such a network representation are referred to here; the average shortest path, the diameter, the clustering coefficient, and the degree distribution. The average shortest path or average geodesic path length is defined as the average number of steps along the shortest paths for all possible pairs of network nodes. The diameter of a network is the length (in number of edges) of the longest shortest path between any two nodes in the network. The clustering coefficient measures the density of triangles in the network, and put simply, is the mean probability that the friend of your friend is also your friend. It can be quantified by defining it as $C = \frac{3\times \text{ number of triangles in the network}}{\text{number of connected triples of vertices}}$, where a "connected triple" means a single vertex with edges running to an unordered pair of others [37]. A clear deviation from the behavior of a random graph can be seen in the clustering property, sometimes called transitivity, and suggests that there is a heightened number of triangles in the network—sets of three vertices each of which is connected to each of the others. The degree distribution is the probability distribution function (PDF) of the node degrees k over the whole network, where k is the number of edges directly connected to a node. The PDF is usually well fitted with a Poisson or a power-law distribution.

2.2. Analysis of the main network types

A great variety of network formation models has been proposed in the past years, however studies in traffic dynamics are mainly based on two simple well-known models; the *ER* [4] for random networks and the *BA* [1] for scale-free networks. The network topology has proved to be critical to the appearance of traffic congestion, with scale-free networks being the less affected at high volumes of traffic [6,12,22]. More recently, the *JR* formation model [3] has been proposed, which can successfully reproduce all of the real networks' basic features including a high cliquishness—a possible congestion determinant. Here, one more class of networks that has experienced a rekindling of interest, the class of networks based on the gravity model [24,26–32], is introduced in order to be examined, additionally to the previous three network classes.

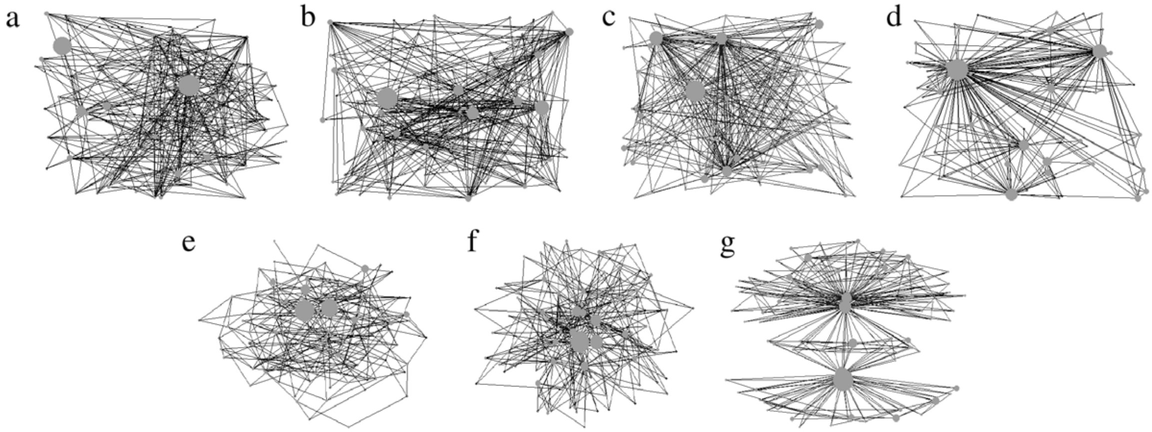

Fig. 1. Typical networks (spatially embedded) of the same size $N = 100$, identical average degree $\langle k \rangle = 6$, equal plane size and different formation model: (a) random, (b) scale-free, (c) *JR*, (d) gravity, (e) random *KK*, (f) scale free *KK*, (g) *JR KK*. The node size is proportional to the fitness value.

Table 1
Basic properties of networks[a] with $N = 100$ nodes, average degree $\langle k \rangle = 6$, fitness distribution factor $\gamma = 1$, spatially embedded in a two-dimensional plane 1000×1000.

Network type	Av. shortest path	Diameter	Clustering coefficient	Power-law PDF exponent	Av. link length
Random	2.699	5.001	0.0527	Poisson distr.	250.04
Random *KK*	2.699	5.001	0.0527	Poisson distr.	114.71
Scale-free	2.385	3.831	0.0893	−1.2741	249.94
Scale-free *KK*	2.385	3.831	0.0893	−1.2741	141.88
Jackson–Rogers	2.031	2.810	0.1093	−0.7239	514.41
Jackson–Rogers *KK*	2.031	2.810	0.1093	−0.7239	338.99
Gravity ($\phi = 1$)	1.859	2.522	0.0967	−0.7251	169.71
Gravity ($\phi = 0$)	1.848	1.905	0.0590	−1.1496	251.41
Gravity ($\phi = 2$)	2.251	3.982	0.1922	−1.1182	100.14
Gravity ($\gamma = 0.5$, $\phi = 1$)	1.848	1.905	0.0709	−0.6009	216.91
Gravity ($\gamma = 1.5$, $\phi = 1$)	2.011	3.381	0.1369	−0.9502	126.14

[a] For the estimation of the characteristics of each network, more than 20 independent realizations are averaged.

Networks generated from the gravity model make use of the Euclidean distance between nodes during network formation, while the *ER*, *BA* and *JR* models do not. As subsequent simulations make use of the Euclidean distance between nodes, a possible bias exists favoring the gravity network models. Therefore, *ER*, *BA* and *JR* networks are separately considered with random node positions and with positions assigned by the Kamada–Kawai (*KK*) spring algorithm [38]. Typical networks (spatially embedded) of all the considered formation models are illustrated in Fig. 1, with the same size $N = 100$, identical average degree $\langle k \rangle = 6$, and equal plane size. Their basic properties are presented in Table 1.

2.2.1. Random networks

In the $G(n, p)$ *ER* random model [4], a network is created by connecting nodes randomly. Each link is included in the network with probability p that is independent from every other link. Equivalently, all networks with n nodes and M links have equal probability of $p^M (1-p)^{\binom{n}{2}-M}$. Here, p is like a weighting function and as it increases from 0 to 1, the model turns more and more likely to include networks with a greater number of links. For example, in the case $p = 0.5$ all $2^{\binom{n}{2}}$ networks on n nodes are chosen with equal probability. In this paper, the ER model is modified to be spatial in a square plane of given size. Moreover, without loss of generality, in the case of the gravity-based traffic flow simulations every node randomly takes a value (fitness).

2.2.2. Scale-free networks

The *BA* model [1] generates scale-free networks using a preferential attachment mechanism. Each new node is connected to a number of existing nodes with a probability proportional to the number of links that the existing nodes already have. The probability that the new node j connects to node i is $P_{ij} = \frac{k_i}{\sum k}$, where k is the degree of the node. For compatibility with the gravity flow simulations that follow, it is deemed necessary to assign an intrinsic attribute (fitness) on every node. Therefore, the Bianconi–Barabási (*BB*) model [39] is chosen as an equivalent of the *BA* model to produce scale-free networks. This model is a variant to the *BA* model and is based on the idea of fitness which varies from node to node, allowing each node a different attraction of links in the network. It is assumed that the existence of fitness modifies the preferential attachment to compete for links. The fitness parameters of the nodes may form a distribution $\rho(\eta)$ characteristic of the studied network. The probability for a new node j to connect to node i is $P_{ij} = \frac{\eta_i k_i}{\sum \eta k}$, in which η is the fitness factor and k is the degree of the node. The *BB* model is also modified to be spatial in a square plane of given size.

