

Bounds on the Mixing Time and Partial Cover of Ad-Hoc and Sensor Networks

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Abstract—In [1], the authors proposed the partial cover of a random walk on a broadcast network to be used to gather information and supported their proposal with experimental results. In this paper, we demonstrate analytically that for sufficiently large broadcast radius r , the partial cover of a random walk on a random broadcast network is in fact efficient and generates a good distribution of the visited nodes. Our result is based on bounding the conductance, which intuitively measures the amount of bottlenecks in a graph. We show that the conductance of a random broadcast network in a unit square is $\Theta(r)$ with high probability, and this bound allows us to analyze properties of the random walk such as mixing time and load balancing. We find that for the random walk to be both efficient and have a high quality cover and partial cover (i.e. rapid mixing), radius at least $r_{\text{rapid}} = \Theta(1/\text{poly}(\log n))$ is sufficient and necessary. Experimental results on the random geometric graphs, namely graphs that represent broadcast networks, that resemble the conductance of the 3-dimensional grid indicate that the analytical bounds on efficiency, namely cover time and partial cover time, are not tight. In particular, $r = \Theta(1/n^{1/3})$ is sufficient radius to obtain optimal cover time and partial cover time.

I. INTRODUCTION

The task of information collection and processing over wireless sensor networks is one of the main challenges in such domains [2] [3] [4]. The strict energy constraints [5] and the dynamics of the network (caused by node mobility, node failure, unreliable communication, etc.) prevent in many cases adopting traditional solutions from related areas like ad-hoc networks and distributed databases. Many such systems rely on state information stored at the nodes and global data structures for proper operation (for example, pointers to cluster heads, routing tables, and spanning trees). As such, those types of systems have critical points of failure and in dynamic environments must adopt failure recovery mechanisms

which impact significantly on the overall performance.

In recent years emphasis has been shifting from deterministic algorithms to randomized algorithms which are often simpler and fast. In particular, we can consider the simple random walk for information processing in a sensor network due to the simplicity of the process, robustness to failures (no critical points of failure), minimal overhead, and locality of computation. This approach is gaining credence and popularity, as recently several authors [1], [6], [7] have considered random walk approaches in various networking settings.

The authors [1] proposed the *partial cover* of a random walk on a sensor network and detailed how it can be used for query processing. As the *cover time* [8] is the expected number of steps taken by a random walk to visit every node in the network, the *partial cover time* (PCT) is the expected time required to cover only a constant fraction of the network. In sensor network applications, for most tasks, it is not necessary to consult every node in the network. In fact, the authors had shown, derived from the well known Matthews bound [9], that the *partial cover time* is up to a factor of $O(\log n)$ times more efficient than the *cover time*. Substantiated by experimental results we showed that the partial cover of 80% of the network is in fact efficient in the number of messages in comparison to other systems and still maintains all the other nice properties of random walks mentioned above.

Kempe *et al.* [6] proposed and analyzed parallel random walk techniques for gossiping in networks. Their approach differs from the [1] approach in the nature of what is expected of the data collection task. The former is concerned with disseminating global information throughout the network to be stored in every node (all-to-all), whereas the latter is concerned with answering a query regarding the global status of the network by the random walk of a single token (all-to-

one). While the [6] approach is much more time-efficient in comparison to the [1] approach, it is energy-inefficient for use in a sensor network or broadcast network due to a number of factors: the total number of messages sent, the correspondingly large associated interference, and the girth of information stored at each node. Therefore, it is necessary to consider the particular nature of the network and the data collection task.

Since then, Gkantsidis *et al.* [7] have proposed random walk techniques for peer-to-peer networks based on the fact that random walks are especially efficient when the underlying topology of the network is an expander [10]. As such, they have advocated expander construction in peer-to-peer network design and random walks to perform search. One nice property of expanders is that they give optimal rate of convergence to stationary distribution, namely mixing rate, due to the smallness of the second largest eigenvalue λ_1 of the weighted transition matrix associated with the underlying network topology. As shall be discussed, λ_1 is essentially related to other relevant and desirable properties of random walks including partial cover time and load balancing.

In this paper, as we are concerned with all-to-one data collection in a sensor or broadcast network, we expand upon our previous work [1]. In particular we investigate the efficiency and quality of the random walk as a function of the broadcast radius and size of the network. Delineating such a relationship is beneficial for network designers to either set the broadcast radius accordingly or to otherwise know what random walk properties are to be expected given their network's broadcast radius and size.

II. OVERVIEW OF THE MODELS, METHODS, AND RESULTS

A well-known model representing sensor networks (or, in general, broadcast networks) and their underlying Markov chain is the class of *geometric graphs* (also known as *Unit Disc Graphs* (UDG) [11]). A *geometric graph* $G(n, r)$ is a graph of n nodes with the property that the nodes may be embedded into the Euclidean plane such that an edge exists between a pair of nodes iff the nodes are within distance r of each other. Let $\mathcal{G}(n, r)$ be a *Random Geometric Graph* (RGG), a graph constructed as follows: Place n nodes uniformly at random in a unit squared area and then connect every pair of nodes at Euclidean distance less than or equal to r . While these graphs have traditionally been studied in relation to subjects such as statistical physics and hypothesis testing [12], random geometric graphs have gained new

relevance with the advent of ad-hoc and sensor networks as they are a model of such networks¹.

Note that, on the one hand, a 2-dimensional grid network in the unit square is a geometric graph of borderline radius to ensure connectivity. And, on the other hand, the complete graph K_n in the unit square is a geometric graph with maximal density and connectivity. As we shall see, the 2-dimensional grid lacks many desirable random walk properties (such as rapid mixing and optimal cover time), whereas the complete graph behaves optimally with respect to such random walk properties. It is easy to see that by increasing r of a geometric graph we are increasing the connectivity and the degree of the nodes and can shift from the grid to the complete graph. Then, intuitively, a question to ask is, do we need to increase r to *maximum* (i.e. $\sqrt{2}$) in order to get these properties, or, rather, is there a more continuous relationship between r and these properties? This question was a primary motivation in investigating the relationship between the radius and the mixing time of a random geometric graph.

As interference and energy grow with increased radius, in particular, we wonder what are small radii for which we may yet obtain good, even optimal, random walk properties. Of course, such bounds on the radii depend on the particular properties desired, and the particular properties desired are related to the following important and relevant questions to consider for a random walk approach in the context of global data collection over broadcast networks:

- 1) How long should we wait for the random walk token to collect information from all of the nodes?
- 2) How long should we wait for the random walk token to collect information from a constant fraction c of the nodes?
- 3) Does the random walk have good load balancing properties?
- 4) What is the quality of a random walk (measured by the distribution of visited nodes) after some t number of steps?
- 5) What is the quality of the partial cover after having collected information from a constant fraction c of the nodes?

