Μοντελοποίηση και Ανάλυση Απόδοσης Δικτύων

Μαθηματικά για Τηλεπικοινωνίες (P/H)

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- (A) Basics of Stochastic Processes
- (B) Basic Network Modeling, Performance Evaluation and Design

2006

PART A
BASICS OF STOCHASTIC PROCESS

- Probability and random variables: Bernoulli trials; Poisson
- Stochastic Processes: independent increments; Wiener & Poisson Processes; stationarity & ergodicity
- Markov Processes

RELATIVE FREQUENCY-BASED probability

$$P(A) = \lim_{n \to \infty} \frac{\text{\# experiments in which A occurs}}{\text{total \# of experiment}}$$

Comments:

1. Requires experimentation
2. n is usually finite ⇨ approximation

3. Convergence implies that as n->∞:

- $\left\{ \left| \frac{n_A}{n} - P(A) \right| > \delta \right\}$ occurs less & less
- Not that : $\left| \frac{n_A}{n} - P(A) \right| < \delta \quad \forall n > n(\delta)$

AXIOMATIC DEFINITION OF PROBABILITY – PROBABILITY SPACE

- Probability Space: $\{\Omega, F, P\}$
- Ω: Sample space
- F: σ-field generated by Ω
- P: A probability measure. It is a set function on F satisfying:

- $P(A) \geq 0 \quad \forall A \in F$
- $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B) \; \forall A, B \in F, A \cap B = \varnothing$

BERNOULLI TRIALS

- Consider a simple experiment with $\Omega=\{s,f\}$, $P(s)=p$ & $P(f)=1-p=q$
- Consider 2 indep. experiments: $\Omega_1=\Omega x \Omega=\{ss,ff,sf,fs\}$, 2^2 possible outcomes
- Consider n indep. experiments: $\Omega_n=\Omega x \ldots x \Omega=\{\ldots\}$, 2^n possible outcomes

$$\overline{\omega}=(\omega_1,\omega_2,\ldots,\omega_n) \quad , \omega_i \in \{s,f\}$$

Joint Probability:

$$P(\overline{\omega}) = P(\omega_1, \omega_2, \ldots, \omega_n) = P(\omega_1)P(\omega_2)\ldots P(\omega_n)$$

(indep. experiments)

Question: Prob. of k successes in n trials = ?

(A) Consider a pattern Π (outcome in Ω_n) counting k successes

$$\Pi = \underbrace{ss\ldots s}_{k-1}\, ff\, \underbrace{s}\, ff\ldots f$$

$$P(\Pi) = P\{ss\ldots sffsff\ldots f\} = p^k q^{n-k}$$

This prob. is the same for ANY pattern with k successes & n-k failures

(B) Consider all possible legitimate patterns. There are:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Call them $\Pi_1, \Pi_2, \ldots, \Pi_{\binom{n}{k}}$

(Binomial coefficients)

Answer:

$$P\{\text{outcome has k successes in n trials}\} = P\left(\bigcup_{i=1}^{\binom{n}{k}} \Pi_i\right) =$$

$$(\text{mutually disjoint}) = \sum_{i=1}^{\binom{n}{k}} P(\Pi_i) = \binom{n}{k} p^k q^{n-k} = b(k, n, p)$$

POISSON PROBABILITY LAW

Assume n->∞, p->0 such that np=a
(a=rate of success)

$$P\{k \text{ successes}\} = \lim_{\substack{n\to\infty \\ p\to 0, np=a}} b(k,n) = \frac{a^k}{k!} e^{-a}$$

Comment:

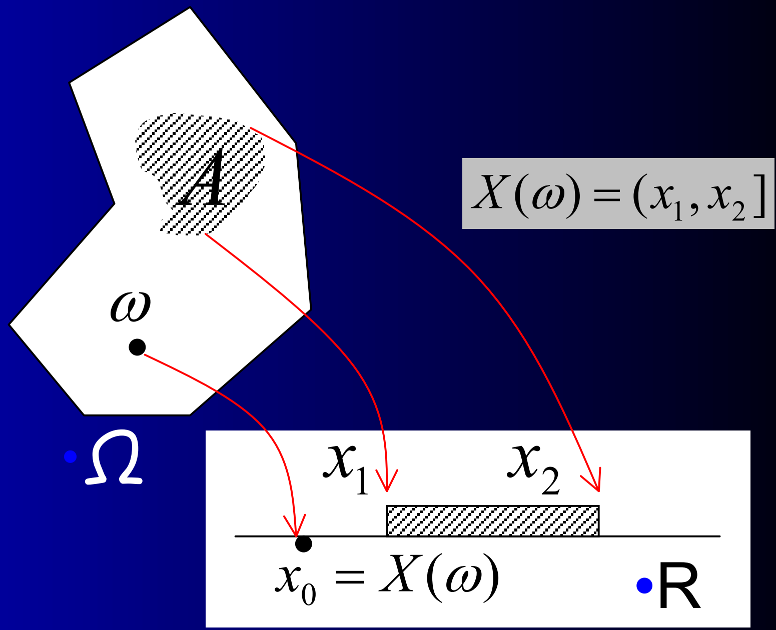
- If n large and p<<1 approximate b(k,n,p)≈e(-np) (np)k /k!
- Cummulative arrivals contributed by a large # of independent sources can be modeled as Poisson
- Above plus other properties of Poisson process, make it a good traffic model