2.2.3. Jackson–Rogers networks

The *JR* model [3] can be used to construct networks that successfully replicate many of the empirical regularities exhibited in real-world networks. Specifically, the model can reproduce all of the following features: (a) small shortest path length, (b) large clustering coefficient, (c) power-law degree distribution, (d) assortativity (tendency for nodes to attach to others that are similar in some way i.e. degree), (e) clustering inversely related to node degree. Nodes are born sequentially. When a new node is born, it meets some of the existing nodes through two processes. First, it meets m_r nodes uniformly at random (parent nodes). Let p_r denote the probability that a new node finds a parent node attractive to link to. Second, the new node then meets m_n of those parent nodes' immediate neighbors. The probability that the new node finds a parent's neighbor attractive to link to, is denoted by p_n. In order for the process to be well-defined upon starting, the initial network begins on a set of at least $m_r + m_n + 1$ nodes, where each node has at least $m_r + m_n$ neighbors. The *JR* model is as well modified to be spatial in a square plane of given size.

2.2.4. Gravity networks

Subsequently, the networks based on the gravity model, or the gravity networks for convenience, are generated using the algorithm proposed in Ref. [24] that maximizes the total expected traffic of the network and thus demonstrate a great interest to investigate their congestion dynamics. The gravity models are simple, intuitive spatial models, taking their name from Newton's law of gravitation and have been applied to describe the movement of people, goods, and information between geographic regions. Their ability to allow the assignment of an intrinsic attribute, called "mass" or fitness to each node and a "distance" representing the connection difficulty, makes them simple and easy network models to apply in a wide range of areas, from trade [31] to transportations [26,32,33] and telecommunications as well [27–30]. Based firstly on the estimations of the traffic flows and thereafter by connecting the node pairs with the highest expected traffic exchange, a topology can be derived. The simplest form of the gravity model is based on the equation

$$E_{ij} = M_i M_j f(d_{ij}) \tag{1}$$

where E is the expected traffic interaction, M is the fitness of each node that measures the importance of location, e.g. its population, and $f(d)$ is the distance decay factor that describes the influence of space. Empirical studies have shown that in transportation and telecommunications networks, such as highways, airlines or wide area networks, the nodes (cities) usually have a power-law distribution of fitness (population) [24,25,40]. Hence, here, the fitness M for node i is considered:

$$M_i = M_0 \cdot \text{rank}_i^{-\frac{1}{\gamma}} \tag{2}$$

where M_0 is a constant that denotes the maximum fitness value for nodes, e.g. $M_0 = 1$, and rank is a randomly assigned unique integer taking values from 1 to N (regardless the node location or degree) that describes the rank of the node if all nodes are sorted in a descending order based on their "mass" or fitness. Concerning γ, it is a constant, e.g. $\gamma = 1$, for which the higher its value is, the more uniform in fitness the nodes are.

The deterrence function f encodes the locality information specific to different types of networks and in many socio-economic systems it is well fitted by a power law:

$$f(d_{ij}) = d_{ij}^{-\phi} \tag{3}$$

where ϕ is a constant, representing the distance sensitivity parameter. In the previous studies traffic types differentiate between whether locality is a large factor or not; hence different ϕ values are chosen. For example, for Internet-related traffic ϕ is set approximately 0, for transaction traffic $\phi = 1$, for voice traffic and transportation $\phi = 2$ and for mobile traffic ϕ is greater than 2 [26,28–30].

Here, a new node j connects to node i with probability P defined by

$$P_{ij} = \frac{E_{ij}}{\sum E} \tag{4}$$

where E is estimated by the gravity equation (1).

This model is also considered in a square plane of given size.

2.2.5. Kamada–Kawai representations

It is clear from all the above characteristics that the gravity network formation makes use of distance, while the *ER*, the *BB* and the *JR* models, as introduced earlier, do not rely on distance at all. Since part of the analysis in next sections will be based on a type of traffic flow (gravity traffic) which uses distance to decide the possibility of traffic exchange, it is natural to expect a possible bias in favor of the gravity network model. Therefore, it seems appropriate to include spatially-dependent modifications of the *ER*, the *BB* and the *JR* models, in order to perform legitimate comparisons and alleviate potential concerns about the networks' equity against spatial advantages. In particular, the *ER*, the *BB* and the *JR* models are modified using the *KK* spring algorithm [38] with the purpose to place the nodes in a spatially "optimized" way, so that they will not be at an inherent disadvantage against the gravity network model. More specifically, the *KK* algorithm is a well-known force-directed algorithm usually applied for the visualization of simple undirected graphs. Using information solely contained within the structure of a graph itself, it calculates the "optimal" layout by constructing a virtual physical model and running an iterative solver to find a low-energy configuration. The approach relies on placing an ideal spring-like force between every pair of nodes such that its length is set to the shortest path distance between the endpoints. The springs push the nodes so their geometric distance in the layout is proportional to their path distance in the graph. The "optimal" layout of vertices is the state in which the positions of the nodes induce minimal total spring energy of the system. In addition to the aforementioned random (*ER*), scale-free (*BB*) and *JR* networks, the modified random *KK*, scale-free *KK* and *JR KK* are included as spatially optimal redrawings of them, in a square plane of given size. However, the focus in the following analysis is not on the evaluation of the *KK* spring algorithm. Actually, it is rather on the comparison among the topological classes, as bounded by their basic statistical properties. The use of the *KK* spring algorithm is only chosen as a convenient method of re-arranging the node locations without altering the validity of the underlying models and the non-geographical properties. As depicted in Table 1, the *KK* versions share the same structural properties with the classic networks, but show significantly lower average link length.

2.3. Analysis of the main traffic flow types

In the previous studies, when modeling the traffic on a given network, packets are generated at each time step with a given rate at homogeneously randomly selected nodes and each packet is given a random destination [7,10,13]. Usually, all the nodes are both hosts and routers and have a limited capacity of packet delivery, per time step. The capacity of packet delivery can be constant for simplicity [13,15] or proportional to either the node's degree or its betweenness centrality [10]. The packets are forwarded following a specified routing strategy. This could be the random walking [41], the shortest path [9], the efficient path [15], the next-nearest-neighbor search strategy [7], the local information [13] or the integration of local static and dynamic information [14,16]. It is also common in these models to define the network capacity measured by a critical generating rate, first presented in Ref. [42], where a transition occurs from uncongested to congested traffic flow [10,13–17,19,20]. The uncongested traffic flow or free-flow state corresponds to the state in which the numbers of created and delivered packets are balanced, while the congested or jammed state corresponds to the state in which packets accumulate on the network.