Question 3 exactly concerns the *mixing rate*, the rate of convergence to stationary distribution of the random walk over $\mathcal{G}(n, r)$. For regular graphs, the more rapidly

¹Although we can think of RGGs in any number of dimensions, in this work we assume only the two dimensional case as that has direct relevance to ad-hoc networks.

mixing the random walk, the more uniformly the nodes are sampled, and thus the more balanced is the load. Intuitively, the fewer the bottlenecks of the random geometric graph, the more rapidly mixing the random walk is. *Conductance* is a measure of bottlenecks that is particularly appealing geometrically and which is directly related to the second eigenvalue λ_1 of the weighted transition matrix [13]. We analytically demonstrate that for sufficiently large n and r , the conductance Φ of $\mathcal{G}(n, r)$ is $\Phi(\mathcal{G}(n, r)) = \Theta(r)$ with high probability, and we give a useful continuous approximation to Φ . Based on the conductance results, we show that for $\mathcal{G}(n, r)$ to be rapidly mixing, radius at least $r_{\text{rapid}} = \Theta(1/\text{poly}(\log n))$ is necessary and sufficient. The idea of using conductance to bound the mixing time of broadcast networks was recently independently proposed in yet unpublished manuscript [14].

Questions 1 and 2 exactly concern the cover time C and, respectively, partial cover time PCT of $\mathcal{G}(n, r)$. Since the random walk should be run C steps or PCT steps to be expected to have collected information from all or almost all of the nodes, we investigate for which broadcast radii r we obtain small C or PCT . Unfortunately, from analytical results based on the conductance we are only guaranteed optimal C and PCT for constant radius r . However, by observing that the k dimensional mesh for $k > 2$ also has optimal C and PCT though it is not even rapid mixing, we hypothesized that perhaps the $\mathcal{G}(n, r)$ with conductance similar to that of the 3-dimensional mesh may also have optimal C and PCT . Since conductance alone is an insufficient measure to analytically tightly bound C and PCT , our hypothesis was also based on the observation that, in the case of random geometric graphs, increase in conductance directly implies increase in average degree and decrease in the hop-diameter of the network as well. Therefore, we formalize the notion of *resemblance* of random geometric graphs to other graphs based on the similarity in conductance, which immediately implies some other comparisons as well. In fact, our experimental results show that the $\mathcal{G}(n, r)$ that resembles the 3-dimensional mesh has C and PCT even lower than that of the 3-dimensional mesh. Therefore, by experimental results we have that radius $r = \Theta(1/n^{1/3})$ is sufficient for optimal C and PCT . It should also be noted that experimental results indicate that the $\mathcal{G}(n, r)$ that resembles the hypercube has slightly better mixing rate and PCT than that of the hypercube. This further demonstrates the usefulness of the notion of resemblance as much is already known about certain graphs such as the grids and hypercube.

For the last two questions, we measure the quality of our random walk in terms of how small a *hole*, namely a contiguous unvisited area, remains after a given number of steps. Such a characterization is eminently sensible due to the geometric characterization of random geometric graphs. We approximate the hole size by its upper bound as the maximum minimum Euclidean distance from unvisited nodes to visited nodes after some number of steps. And, so the last two questions exactly concern how small a hole remains after some amount of time. Our experimental results indicate that the quality of the random walk is significantly improved by rapid mixing. This intuitively makes sense as the quality of a random walk is related to load-balancing properties. Thus, although for efficiency radius $r = \Theta(1/n^{1/3})$ suffices, to achieve both efficiency and high quality geometric distribution of visited nodes, radius $r = \Theta(1/\text{poly}(\log n))$ is recommended.

Note that a high quality partial cover may not be necessary depending upon the nature of the data collection task. For example, to collect a majority vote, with very high probability after 90% PCT , the majority vote will have been obtained regardless of the distribution quality. Another example is searching the network. However, for a temperature sensing system, for example, the quality of the random walk may be eminently important.

III. PRELIMINARIES

A. Markov chains and the Simple Random Walk

The probabilistic rules by which a random walk operates is defined by the corresponding *Markov chain*. Let $\mathfrak{M}$ be a Markov chain over state space Ω and probability transition matrix P (i.e $P(x, y)$ is the probability to move from x at time t to y at time $t + 1$). In such terms, the stationary distribution of $\mathfrak{M}$, if such exists, is then defined as the unique probability vector π such that

$$\pi P = \pi$$

A primary motivation in considering a random walk approach as opposed to a deterministic protocol is simplicity and locality of computation. So, if the random walk is currently at node q , then the simplest probabilistic rule by which to choose the next node is simply to choose a node uniformly at random from among the set of neighbors of q . We call the Markov chain $\mathfrak{M} = (\Omega, P)$ corresponding to such a random walk the *simple random walk*. Note that we may just as well define such $\mathfrak{M}$ by its underlying graph $G = (V, E)$. For such G , for any node $v \in V$, let $\delta(v)$ denote the degree of v , that is the

number of neighbors of v in G and let $P(v, u) = \frac{1}{\delta(v)}$ for $(v, u) \in E$ and 0 otherwise. It is well known that the simple random walk $\mathfrak{M} = (\Omega, P)$ over a connected graph $G = (V, E)$ has a stationary distribution π such that, for any node $q \in V$ [15],

$$\pi(q) = \frac{\delta(q)}{2m} \quad (1)$$

where $m = |E|$. Further, when the underlying graph G is regular, that is when there is d such that for all q in $\mathfrak{M}$, $\delta(q) = d$, the stationary distribution is the uniform distribution [15]

$$\pi(q) = \frac{d}{2m} = \frac{1}{n} \quad \forall q \in \Omega$$

where $n = |\Omega| = |V|$. It is also easy to confirm that the chain is *reversible*, that it satisfies the detailed balance condition with respect to π

$$Q(u, v) = \pi(v)P(v, u) = \pi(u)P(u, v) \quad \forall v, u \in V$$

If P is also aperiodic (i.e. G is non-bipartite, which we assume true in our case ²) then the chain is *ergodic* and the distribution of the states at time t approaches π as $t \rightarrow \infty$, regardless of the starting state.

At stationary distribution, it is clear that the random walk has optimal load-balancing qualities for regular graphs G . Similarly, it is clear that the faster the random walk on a regular graph converges to stationarity, the greater its load-balancing qualities and the quality of the partial cover as mentioned regarding question 4.