In the aforementioned traffic dynamics models, the transition to congestion is examined while increasing the generating rate. However, the main goal of this paper is rather determining the congestion tolerance as traffic flows increase than observing the phase transition. Furthermore, in traditional models each link is not restricted to handling at most a number of flows. Usually, the bandwidth of the links is neglected and no maximum capacity for bearing flows is assigned on links. Obviously, in real-world traffic networks, the capability of each link is limited and differs among links.

Thus another simple model is applied, that considers traffic flows rather than packets, similar to that introduced in Refs. [12,18,22,43]. In that traffic model it is assumed that at each time step, unit traffic flow is generated between any two nodes belonging to the same connected component and capacities are randomly assigned on the links. Every origin–destination node pair i, j demands Q_{ij} traffic flows, presumably following different paths in the network. The capacity U_{ij} on the link (i, j) is randomly assigned in the range [20–60], which shows the maximum possible crossing flows on that link. A case that all the link capacities are entirely equal is presented in Refs. [18,22], which cannot reflect the conditions in real networks, however. Costs are also put as weights on the links using a special cost function, the US Bureau of Public Roads formula [44], also used in Refs. [12,18,22,43,45]. The link cost is not a constant or a random value, but a function of the flows with congestion effects. Units of flows, one by one, follow the shortest path routing strategy (Floyd–Warshall algorithm) [46] in terms of travel cost, but as flows accumulate on the shortest path, time step after time step, congestion develops on it. Then, for subsequent flows another path becomes the shortest in terms of travel cost, which again would become congested, and so on. Since all traffic flows in each time step are to be assigned simultaneously, a game is developed among traffic flows for the selection of the feasible paths with minimum cost.

2.3.1. User-equilibrium considerations

In order to guarantee convergence to equilibrium an iterative procedure has been employed, based on the Floyd–Warshall algorithm, which ensures the compliance to the user-equilibrium (UE) conditions. This procedure distributes the traffic flows in the feasible paths by treating them as individual units of traffic flows and not as a whole of demanded traffic flow for an origin–destination pair. The traffic flow assignment for each unit requires a continuous update of the routes' costs and identification of the congested links. This procedure can cause a UE flow assignment [47]. The UE model of traffic assignment is actually based on the fact that flows choose their own route towards their destination so as to minimize their travel cost and on the assumption that such a behavior on the individual level creates equilibrium at the network level. Flows on links are said to be in equilibrium when no flow can improve its travel cost by unilaterally shifting to another route. The traffic is split across available routes in such a way that it equalizes the cost across all the available routes between a given origin–destination pair. The UE concept implies equilibrium with traffic flows served at a certain minimum travel cost, including the effects of packet queuing and with no need to specify any queue details.

This notion of equilibrium flows is generally referred to as Wardrop's principle [48] and it is commonly used for the prediction of traffic patterns in transportation and Internet-like networks that are subject to congestion. With respect to telecommunications applications, there is a rich literature summarized in Ref. [49] and references therein. For example, Internet TCP (Transmission Control Protocol) data packet flows are controlled by a simple feedback loop informing the source in the case of data loss as an indication of congestion. This information enables the source computers to discover and use selfishly the available capacity, by sharing it among different flows of packets that leads to a UE flow assignment [50,51]. It has been found that in Internet-like environments Wardrop's selfish routing achieves close to optimal average latency [52,53]. However, the UE hypothesis should be handled with caution. There are routing scenarios that may not satisfy common assumptions for the motivation of Wardrop equilibrium such as accurate knowledge of the network and its cost functions.

Formally, the UE conditions of Wardrop are here defined as follows:

$$Q_r = 0 \quad \text{if } C_r(Q) > \pi_{ij}\ \forall r, i, j \tag{5}$$

$$Q_r > 0 \quad \text{if } C_r(Q) = \pi_{ij}\ \forall r, i, j \tag{6}$$

$$\sum_{Q_r} = t_{ij} \quad \forall r, i, j \tag{7}$$

where $\pi_{ij} = \min C_r(Q)\ \forall i, j, r$ stands for the considered path, i, j denote the pair of origin and destination, Q_r represents the flow over path r, $C_r(Q)$ refers to the cost of path r and t_{ij} indicates the traffic demand for pair ij. The interpretation of conditions is that all used paths connecting an origin–destination pair ij have equal and minimal costs, denoted by π_{ij}.

The US Bureau of Public Roads cost function used is described by

$$C_{ij}(Q_{ij}) = C_{ij}^0[1 + a(Q_{ij}/U_{ij})^\beta] \tag{8}$$

where C_{ij}^0 is the uncongested travel cost from origin i to destination j, U_{ij} is the corresponding capacity of the edge, and α and β are positive constants. The minimum function value is the uncongested cost at $Q_{ij} = 0$. The initial cost on the link (i, j) is randomly assigned in the range $(0, 0.1]$. The quantities α and β are model parameters, for which the values $\alpha = 0.15$ and $\beta = 4$ are typically used [12,18,22,44,47].

When $Q_{ij} > 0$, the travel cost increases with the traffic flow. A link can accommodate traffic flows properly if its capacity is not exceeded $Q_{ij} < U_{ij}$ [12,22,47], otherwise the link is considered as congested. A congested link is supposed to have infinite cost. The cost is artificially imposed as infinite in order to inform other flows not to choose a path that includes a congested link. However, the congested links are not removed from the network; they remain functional for the traffic flows already assigned but are not available for additional assignments.

2.3.2. Random and gravity-based traffic exchange

Regarding the flow interaction between nodes in the network, it is common for the previous studies to homogeneously randomly select origin and destination nodes, which is a rather simplistic than reasonable assumption, however useful for evaluating simple networks. Nevertheless, there are limited cases, such as Refs. [17,20], where *BA* scale-free unweighted networks are studied, with the probability of generating/receiving traffic proportional to the degree of the nodes. In addition to the standard paradigm of traffic exchange, the approach followed in this paper is again based on the gravity model perspective that allows for a realistic traffic generation for the whole network.

From a modeling viewpoint, gravity models have long been used to model flows in spatial interaction networks, in order to estimate the level of traffic in each link of a network [34,35] on a macroscopic scale. These models focus on the intensity of interaction between origin and destination nodes separated by a certain distance (financial or temporal cost). It has been shown that for human migration, international trade, transportation or communications between cities the volume of traffic between two population centers i and j is successfully modeled by Eq. (1). For example, in Ref. [27] it is shown that inter-city communication intensity between two cities is proportional to the product of their sizes divided by the square of their distance; thus it is characterized by a gravity model. Besides, in Ref. [26] both public and private transportation traffic flows between cities obey the gravity equation. In Ref. [33] the estimation of origin–destination flows distribution also forms a gravity model. As well, the concept of gravity laws has a long tradition in the field of economic geography. For instance, in the multi-regional core–periphery (CP) model, the decay of the agglomeration/dispersion forces with distance (i.e. these forces are spatially discounted) is common to be described by gravity laws, as in Ref. [54].

Here, gravity-based traffic flows are generated in each time step, in the sense that every unit traffic flow is exchanged between node pairs, origin i and destination j, with probability P defined by Eq. (4).

3. Results and discussion

Finally, four classes of networks are considered here; the random, the scale-free, the *JR* and the gravity one. In addition, three redrawings of random, scale-free and *JR* networks, the random *KK*, the scale-free *KK* and the *JR KK* are considered in the analysis. All of them are adjusted to be spatial in a two-dimensional plane (1000×1000) by assigning coordinates to each node $\{x_i, y_i\}_{i=1,\ldots,N}$ and all of them include nodes that carry a fitness value $\{f_i\}_{i=1,\ldots,N}$. The locations of the nodes in the

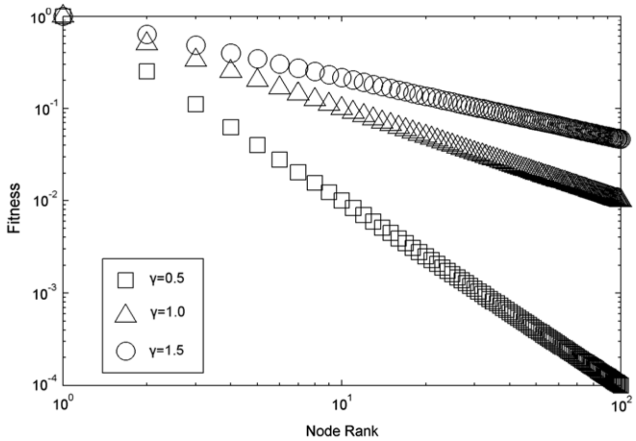

Fig. 2. Fitness distribution for different fitness distribution factors.

plane are randomly chosen in every realization of the simulations. However, for the *KK* network versions, the node locations are "optimally" chosen using the *KK* algorithm. The constructed networks have size $N = 100$ nodes, average degree $\langle k \rangle = 6$, and dynamic weights to describe the cost, the capacity and the traffic flows on the links. Only the nodes that consist of the largest component are taken into account to ensure a single connected cluster of size N. While the size of the considered networks is quite small because of computational complexity and constraints, this is a moderately sufficient size to extract statistically significant results. Throughout the paper, for the characteristics of each network, more than 20 independent realizations are averaged.

3.1. Quantities of interest and parameter values

The primary goal of the following simulations is to understand the influence of the topology – random, scale-free, *JR* and gravity networks – on the dynamics of traffic flow. Another goal is to explore the tolerance of topologies to congestion for different traffic flow models—the random and the gravity-based one. Thus, the simulations focus on two quantities of interest that can characterize the network performance and the cost budget; the congestion factor, J, and the total network cost, *TNC*.

The congestion factor J is defined as the percentage of congested links out of the total links, as introduced in Ref. [12]:

$$J = \frac{T}{S} \tag{9}$$

where T is the total number of congested links and S is the total number of links in the network. Obviously, $J = 0$ corresponds to uncongested traffic on the network, and $J = 1$ indicates the worst case of network congestion.

Total network cost is discussed in both [18,43] as total system cost but here it is modified to include the link length as a cost parameter, which is important in the case of spatial traffic networks:

$$TNC = \sum_{i,j} \left(Q_{ij} \times C_{ij}(Q_{ij}) \times L_{ij} \right) \tag{10}$$

where Q_{ij} is the link flow on the link, $C_{ij}(Q_{ij})$ is the cost of flow on the link and L_{ij} is the length of the link on the path that connects the two nodes i and j.

In the following, the simulations are performed by considering a variety of fitness distributions and distance functions. Three different power-law distributions are chosen for fitness assignment as they are very frequent in physical and social systems. There are constructed and compared networks for $\gamma = 0.5$, $\gamma = 1.0$ (Zipfian distribution) and $\gamma = 1.5$, which are approximately the lower, the average, and the upper limit of γ, respectively, in the case of city population distributions [40]. These fitness distributions are illustrated in Fig. 2. The behavior of J and *TNC* is also examined for three indicative values of ϕ, namely 0, 1, and 2. The distance decay factor for these ϕ values is presented in Fig. 3.

3.2. Differences among network models and between traffic models

Both the standard paradigm of uniform traffic volumes between randomly interacting node pairs (random traffic) and the more realistic gravity-based interactions (gravity traffic) are depicted in Fig. 4. It is worth noting that in all network types the congestion factor becomes stationary for large volumes of traffic Q. It is obvious that random, scale-free and *JR* networks are more prone to suffering from congestion than gravity ones. In agreement with other findings [12,22] scale-free networks are equivalently – or more – susceptible to traffic congestion compared to the random ones when the total system flow Q is

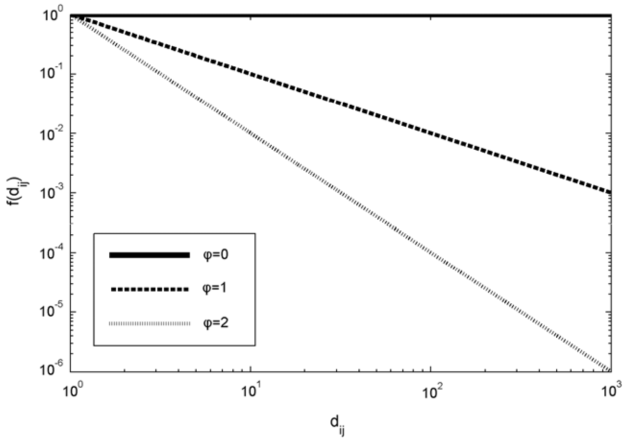

Fig. 3. The distance decay factor $f(d_{ij})$ for different ϕ values against distance d_{ij}.

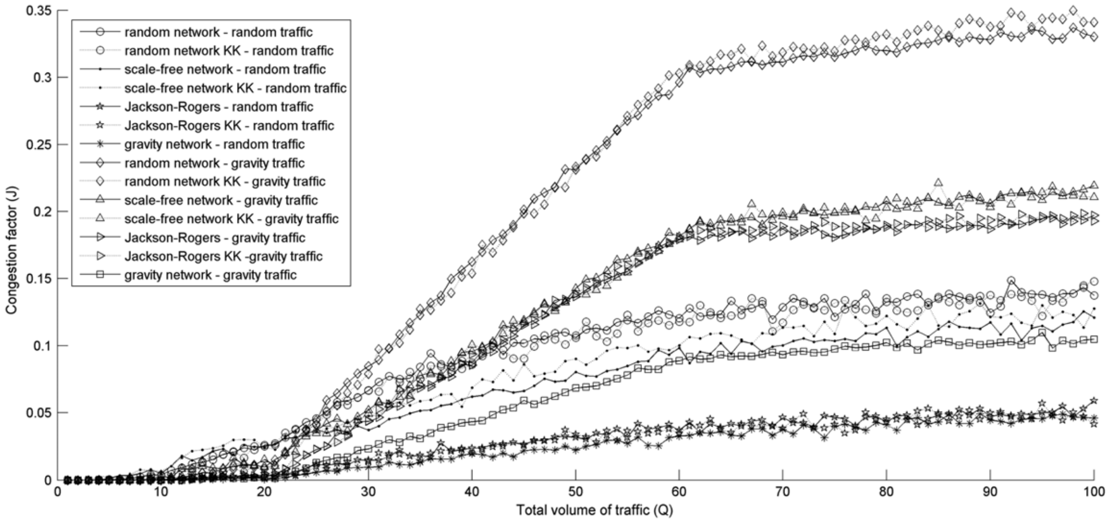

Fig. 4. Congestion factor J as a function of total volume of traffic Q for random, random *KK*, scale-free, scale-free *KK*, *JR*, *JR KK* and gravity networks for both random and gravity-based traffic patterns. Each curve corresponds to an average over 20 independent realizations of the networks with $N = 100$, $\langle k \rangle = 6$. It is assumed for all networks' fitness distribution: $\gamma = 1$, for the construction of gravity networks: $\phi = 1$, and for the gravity-based traffic flows: $\phi = 1$.