B. Mixing Time and the Spectral Gap ($1 - \lambda_1$)

The efficiency with which a random walk of $\mathfrak{M}$ may be used to sample over state space Ω with respect to stationary distribution π is precisely given by the rate at which the distribution of the states at time t converges to π as $t \rightarrow \infty$. In order to speak of convergence of probabilities, one must have a notion of distance over time. Let x be the state at time $t = 0$ and denote by $P^t(x, \cdot)$ the distribution of the states at time t . The *variation distance* at time t with respect to the initial state x is defined to be [16]

$$\Delta_x(t) = \max_{S \subseteq \Omega} |P^t(x, S) - \pi(S)| = \frac{1}{2} \sum_{y \in \Omega} |P^t(x, y) - \pi(y)|$$

The rate of convergence to stationary may be measured by the *mixing time*, the function [16]

$$\tau_x(\epsilon) = \min\{t \mid \Delta_x(t') \leq \epsilon, \forall t' \geq t\}$$

²One odd length cycle is sufficient to guarantee that G is non-bipartite.

which intuitively is the minimum number of steps t required, starting from node x , to guarantee that for any node y the probability of being at y after t or more steps is at most ϵ away from the probability of being at y under the stationary distribution (i.e. $\pi(y)$). A chain $\mathfrak{M}$ is considered *rapidly mixing* iff $\tau_x(\epsilon)$ is $O(\text{poly}(\log(n/\epsilon)))$. For $\mathfrak{M}$ to be used for efficient sampling (according to its stationary distribution), we want $\mathfrak{M}$ to be rapidly mixing.

As the stationary distribution π is defined to be such that $\pi P = \pi$, it corresponds to the eigenvalue $1 = \lambda_0$ of P . Let the rest of the eigenvalues of P in decreasing order be: $1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq -1$. Since $\mathfrak{M}$ is ergodic $\lambda_{n-1} > -1$, and it is well known that the rate of convergence to π is governed by the second largest eigenvalue in absolute value $\lambda_{\max} = \max\{\lambda_1, |\lambda_{n-1}|\}$, and in particular by the *spectral gap* $1 - \lambda_{\max}$ [16]:

Proposition 1: For an ergodic Markov chain, the quantity $\tau_x(\epsilon)$ satisfies

$$(i) \quad \tau_x(\epsilon) \leq (1 - \lambda_{\max})^{-1} (\ln \pi(x)^{-1} + \ln \epsilon^{-1})$$

$$(ii) \quad \max_{x \in \Omega} \tau_x(\epsilon) \geq \frac{1}{2} \lambda_{\max} (1 - \lambda_{\max})^{-1} \ln(2\epsilon)^{-1}$$

As we want the starting state of a random walk to be arbitrary, the statement above implies that a large spectral gap ($1 - \lambda_{\max}$) is both a necessary and sufficient condition for rapid mixing. In practice the smallest eigenvalue is not important since by simply adding self-loop probabilities of $\frac{1}{2}$ (“staying” probability) at each node, we create a new chain that has the same stationary distribution, and its eigenvalues, $\{\lambda'_i\}$, are similarly ordered and satisfy $\lambda'_{n-1} > 0$ and $\lambda'_{\max} = \lambda'_1 = \frac{1}{2}(1 + \lambda_1)$ [17]. This shows that it is sufficient to bound λ_1 to prove rapid mixing. A well-known method for bounding λ_1 to prove rapid mixing when the underlying graph has a geometric interpretation is a *Conductance argument* [13]. This is the method we shall use, as random geometric graphs have a strong geometric interpretation.

C. Conductance

Intuitively, one would expect that when the graph that underlies the Markov chain $\mathfrak{M}$ doesn't have *bottlenecks*, the lower the probability of getting stuck in any particular set of states, and thus the more rapidly mixing $\mathfrak{M}$ is. The property of “no bottlenecks” is formalized in a continuous manner with the notion of *conductance*.

The *conductance* of a reversible Markov chain $\mathfrak{M}$ is defined by [17]

$$\Phi = \Phi(\mathfrak{M}) = \min_{S \subseteq \Omega, 0 < \pi(S) \leq 1/2} \frac{Q(S, \bar{S})}{\pi(S)}$$

where $\bar{S} = \Omega - S$, $\pi(S)$ is the probability density of S under the stationary distribution π , and $Q(S, \bar{S})$ is the sum of $Q(v, u)$ over all $(v, u) \in S \times \bar{S}$.

In graph-theoretic terms, the conductance of $\mathfrak{M}$ is the minimum over all subsets $S \subset \Omega$ of the ratio of the weighted flow across the cut $Cut(S, \bar{S})$ to the weighted capacity of S . The higher the conductance of $\mathfrak{M}$, there are fewer bottlenecks in $\mathfrak{M}$, and the more rapidly mixing $\mathfrak{M}$ is. This intuition is confirmed by the following theorem [16]:

Theorem 1: The second eigenvalue λ_1 of a reversible Markov chain $\mathfrak{M}$ satisfies

$$1 - 2\Phi \leq \lambda_1 \leq 1 - \frac{\Phi^2}{2}$$

The above Theorem along with Proposition 1 yield the following powerful corollary bounding the mixing time via conductance [13]:

Corollary 1: Let $\mathfrak{M}$ be a finite, reversible, ergodic Markov chain with loop probabilities $P(x, x) \geq \frac{1}{2}$ for all states x . Let Φ be the conductance of $\mathfrak{M}$. Then, for any initial state x , the mixing time of $\mathfrak{M}$ satisfies

$$\tau_x(\epsilon) \leq 2\Phi^{-2}(\ln \pi(x)^{-1} + \ln \epsilon^{-1})$$

IV. MIXING TIME OF RGGs

A. Bounding the Conductance of the k dimensional Grid

To begin with a simple example of a conductance argument with similarities to the conductance argument for general random geometric graphs, we consider the case of the 2-dimensional grid which is a sub-class of the class of regular geometric graphs.

Let $M(2, n)$ denote the two dimensional grid of n nodes. It is easy to see that, since the graph has a regular geometric structure, the minimum conductance occurs when we consider $\min(|S|, |\bar{S}|)$ of maximum capacity, that is when $\pi(S) = \pi(\bar{S}) = \frac{1}{2}$ so that S has half of the nodes of $M(2, n)$. Furthermore, as there are many possible ways of separating the nodes of $M(2, n)$ into two halves S and $\bar{S}$, we need to consider the separation that gives the minimum flow across $Cut(S, \bar{S})$, which occurs when the length of the boundary between S and $\bar{S}$ is minimized (since every edge has the same weight due to regularity). It is easy to check that the separation satisfying this is with a separating line l parallel to one of the axis. Since there are $n^{\frac{1}{2}}$ edges crossing such a cut and each edge has weight $w = \frac{1}{4}$, the conductance of

the two dimensional grid of n nodes is³

$$\begin{aligned} \Phi(M(2, n)) &= 2Q(x, y) = 2 \sum_{\substack{x \in S \\ y \in \bar{S}}} \frac{1}{n} \frac{1}{4} \\ &= 2n^{\frac{1}{2}} \frac{1}{4n} = (2n^{\frac{1}{2}})^{-1} \end{aligned}$$

This argument easily generalizes to the k dimensional grid $M(k, n)$, and we obtain the following by Theorem 1 and Corollary 1 above:

Lemma 2: For the k dimensional grid $M(k, n)$ of n nodes we have the following:

- 1) $\Phi(M(k, n)) = (kn^{\frac{1}{k}})^{-1}$
- 2) $\frac{1}{2}(kn^{\frac{1}{k}})^{-2} \leq 1 - \lambda_1 \leq 2(kn^{\frac{1}{k}})^{-1}$
- 3) $\tau_x(\epsilon) \leq 2k^2 n^{\frac{2}{k}} (\ln n + \ln \epsilon^{-1})$

B. Bounding the Conductance of $\mathcal{G}(n, r)$

Let $\mathcal{G}(n, r)$ be a random geometric graph constructed as mentioned earlier. Our main analytical results are the following:

Theorem 2 (Conductance of RGG): For $c > 1$, if $r^2 \geq \frac{c4 \log n}{n}$, then w.h.p.⁴:

$$\Phi(\mathcal{G}(n, r)) = \Theta(r)$$

From Theorem 2, Theorem 1 and Corollary 1 we obtain these bounds:

Corollary 3: For $c > 1$, if $r^2 \geq \frac{c4 \log n}{n}$, then w.h.p. the mixing time of $\mathcal{G}(n, r)$ is as follows:

- 1) $\tau_x(\epsilon) = O(r^{-2}(\ln n + \ln \epsilon^{-1}))$
- 2) $1 - \lambda_1 = \Omega(r^2)$ and $1 - \lambda_1 = O(r)$

Together with Proposition 1 (ii) we also obtain the necessary condition:

Corollary 4: Radius $r = \Omega(1/\text{poly}(\log n))$ is w.h.p. necessary and sufficient for $\mathcal{G}(n, r)$ to be rapidly mixing.

Our analytical results for random geometric graphs are based on certain “nice” properties such graphs possess. These properties include the uniformity of node distribution and the regularity of node degree. Before we begin the proof of Theorem 2, we formalize this with the *geo-dense* property and introduce the notion of *bins*, equal size areas that partition the unit square.

³We ignore the two nodes on the borders which have only 3 neighbors.

⁴Event $\mathcal{E}_n$ occurs with high probability if probability $P(\mathcal{E}_n)$ is such that $\lim_{n \rightarrow \infty} P(\mathcal{E}_n) = 1$.

Definition 1: Let $\mathcal{F}(n, r(n))$ be a class of geometric graphs⁵. For a constant $\mu \geq 1$ we say that such a class is μ -**geo-dense** if every square bin of size at least $A = r^2/\mu$ (in the unit square) has $\Theta(nA) = \Theta(nr^2/\mu)$ nodes as $n \rightarrow \infty$.

In *geo-dense* graphs there are no large areas that fail to contain a sufficient number of nodes. We show that *random* geometric graphs are *geo-dense* for radius $r_{\text{reg}} = \Theta(r_{\text{con}})$, and all nodes have the same order degree.

Let $\mathcal{F}(n, r(n))$ be a class of geometric graphs that is μ -*geo-dense* for a constant $\mu \geq 4$. Let V and E be the set of nodes and edges of $G(n, r) \in \mathcal{F}$ and $\delta(v)$ denote the degree of $v \in V$:

Lemma 5: $\forall v \in V \ \delta(v) = \Theta(nr^2)$

Proof: First note that the 4-*geo-dense* property guarantees that if we divide the unit square to square bins of size $\frac{r}{\sqrt{2}} \times \frac{r}{\sqrt{2}}$ each, then the number of nodes in every bin will be $\Theta(nr^2)$ as $n \rightarrow \infty$. Since the diagonal of each bin is r , for every bin the set of nodes in the bin forms a clique, and since every node $v \in V$ is in some bin, we have that $\delta(v) = \Omega(nr^2), \forall v \in V$. Similarly, when we divide the area into bins of size $r \times r$ every node may be connected to the nodes of at most nine bins (that is its own bin and the bordering bins), and we have that $\delta(v) = \Theta(nr^2), \forall v \in V$. ■

As a result we can claim the following about the number of edges

$$m = |E| = \Theta(n^2 r^2)$$

To prove that $\mathcal{G}(n, r)$ can be μ -*geo-dense* for μ sufficient for our needs, we utilize the following result [10], [18], [19]

Lemma 6 (Balls in Bins): For a constant $c > 1$, if one throws $n \geq cB \log B$ balls uniformly at random into B bins, then w.h.p. both the minimum and the maximum number of balls in any bin is $\Theta(\frac{n}{B})$.

Following Balls in Bins Lemma we can now make the claim about the geo-density of $\mathcal{G}(n, r)$ precise:

Lemma 7 (Geo-density of RGG): For constants $c > 1$ and $\mu \geq 1$, if $r^2 = \frac{c\mu \log n}{n}$ then w.h.p. $\mathcal{G}(n, r)$ is μ -*geo-dense*, that is, any bin area of size r^2/μ in $\mathcal{G}(n, r)$ has $\Theta(\log n)$ nodes w.h.p.

Proof: Let an area of r^2/μ be a bin. If we divide the unit square into such equal size bins we have $B = \frac{n}{c \log n}$ bins. For the result to follow we check that Lemma 6 holds by showing that $n \geq c'B \log B$ for some constant

$c' > 1$:

$$\begin{aligned} B \log B &= \frac{n}{c \log n} \log\left(\frac{n}{c \log n}\right) \\ &= \frac{n}{c \log n} (\log(n) - \log(c \log n)) \\ &= \frac{n}{c} - \left(\frac{n}{c \log n}\right) (\log(c \log n)) \\ &\leq n/c \end{aligned}$$

Now combining the results of Lemmas 5 and 7 we can also claim the following about $\mathcal{G}(n, r)$:

Corollary 8: For $c > 1$, if $r^2 \geq \frac{c^4 \log n}{n}$, then w.h.p. $\forall v \in \mathcal{G}(n, r), \delta(v) = \Theta(nr^2)$ and $m = |E| = \Theta(n^2 r^2)$

Note that the critical radius for connectivity in $\mathcal{G}(n, r)$, r_{con} , is s.t. $\pi r_{\text{con}}^2 = \frac{\log n}{n}$ [20], [21]. We have just shown that for $r_{\text{reg}} = \Theta(r_{\text{con}})$ w.h.p. $\mathcal{G}(n, r_{\text{reg}})$ will have the nice properties mentioned above⁶.