small. The reason for this lies in the fact that in scale-free networks edges connected to hub nodes are assigned heavier flows, which leads to the congestion on these edges easily at the beginning. However for random networks, flows assigned to each edge are relatively even. The *KK* versions of random and scale-free networks do not show big differences from the classic ones neither under random traffic, nor under gravity traffic. The *JR* networks are found quite resistant to congestion, more than the scale-free ones but less (or marginally less) than the gravity ones. In the meanwhile, the gravity networks can support all volumes of traffic with considerably lower congestion factor. This may be explained by the structural properties of the derived gravity networks; they are neither uniform, nor power-law distributed. Although gravity networks can reproduce a scale-free behavior under particular circumstances, the spatial constraints can make the network more homogeneous and its degree distribution deviate from the power-law form [24,26], allowing for a more distributed flow exchange. This is additionally confirmed by the statistical properties of gravity topologies as shown in Table 1 indicating small average shortest path, small diameter and high clustering coefficient, respectively. Apparently, the congestion degree is more severe when the gravity-based patterns are used to simulate the flows in the network.

A similar illustration in Fig. 5 also applies for the total network cost. It is as well apparent that the gravity network topology is more ''economic'' than the other topologies, mostly because of both small J and average link length (see Table 1). Although, the cost of the *JR KK* for the random traffic scenario can well compete with the cost of the corresponding gravity network. This may be due to a more even traffic distribution in the *JR KK*, since the *JR KK* is associated with a higher clustering coefficient which means more alternative paths and overcoming the larger average link length. Generally, the lower values in the average link length of *KK* versions are also responsible for the lower cost in the *KK* versions of random, scale-free and *JR* networks, as opposed to the native models' cost. Concerning the total network cost in the case of gravity traffic for

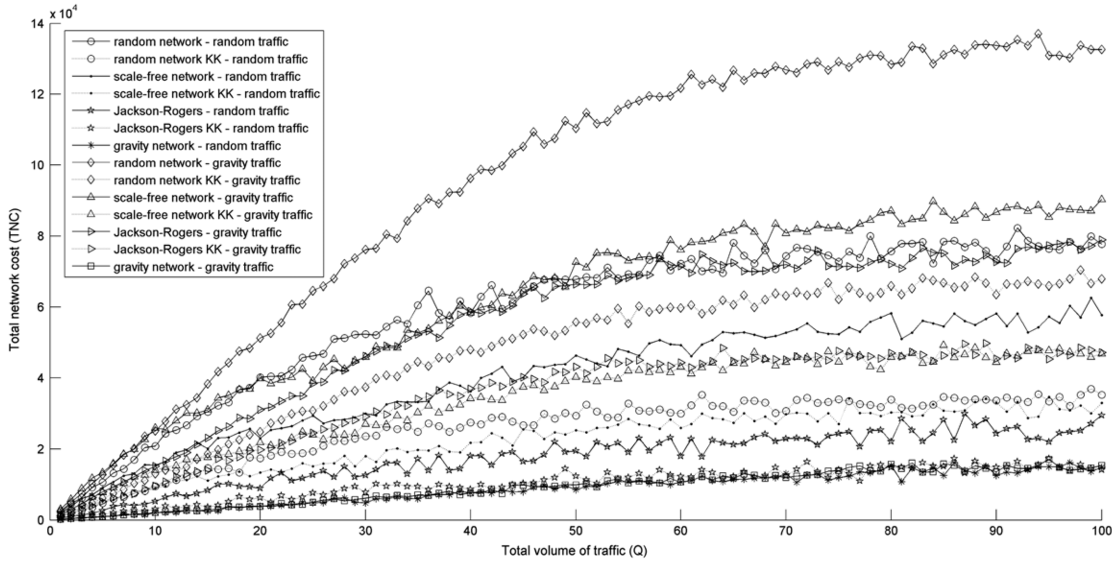

Fig. 5. Total network cost TNC as a function of total volume of traffic Q for random, random *KK*, scale-free, scale-free *KK*, *JR*, *JR KK* and gravity networks for both random and gravity-based traffic patterns. Each curve corresponds to an average over 20 independent realizations of the networks with $N = 100$, $\langle k \rangle = 6$. It is assumed for all networks' fitness distribution: $\gamma = 1$, for the construction of gravity networks: $\phi = 1$, and for the gravity-based traffic flows: $\phi = 1$.

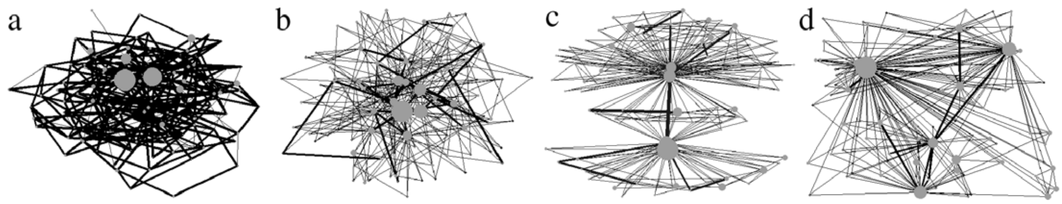

Fig. 6. Typical networks under random traffic flows: (a) random *KK* topology, (b) scale-free *KK* topology, (c) *JR KK* topology, (d) gravity topology. The node size is proportional to the fitness value and the thick edge width indicates congestion.

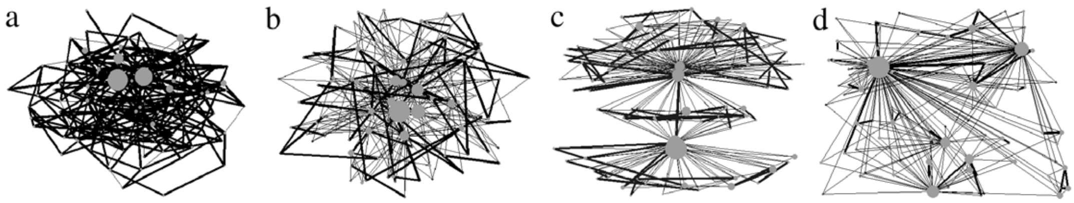

Fig. 7. Typical networks under gravity traffic flows: (a) random *KK* topology, (b) scale-free *KK* topology, (c) *JR KK* topology, (d) gravity topology. The node size is proportional to the fitness value and the thick edge width indicates congestion.

random, scale-free and *JR* networks, it appears to be quite higher than under random traffic flows. However, in the case of the gravity network the total network cost remains at the same level despite the modification in the traffic patterns. It is clear from the above findings that the more realistic gravity-based traffic patterns induce different tolerance to congestion in all topologies, with the gravity topologies again suffering less from congestion while retaining the lowest total network cost (see Figs. 6 and 7).

3.3. The role of clustering coefficient

The findings indicating the superiority of gravity network formation model may, though, be justified to some extent by the model's basic properties. As supported by Watts and Strogatz in Ref. [2], the network structure influences the functionality of processes running over the topology, such as the speed and extent of transmission. Moreover, in the related literature, the high clustering coefficient of gravity networks (see Table 1) has been already noticed and associated with the ability to overcome traffic congestion. More specifically, the network model in Ref. [55], which is based on the assumption that the benefit of a link is decreasing in the distance between two agents, is actually a form of gravity law. In that paper it is

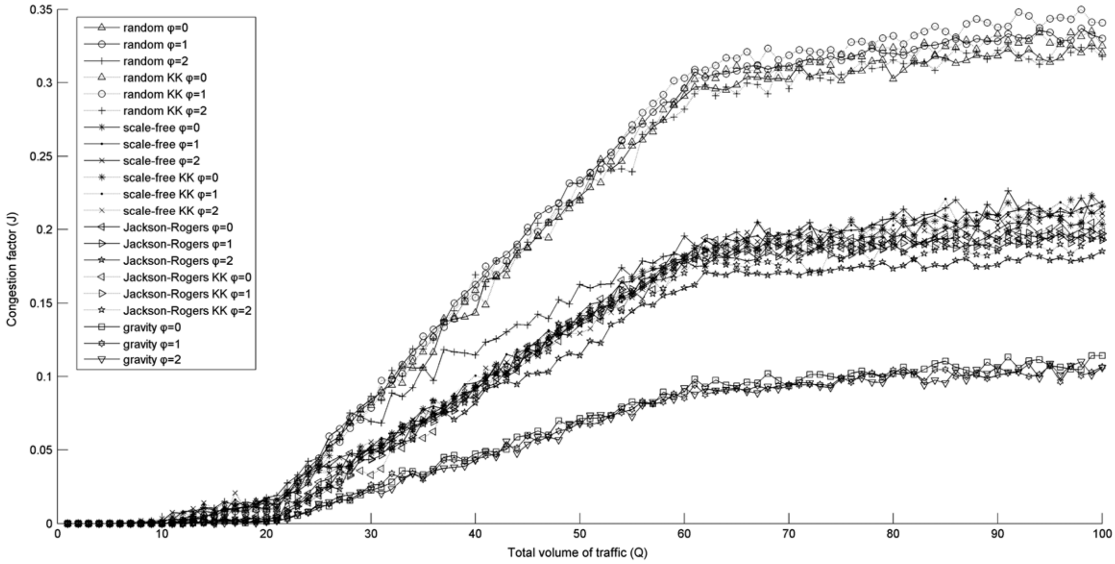

Fig. 8. Congestion factor J as a function of total volume of traffic Q for random, random *KK*, scale-free, scale-free *KK*, *JR*, *JR KK* and gravity networks for gravity-based traffic flows with different values of ϕ. Each curve corresponds to an average over 20 independent realizations of the networks with $N = 100$, $\langle k \rangle = 6$. It is assumed for all networks' fitness distribution: $\gamma = 1$ and for the construction of gravity networks: $\phi = 1$. Curves for random, scale-free, *JR* and gravity networks appear to overlap.

found that the high clustering coefficient of a co-authorship network can be explained by the suggested gravity-like concept. Furthermore, authors in Ref. [56] discover a high clustering coefficient in world trade networks which are there artificially constructed on the basis of gravity laws. Irrefutably, the high clustering is not a characteristic limited to the gravity networks, but a well observed universal feature representing the cliquishness of most real-world networks, as primarily discussed in Refs. [2,3]. Therefore, and since clustering is expected to be correlated to the congestion level, as pointed out in Ref. [12], it is useful to consider the extent of the effect of the high clustering in the resulting congestion factor. Specifically, in Ref. [12] networks of different clustering coefficients are generated and their congestion factor is found to be inversely correlated with the clustering coefficient. Additionally, it is there witnessed that the scale-free networks have a lesser congestion factor than small-world networks for the same clustering coefficient.

However, the measurements in this paper do not fully confirm these outcomes, on the grounds that the present analysis goes beyond the random traffic scenario which has been a main assumption in Ref. [12]. Indeed, as observed in Fig. 4 and Table 1, in the case of random traffic simulations, the congestion factor is approximately inversely correlated with the clustering coefficient; the higher the clustering coefficient, the lower the congestion factor (with the slight exception of *JR* networks which have roughly the same congestion factor as gravity networks have). Nevertheless, in the case of gravity traffic simulations, this does not hold true; for instance, even though the *JR* model and its *KK* version both have the highest clustering coefficient among the networks presented in Fig. 4, the *JR* congestion factor under gravity traffic type appears sufficiently higher than the gravity networks with similar clustering coefficient. It is particularly observed that under the more realistic scenario of gravity traffic type, the gravity networks sustain an adequately lesser congestion factor than scale-free and *JR* networks for approximately the same clustering coefficient ($C \approx 0.1$). Moreover, this finding against correlation between clustering coefficient and congestion level under gravity traffic type is corroborated with the gravity networks constructed with parameter $\varphi = 2$. Despite the fact that they have the highest clustering coefficient (Table 1) among the investigated gravity network cases with $\varphi = 0, \varphi = 1$, they yet seem to have a rather higher congestion factor than the compared networks (see Fig. 10 later in the paper). Drawing to a conclusion, the above observations confirm that the clustering coefficient does correlate with the congestion level, but not strictly and only for the random traffic type scenarios. Instead, results indicate that under gravity traffic type scenarios, such a correlation is not a trivial one. Last but not least, it is notable to mention the emergence of a low congestion level correlation with both low average shortest path and low diameter measurements (see Table 1), with these being actually some well observed properties of real-world networks.

3.4. Dependence on distance parameter

When examining the dependence of J and *TNC* on the distance decay function parameter ϕ of the gravity traffic flows, as seen in Figs. 8 and 9, there is no great influence generally. The locality of the gravity traffic patterns is discovered not to act upon both the congestion factor and the total network cost. The only exception lies in the case of $\phi = 2$ for random topologies which actually results in lower J and *TNC*, resembling the congestion tolerance behavior of scale-free topologies. This may be explained by the large locality factor that represents a tendency for higher flow interaction between geographical neighbors.