Now we may begin the proof of our main result:

Proof:[of Theorem 2] Let $\text{Cut}(S, \bar{S})$ denote the cut size between S and $\bar{S}$ in $\mathcal{G}(n, r)$ (the total number of edges crossing from S to $\bar{S}$). Since $\mathcal{G}(n, r)$ is 4-*geo-dense* and “almost regular” w.h.p. by Lemma 7 and corollary 8 we can observe that the minimum conductance is when we divide the area into two halves S and $\bar{S}$ with $\pi(S) \approx \pi(\bar{S}) \approx \frac{1}{2}$ and such that the length of the boundary between S and $\bar{S}$ is minimized. Similarly to the regular grid case, the separation satisfying this is with a separating line l parallel to one of the axis. Let $\text{Cut}_\Phi(S, \bar{S})$ be the above cut, the one that minimizes $\Phi(\mathcal{G}(n, r))$. Next we bound $\text{Cut}_\Phi(S, \bar{S})$.

For the lower bound of $\text{Cut}_\Phi(S, \bar{S})$, partition the area into bins of size $\frac{r}{2\sqrt{2}} \times \frac{r}{\sqrt{2}}$ as in Figure 1 (a). By the 4-*geo-dense* property w.h.p. the number of nodes in any bin is $\Theta(nr^2)$. Notice that the set of nodes in any two horizontally adjacent bins (such as B_0 and B_1 in Fig 1 (a)) forms a clique. Therefore, to lower bound $\text{Cut}_\Phi(S, \bar{S})$, we are only considering the crossing edges within each separate such clique along the dividing line l . Since there are at least $\frac{\sqrt{2}}{r}$ cliques along the dividing line l , and for each bin on the left side of l we have $\Omega(n^2 r^4)$ such edges crossing to the right of l , we obtain the desired lower bound $\text{Cut}_\Phi(S, \bar{S}) = \Omega(r^3 n^2)$.

For the upper bound partition the area into bins of size $r \times r$ as in Figure 1 (b). Note that for each edge (u, v) crossing l , v must be in some left bin B_0 adjacent to l , and so u must be in one of three possible bins

⁶Note, however, that even though $\mathcal{G}(n, r_{\text{reg}})$ is *geo-dense* in our terms, it is *not* a dense graph in graph theoretic terms (i.e a graph with $\Theta(n^2)$ edges), but is a sparse graph with expected number of $\Theta(n \log n)$ edges.

⁵either random or deterministic

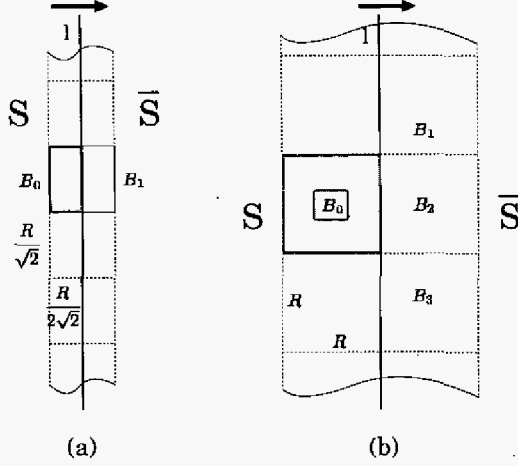


Fig. 1. (a) Lower bound for the Conductance in $\mathcal{G}(n, r)$. (b) upper bound for the Conductance in $\mathcal{G}(n, r)$

B_1, B_2, B_3 that are on the right of l and touching B_0 as shown in the picture. To upper bound $Cut_\Phi(S, \bar{S})$, we consider the maximum number of crossing edges from any $r \times r$ sized bin B_0 in S to three $r \times r$ sized bins B_1, B_2 and B_3 in $\bar{S}$. As there are $\frac{1}{r}$ such bins as B_0 , and from the 4-geo-dense property, w.h.p. the number of nodes in any bin is $\Theta(nr^2)$, we get the desired upper bound as follows:

$$Cut_\Phi(S, \bar{S}) = O\left(\frac{1}{r} \cdot nr^2 \cdot 3nr^2\right) = O(r^3 n^2)$$

So, combining the upper and lower bounds, we have that w.h.p.,

$$Cut_\Phi(S, \bar{S}) = \Theta(r^3 n^2)$$

And, thus, by corollary 8, equation (1), and the definition of $P(x, y)$ we complete the proof:

$$\begin{aligned} \Phi(\mathcal{G}(n, r)) &= \frac{Q(S, \bar{S})}{\pi(S)} \\ &= \frac{1}{\pi(S)} \sum_{\substack{x \in S \\ y \in \bar{S}}} \pi(x) P(x, y) \\ &= \Theta(2) \sum_{\substack{x \in S \\ y \in \bar{S}}} \Theta\left(\frac{1}{n}\right) \Theta\left(\frac{1}{nr^2}\right) \\ &= Cut_\Phi(S, \bar{S}) \frac{1}{\Theta(r^2 n^2)} \\ &= \Theta(r^3 n^2) / \Theta(r^2 n^2) \\ &= \Theta(r) \end{aligned}$$

V. COVER TIME AND PARTIAL COVER TIME OF RANDOM GEOMETRIC GRAPHS

Known results for the cover time of specific graphs vary from the best case of $\Theta(n \log n)$ to the worst case of $\Theta(n^3)$. The best cases correspond to dense, highly connected graphs, for example, the complete graph, d -regular graphs with $d > \frac{n}{2}$, and the hypercube. When connectivity decreases and bottlenecks exist in the graph, the cover time increases, as exemplified by the lollipop graph and, to a lesser degree, by the line graph which has cover time $\Theta(n^2)$.

Therefore, intuitively, one would anticipate a relationship between the spectral gap $(1 - \lambda_1)$ and small cover time. In confirmation of this intuition, a bound for the cover time for regular graphs G that is based on the spectral gap $(1 - \lambda_1)$ is given by [22] and [23]:

Theorem 3: For regular graph $G = (V, E)$ with $n = |V|$ and second largest eigenvalue λ_1 the cover time of G is bounded as follows: $C(G) = O(n \log n / (1 - \lambda_1))$

From the same analysis of [23] and [22] one may directly obtain the partial cover time of a regular graph via a partial balls in bins argument:

Lemma 9: For regular graph $G = (V, E)$ with $n = |V|$ and second largest eigenvalue λ_1 the partial cover time of G is bounded as follows for any constant c such that $0 < c < 1$: $PCT_c(G) = O(n / (1 - \lambda_1))$

Therefore, it follows from Theorem 3, Lemma 9, and Corollary 3 that we can bound the cover time and PCT of $\mathcal{G}(n, r)$ as follows:

Corollary 10: For $c > 1$, if $r^2 \geq \frac{c4 \log n}{n}$, then w.h.p

$$C(\mathcal{G}(n, r)) = O(r^{-2} n \log n)$$

and, for any constant c , $0 \leq c < 1$,

$$PCT_c(\mathcal{G}(n, r)) = O(r^{-2} n)$$

If these bounds on cover time and PCT were tight, then the only way to achieve optimal cover time for random geometric graphs would be by choosing a radius r that is constant irrespective of the network size n . Recalling that our definition of $\mathcal{G}(n, r)$ is normalized to a unit area, this would mean that only broadcast networks of constant hop diameter may have optimal cover and partial cover. Even the minimum radius required for rapid mixing, which is $r_{\text{rapid}} = \Theta(1/\text{poly}(\log n))$, is several orders lower than such a radius.