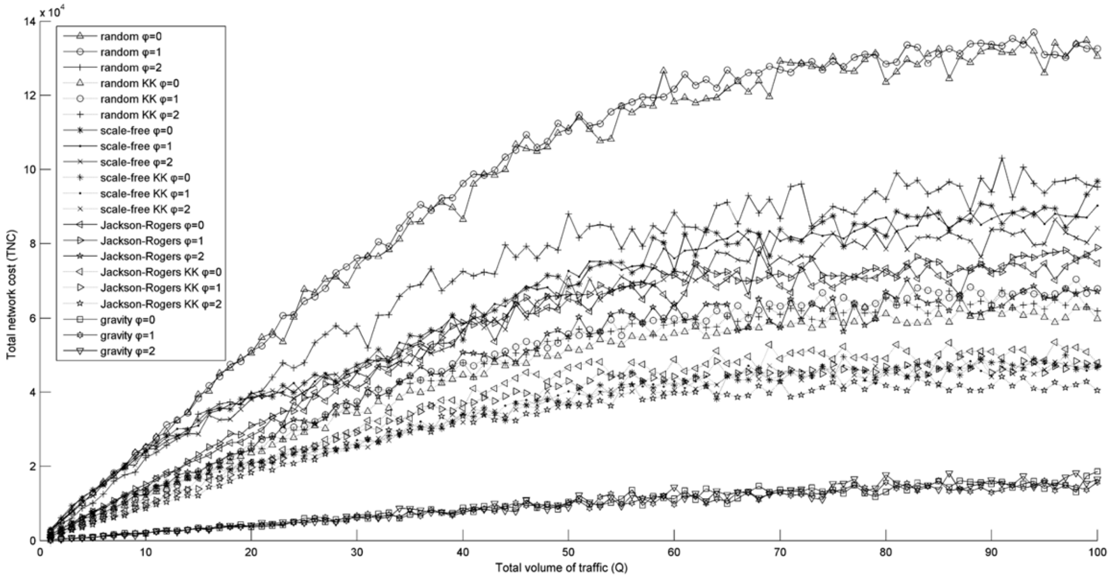

Fig. 9. Total network cost *TNC* as a function of total volume of traffic Q for random, random *KK*, scale-free, scale-free *KK*, *JR*, *JR KK* and gravity networks for gravity-based traffic flows with different values of ϕ. Each curve corresponds to an average over 20 independent realizations of the networks with $N = 100$, $\langle k \rangle = 6$. It is assumed for all networks' fitness distribution: $\gamma = 1$ and for the construction of gravity networks: $\phi = 1$. Curves for random, scale-free, *JR* and gravity networks appear to overlap.

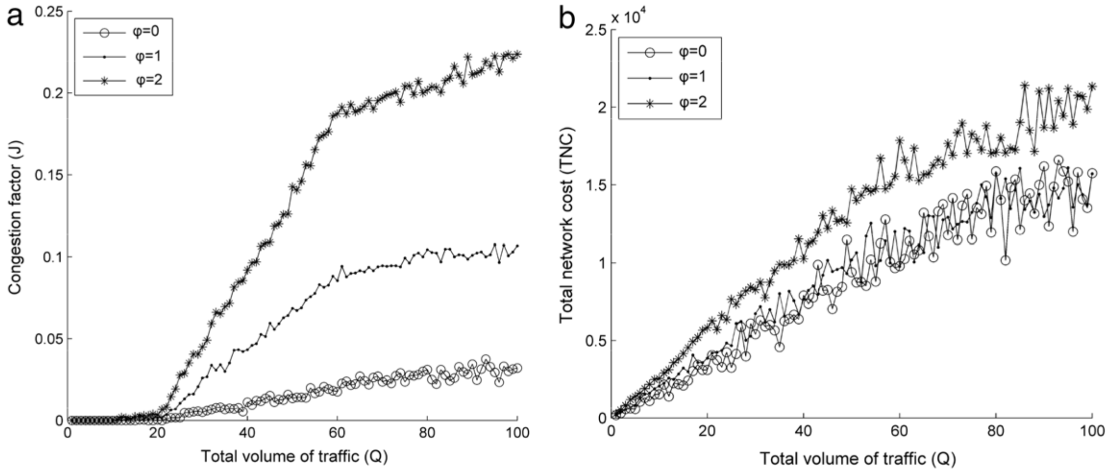

Fig. 10. (a) Congestion factor J as a function of total volume of traffic Q for gravity networks constructed with different values of ϕ and under gravity-based traffic flows. (b) Total network cost *TNC* as a function of total volume of traffic Q for gravity networks constructed with different values of ϕ and under gravity-based traffic flows. Each curve corresponds to an average over 20 independent realizations of the networks with $N = 100$, $\langle k \rangle = 6$. It is assumed fitness distribution: $\gamma = 1$ and for traffic flows: $\phi = 1$.

Indeed, this behavior is not visible in the random *KK*, where the node locations are spatially "optimized". Thus, congestion is probably limited to occur only partially at certain paths that connect geographical neighbors excluding the burden of paths that connect geographically distant nodes.

3.5. Adjustment of the gravity model parameters

By all means, further exploring how the more optimal gravity topologies perform under the more realistic gravity traffic patterns is of considerable importance. Under gravity-based traffic flows, the construction parameter ϕ of the distance decay function is observed to be especially sensitive to traffic congestion, as seen in Fig. 10, where gravity topologies are generated using a series of ϕ values. For the traffic flows the ϕ value is fixed to 1, but as seen earlier in Figs. 8 and 9 it does not influence strongly the considered indices. It is found that the larger the construction parameter ϕ is, the higher the congestion factor J is. This outcome is expected since a high ϕ represents a strong interaction between geographical neighbors due to spatial constraints. As a result, many short links are constructed, as confirmed in Table 1, which are congested easily in the beginning

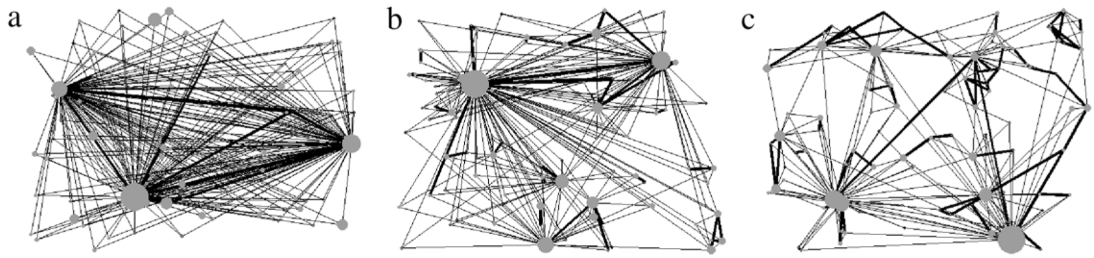

Fig. 11. Typical gravity networks under gravity traffic flows: (a) gravity topology $\phi = 0$, (b) gravity topology $\phi = 1$, (c) gravity topology $\phi = 2$. The node size is proportional to the fitness value and the thick edge width indicates congestion.