However, fortunately, the bounds given by Theorem 3, Lemma 9, and correspondingly by Corollary 10 are not tight. An especially demonstrative case of this is for k dimensional grid $M(k, n)$ of n nodes for any $k > 2$. The cover time and PCT for $M(k, n)$ are known to

be optimal, that is $\Theta(n \log n)$ and $\Theta(n)$ respectively [24]. Yet, recalling Lemma 2, Theorem 3 and Lemma 9 yield only that $C(M(k, n)) = O(k^2 n^{1+\frac{2}{k}} \log n)$ and that $PCT_c(M(k, n)) = O(k^2 n^{1+\frac{2}{k}})$.

The following theorem provides, in many cases, tighter bounds on the cover time C in term of the maximum hitting time h_{max} . For arbitrary nodes $i, j \in V$, let h_{ij} be the expected number of steps for the random walk to move from i to j , namely the *hitting time* between i and j . Then h_{max} (h_{min}) is defined as the maximum (minimum) h_{ij} over all ordered pairs of nodes.

Theorem 4 (Matthews' Theorem [9]): For any graph G ,

$$h_{min} \cdot H_n \leq C(G) \leq h_{max} \cdot H_n$$

where $H_k = \ln(k) + \Theta(1)$ is the k -th harmonic number.

Similarly, in [1] the authors showed the following bound of the partial cover time, PCT, in terms of the hitting time:

Lemma 11: For any graph G , and $0 \leq c \leq \frac{n-1}{n}$

$$PCT_c(G) < 2 \cdot h_{max} \cdot \left\lceil \log_2 \left(\frac{1}{1-c} \right) \right\rceil = O(h_{max})$$

In particular, as the maximum hitting time for $M(k, n)$ is known to be $O(n)$ for $k > 2$ [24], Matthew's Theorem and Lemma 11 yield that the cover time and PCT of $M(k, n)$ is $O(n \log n)$ and $O(n)$, respectively, which are tight. In future work, we plan to investigate the maximum hitting time of $\mathcal{G}(n, r)$ to obtain tighter analytical bounds on the cover time and PCT of random geometric graphs based on Matthew's Theorem and Lemma 11. In this work, we have investigated the cover time and PCT of random geometric graphs via experiments.

Notice that whether one bounds the cover time or partial cover time based on the hitting time or based on the spectral gap, an $O(\log n)$ efficiency is gained in moving from the cover time to the partial cover time. In fact, as can be observed from the case of the k dimensional grid $M(k, n)$, the $O(\log n)$ gain in efficiency is often tight. Experimental results comparing cover times and PCTs of random geometric graphs from [1] also support this tightness for the case of random geometric graphs, further justifying consideration of the partial cover for information collection in broadcast networks.

VI. RESEMBLANCE OF RANDOM GEOMETRIC GRAPHS

We are concerned with small radii (as a function of n) for which optimal cover time and PCT may yet be achieved. From above discussion we have seen that

connectivity and the spectral gap are closely related to the cover time and PCT, and we expect that, for fixed n , as r increases, the cover time and PCT of $\mathcal{G}(n, r)$ decrease. We know that $M(3, n)$ has optimal cover time and PCT even though it has constant degree and is not even rapidly mixing. So, we wonder for which radius r would the cover time and PCT of $\mathcal{G}(n, r)$ be on the same order as the cover time and PCT, respectively, of $M(3, n)$. As conductance is a measure of connectivity and spectral gap, and $\Phi(\mathcal{G}(n, r)) = \Theta(r)$ from Theorem 2, we ask: If $r = \Phi(M(3, n))$ does $\mathcal{G}(n, r)$ also have optimal cover time and PCT w.h.p.?

To formalize the implicit notion of resemblance of graphs underlying that question, let us define the following: A random geometric graph $\mathcal{G}(n, r)$ *resembles* another graph $G = (V, E)$, with $n = |V|$, iff $r = \Theta(\Phi(G))$. And, as abbreviation, we may write, $\mathcal{G}(n, r) \equiv_{\Phi} G$ to mean that $\mathcal{G}(n, r)$ resembles G .

As the spectral gap and mixing time may both be upper bounded and lower bounded by functions of conductance, it is easy to see that two graphs which resemble each other will also have spectral gap and mixing time within close range of each other. For example, if $\mathcal{G}(n, r) \equiv_{\Phi} G$, then $\mathcal{G}(n, r)$ is rapidly mixing iff G is rapidly mixing. However, precisely because our notion of resemblance is based essentially on spectral gap, which in general is not sufficient to characterize the cover time and PCT due the non-tightness of Theorem 3 and Lemma 9 for the case of $M(k, n)$ for $k > 2$, we do not necessarily expect that any two graphs with the same conductance also have the same cover time or PCT.

But, in the case of random geometric graphs, other special properties improve with increased radius. In particular, it is easy to see that both the average degree δ and the hop diameter D of random geometric graphs are functions of the radius. Specifically, it is easy to see the following:

Corollary 12: If $\mathcal{G}(n, r) \equiv_{\Phi} M(k, n)$ for some $k \geq 2$ then w.h.p.

- 1) $r = \Theta(n^{-\frac{1}{k}})$
- 2) $D(\mathcal{G}(n, r)) = \Theta(n^{\frac{1}{k}}) = D(M(k, n))$
- 3) $\forall v \in V, \delta(v) = \Theta(n^{\frac{k-2}{k}})$

So, when $\mathcal{G}(n, r) \equiv_{\Phi} M(k, n)$, for any constant $k \geq 2$: not only do the two graphs have similar conductance and diameter, but also the average degree of the nodes of $\mathcal{G}(n, r)$ is orders higher than the degree of any node of $M(k, n)$ (which is just $d = 2k$). This motivates our

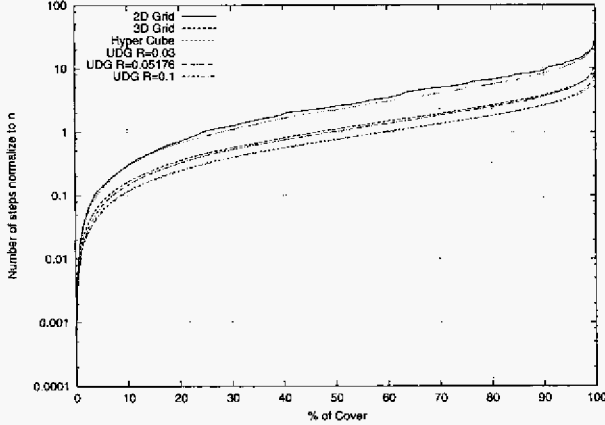


Fig. 2. The progress of partial cover time as function of number of steps normalized to n for different graphs

hypothesis above, paraphrased in terms of resemblance:

Hypothesis 1: If $\mathcal{G}(n, r) \equiv_{\Phi} M(k, n)$ for some $k > 2$ then, for any constant $0 \leq c < 1$, w.h.p.