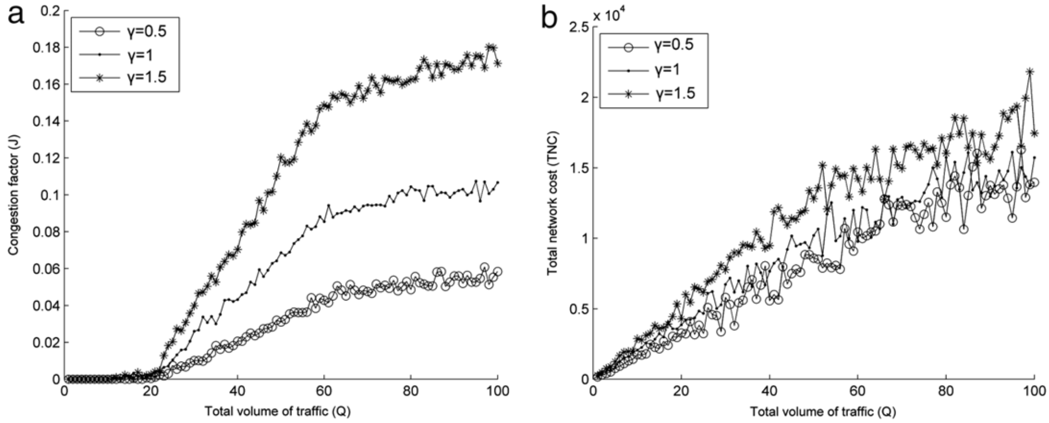

Fig. 12. (a) Congestion factor J as a function of total volume of traffic Q for gravity networks constructed with different values of γ and under gravity-based traffic flows. (b) Total network cost *TNC* as a function of total volume of traffic Q for gravity networks constructed with different values of γ and under gravity-based traffic flows. Each curve corresponds to an average over 20 independent realizations of the networks with $N = 100$, $\langle k \rangle = 6$. It is assumed for network topologies: $\phi = 1$ and for traffic flows: $\phi = 1$.

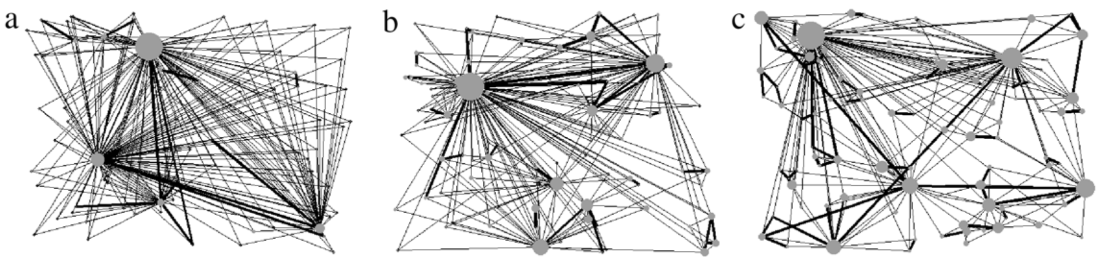

Fig. 13. Typical gravity networks under gravity traffic flows: (a) $\gamma = 0.5$, (b) $\gamma = 1$, (c) $\gamma = 1.5$. The node size is proportional to the fitness value and the thick edge width indicates congestion.

(see Fig. 11(c)). This finding is also in accordance with the larger average shortest path and diameter, as observed in Table 1, meaning a longer travel path for traffic flows. Regarding the total network cost, it appears to be not rather sensitive to the distance sensitivity parameter ϕ of the distance decay function. Despite the large differences in the behavior of J, there are no great deviations in *TNC*. In particular, for low ϕ values there is an overlap on the *TNC* as traffic volume increases, while for $\phi = 2$ there is a slight differentiation, probably due to the larger number of congested edges.

Modifying the fitness distribution factor γ turns out to be particularly sensitive to traffic congestion. Here, the gravity topologies are generated with different fitness distribution factor γ. Fig. 12 shows the relationship between the fitness distribution factor γ and the congestion factor J. It can be noticed that a larger γ corresponds to a higher congestion factor. A large γ actually implies a more uniform fitness distribution and therefore a more distributed and homogeneous interaction between almost equivalent nodes in terms of fitness (see Fig. 13(c)) provoking the construction and utilization of shorter links. This leads to a larger average shortest path and diameter – already pointed out in Table 1 – that contribute in causing intense congestion. On the other hand, a small γ indicates a topology with a more centralized concept consisting of a few hubs and many non-hubs with many alternative multi-hop paths (see Fig. 13(a)). Regarding the total network cost, it appears to be relatively insensitive to traffic congestion. Again, there are no great deviations in *TNC*.

Shortly, the abovementioned scenarios of varying the ϕ or the γ parameters provide insights for the congestion dynamics in different assumptions of gravity-based topologies along with gravity-based traffic patterns. As a result, the congestion level depends on the distance decay function of the gravity equation, that is to say on the spatial details. In addition, the congestion appears to depend on the nodes' fitness distribution factor, in other words on the nodes' attributes details. For example, a gravity topology connecting nodes following a fitness distribution with a low factor γ and with the presence of a distance decay function with a low parameter ϕ is expected to be extremely tolerant to congestion. On the other hand, cases of gravity topologies constructed over nodes of more uniform fitness distribution and of larger locality factor are anticipated to be more prone to congestion. Lastly, the parameter ϕ of the distance decay function of the gravity traffic flows is found to be almost irrelevant to the deviations of the congestion factor. Drawing to a conclusion, with regard to real-world traffic networks, such as networks carrying Internet traffic flows which appear to obey gravity patterns, the construction of a gravity topology with $\phi = 0$ is optimally expected. This structure is found to be associated with the lowest congestion factor and the minimal total network cost, as defined in the present paper.

4. Conclusions

The influence of the network topology on the congestion dynamics under different network types and traffic flow patterns has been addressed and discussed. The class of gravity networks, recently attracting research attention again, is investigated and in contrast to the standard paradigm of randomly interacting node pairs, more realistic gravity traffic patterns are used to simulate the flows in the network. It is shown that depending on the traffic pattern, the networks have different tolerance to congestion, with the gravity traffic causing more severe congestion to all networks. Moreover, the study demonstrates that the topologies created on the basis of the gravity model suffer less from congestion than the random, the scale-free or the *JR* ones, plus at a lower cost. The congestion level is found to be approximately correlated with the network clustering coefficient in the case of random traffic, whereas in the case of gravity traffic such a correlation is not a trivial one. Other basic network properties such as the average shortest path and the diameter are seen to correlate fairly well with the congestion level. Further investigation on the adjustment of the gravity model parameters indicates particular sensitivity to the traffic congestion, whereas only minor sensitivity to the total network cost. Although the results may not generalize beyond the UE routing scheme, they yet represent a first attempt to include gravity networks and gravity traffic flows in the network congestion analysis. The findings of this work could be exploited by traffic network designers in order to construct cost efficient networks with a congestion minimization functioning. Studying the underlying communicating populations and the unevenness of their interaction may lead to a network topology based on the expected traffic exchange, i.e. gravity-like, which subsequently may lead to low congestion effects and low total cost. In future, a more exhaustive observation of the gravity network statistical properties would unveil the explanation for their dominance under congestion.

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