- 1) $C(\mathcal{G}(n, r)) = O(C(M(k, n))) = \Theta(n \log n)$
- 2) $PCT_c(\mathcal{G}(n, r)) = O(PCT_c(M(k, n))) = \Theta(n)$

In particular, this would imply that the radius required for optimal cover time and PCT is $\Theta(n^{-\frac{1}{3}})$, which is significantly lower than $r_{\text{rapid}} = \Theta(1/\text{poly}(\log n))$, the minimum radius required for rapid mixing. In the next section, we demonstrate experimental results in support of this hypothesis.

VII. EXPERIMENTAL RESULTS

In this section we validate our analytical results and hypotheses using simulations of simple random walks on different graphs. Our random geometric graphs were constructed by placing n random nodes in a unit area and connecting every two nodes at distance less than or equal to r . For each experiment we took the average of 100 runs. Unless otherwise specified, the graphs are of size $n = 4096$ nodes. In our experiments based on resemblance, we used the continuous approximation to the conductance as given in the Appendix.

A. Cover Time and PCT

Fig. 2 represents the number of steps (time) normalized to n (on log scale) as a function of the progress of the partial cover. The Figure presents three different well known graphs: 2-dimensional-grid $M(2, n)$, 3-dimensional grid $M(3, n)$, and the hypercube $H(n)$, and three different random $\mathcal{G}(n, r)$'s with $r =$

TABLE I
NUMBER OF STEPS REQUIRED FOR 80% COVER OF
3-DIMENSIONAL GRID AND CORRESPONDING RGG

n	r	$\mathcal{G}(n, r)$	3D Grid
1000	0.0859	2.7142	2.7016
1728	0.0704	2.5828	2.6947
2744	0.05965	2.5697	2.7090
4096	0.05176	2.4693	2.6874

0.03, 0.05176, 0.1 which closely achieve the same partial cover as the respective known graphs. This validates our intuition that we can achieve any cover time up to an optimal one by increasing r and legitimizes the notion of resemblance for random geometric graphs. Importantly, note that $r = 0.05176$ corresponds to the radius of the $\mathcal{G}(4096, r)$ that resembles $M(3, 4096)$, obtained using the approximation to conductance in equation (2). One may observe from Fig. 2 that the PCT and cover time of the $\mathcal{G}(4096, r)$ with $r = 0.05176$ behave very similarly to the PCT and cover time of $M(3, 4096)$. One may further observe from Table I, that for varying network size n , the 80% PCT of $M(3, n)$ and the $\mathcal{G}(n, r)$ that resembles it has very similar PCT. Therefore, Fig. 2 and Table I support our Hypothesis 1 and demonstrate that the PCT for the $\mathcal{G}(n, r)$ that resembles $M(3, n)$ is almost the same as the PCT of $M(3, n)$, which is optimal, namely $O(n)$ instead of $O(n^{\frac{2}{3}}n)$ which follows from Corollary 12 and Corollary 10.

In Fig. 2, one may directly observe the sharp increase in the number of steps (time) for every graph as the partial cover approaches the full cover. This confirms the non-negligible gap previously observed by [1] between the order of the cover time and the order of the PCT for random geometric graphs, further justifying consideration of the partial cover.

B. Quality of the Random Walk based on Hole Size

As previously stated, the load balancing properties of an almost regular graph are directly related to the graph's mixing rate, the measure of variation distance over time. Fig. 3 compares the variation distance as a function of number of steps (time) up to 95% partial cover for different $\mathcal{G}(n, r)$'s and a 3-dimensional grid $M(3, n)$. Again, as expected, when r increases the variation distance drops faster. Also note that with r less than the radius of the $\mathcal{G}(n, r)$ resembling the 3-dimensional grid (0.05176) we get even better mixing time than the 3-dimensional grid. Since this was also the case in Fig. 2 and Table I it

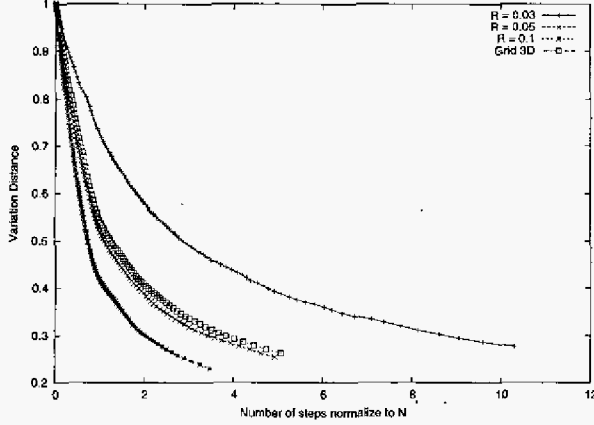


Fig. 3. The decrease in variation distance as function of steps up to 95% partial cover

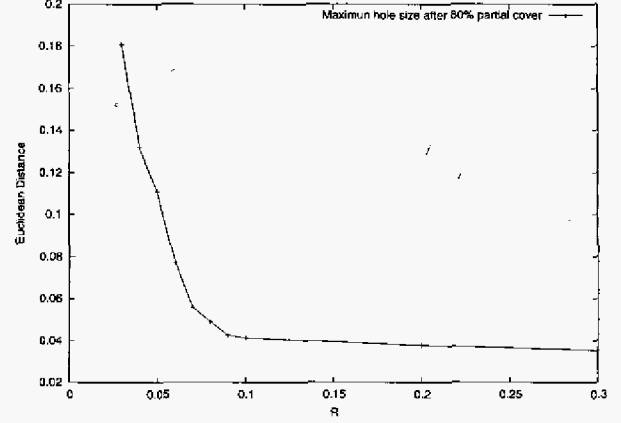


Fig. 5. Hole Size at 80% Partial Cover as a Function of radius r

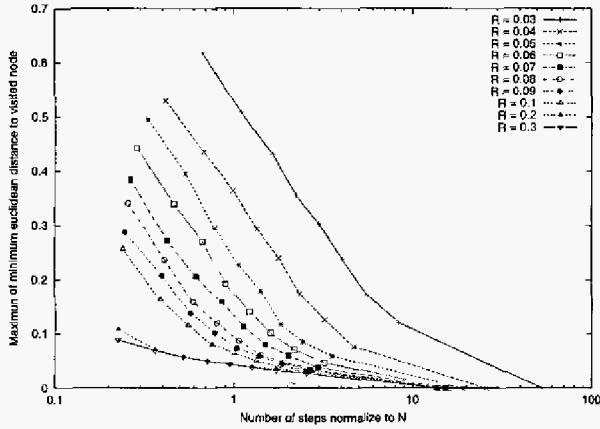


Fig. 4. Hole size as a function of the number of steps normalized to n for random geometric graphs with different radii r

seems that the random geometric graph that resembles $M(3, n)$ actually has better cover time and mixing time than $M(3, n)$.

When the mixing time is better one would expect that the quality of the partial cover will improve, meaning that the random walk will not leave large contiguous areas in the network uncovered. To make this measurement more precise let $\min(v)$ be the minimum distance from v to a visited node in a (partial) random walk. We define the *hole size* of a random walk as the maximum of $\min(v)$ over all the nodes in the graph. Note that the hole size is decreasing as the random walk proceeds and more nodes are visited; after cover time it is 0. Fig. 4 presents the decrease in the hole size as a function of number of steps of the random walk for $\mathcal{G}(n, r)$ with increasing r . The figure shows that the rate of improvement in the

quality is strongly dependent upon r , similarly to the mixing rate. Note that each walk was sampled at 10%, 20%, and up to 100% cover where the hole size is 0. An interesting point to discuss is that this experiment also validates the fact that graphs with different spectral gap (and mixing time) can have the same cover time (such as the 3-dimensional grid and the hypercube). For example the graphs with $r > 0.06$ seems to have very similar cover time but nevertheless very different partial cover quality.

Fig. 5 shows the improvement in the hole size after 80% cover as a function of r . Note that it behaves like the function r^{-2} , which further supports the intuition that the hole size is directly related to the mixing rate.

VIII. CONCLUSIONS AND FUTURE WORK

We have analytically obtained bounds on the mixing time of $\mathcal{G}(n, r)$ which is $O(r^{-2} \ln n)$. In particular, the analytical bounds show that the minimum radius required for rapid mixing is $r_{\text{rapid}} = \Theta(1/\text{poly}(\log n))$. Although we also obtained analytical bounds on the cover time and partial cover time of $\mathcal{G}(n, r)$ based on the spectral gap, experimental results on the random geometric graphs that resemble $M(3, n)$ indicate that those bounds are not tight. In particular, $r = \Theta(1/n^{1/3})$ is sufficient radius to obtain optimal cover time and partial cover time. However, experimental results on hole size and intuition on load-balancing indicate that for the random walk to have good quality, a short mixing time, or rapid mixing, is needed. Therefore, to be both efficient and have good quality cover and partial cover with high probability, any radius r such that $r = \Omega(1/\text{poly}(\log n))$ is necessary and sufficient.

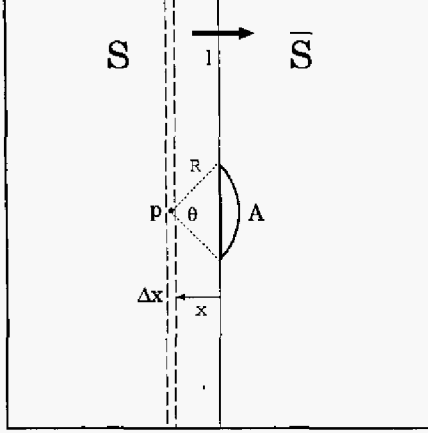


Fig. 6. Approximating the Conductance in RGG

Since the submission of this paper, the authors have theoretically validated the experimental result of this paper that corroborated the hypothesis that $r = \Theta(1/n^{1/3})$ is sufficient radius to obtain optimal cover time and partial cover time [19]. Moreover, using the technique of bounding resistance, as proposed in a preliminary version of this paper, the authors showed that *any radius r that is on the order of guaranteeing connectivity r_{con} , as well as any larger radius, yields that $\mathcal{G}(n, r)$ has optimal cover time and partial cover time, thus positively closing the question on the efficiency of the partial cover.*

Concerning further questions on the quality of the cover, we also plan to analytically investigate the exact relationship between mixing time and the quality of the random walk in terms of hole size.

Finally, although our random walk does not pose any interference in and of itself, in sensor networks there is always the issue of energy optimization. Whereas finding small radii for which optimal random properties are exhibited implicitly incorporates the idea that large radii impose interference and energy constraints, we would like to explicitly incorporate energy and interference into our current random walk model and find the optimum radius under the new model.

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APPENDIX

CONTINUOUS APPROXIMATION OF CONDUCTANCE

Follow Fig. 6 Let l be the dividing. A point p in S that is at distance $x < r$ from l neighbors the nodes in

the gray area A in the Figure. The size of A is given by $\frac{1}{2}r^2(\theta - \sin \theta)$. (Observe that $\theta = \arccos(\frac{x}{r})$ and A is a function of x .) So p has an expected number of nA edges crossing to $\bar{S}$. Taking the integral over all the points in distance $0 \leq x \leq r$ and assuming that there are $n\Delta x$ nodes in the area $1 \cdot \Delta x$ we get that the expected number of edges crossing from S to $\bar{S}$ is (ignoring the effect of the borders) ⁷

$$\begin{aligned} E[Cut(S, \bar{S})] &\leq \int_0^r \frac{1}{2}r^2 n A n \Delta x \\ &= \int_0^r \frac{1}{2}r^2 n \left[\arccos\left(\frac{x}{r}\right) - \sin\left(\arccos\left(\frac{x}{r}\right)\right) \right] n \Delta x \\ &= \frac{1}{2}r^2 n^2 \left[-2r \sqrt{1 - \frac{x^2}{r^2}} + \frac{2}{3}r \left(1 - \frac{x^2}{r^2}\right)^{\frac{3}{2}} + 2x \arccos\left(\frac{x}{r}\right) \right]_0^r \\ &= \frac{1}{2}r^2 n^2 (0 - (-2r + \frac{2}{3}r)) \\ &= \frac{2}{3}r^3 n^2 \end{aligned}$$

To approximate the conductance we use this upper bound and improve it by assuming that the expected degree is $\pi r^2 n$ and by taking out part of the border effect as we take the integral over the area $(1-r) \cdot \Delta x$ (assuming $r \ll 1$)

$$\begin{aligned} \Phi(\mathcal{G}(n, r)) &= \frac{Q(S, \bar{S})}{\pi(S)} \\ &= 2 \sum_{\substack{x \in S \\ y \in \bar{S}}} \pi(x) P(x, y) \\ &\approx 2 \sum_{\substack{x \in S \\ y \in \bar{S}}} \frac{1}{n} \frac{1}{\pi r^2 n} \\ &\approx \frac{2}{3} n^2 r^3 (1-r) \frac{1}{\pi r^2 n^2} \\ &= \frac{4}{3\pi} r (1-r) \end{aligned} \tag{2}$$

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⁷Note that as $r \rightarrow \infty$ and $r \rightarrow 0$ the above bound tightens and approaches equality.

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