RANDOM VARIABLES (RV)

A mapping X from Ω into the real numbers satisfying:

$$(1)\quad X^{-1}\{(-\omega, x)\} = C \in F$$

$$(2)\quad P(X = \pm\infty) = 0$$



Measures:

$$\text{PDF}: F_X(x) = P\{X \le x\}$$

$$\text{pdf}: f_X(x) = \frac{\mathrm{d}F_X(x)}{dx}, F_X(x) = \int_{-\infty}^{x} f_X(y)dy$$

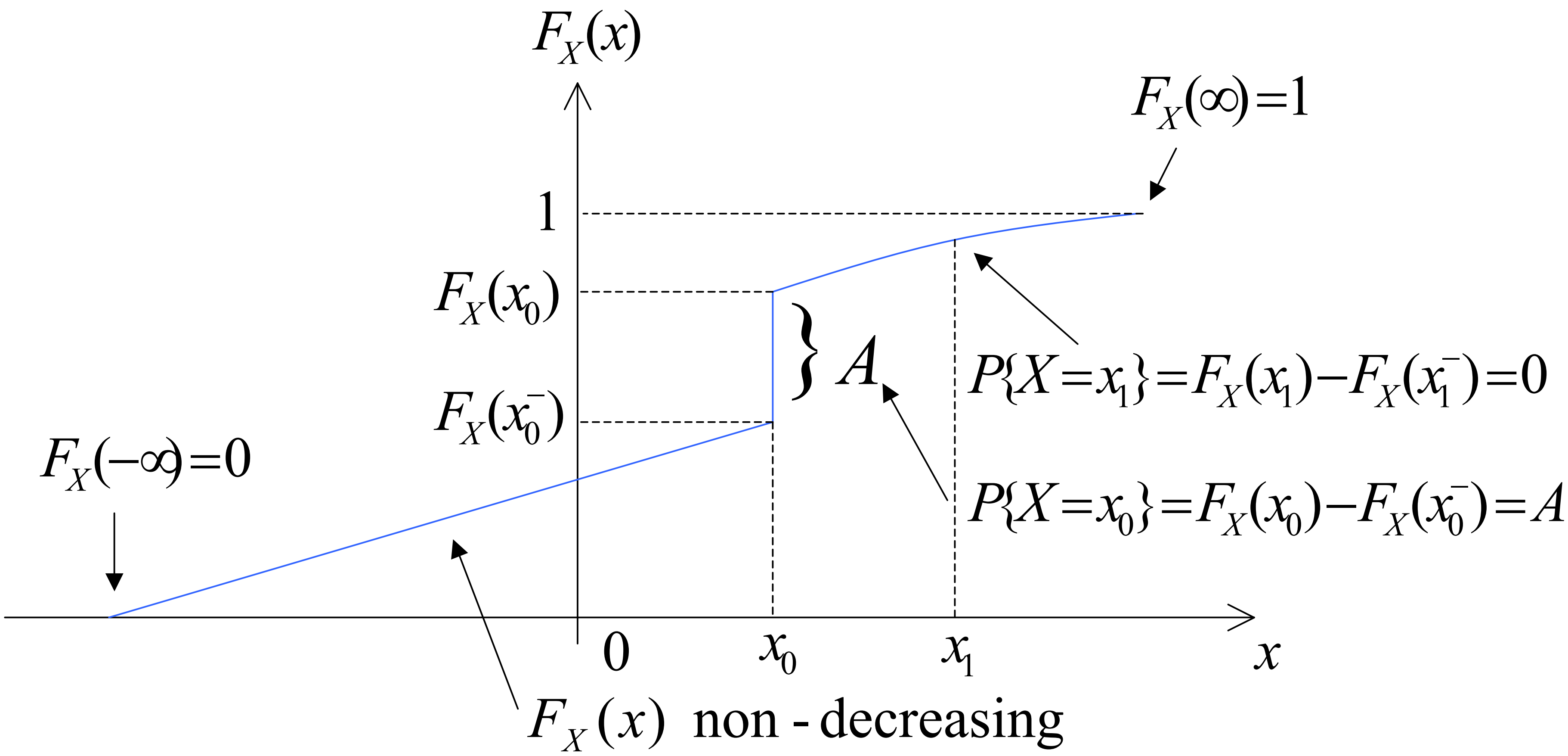
$$\text{PMF}: P\{X = x_i\} \quad \forall i \quad \text{for discrete-value RV}$$

SOME PROPERTIES OF PDF

- $F_X(\infty) = 1\,,\; F_X(-\infty) = 0$
- $F_X(x)$ non-decreasing
- $F_X(x)$ is right continuous
- $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$
- $P\{X = x\} = F_X(x) - F_X(x^-)$

  $(P\{X = x\} = 0$ unless $F_X(x) \neq F_X(x^-), F_X(x)$ has a jump)
- $P\{x_1 \le X \le x_2\} = F_X(x_2) - F_X(x_1^-)$

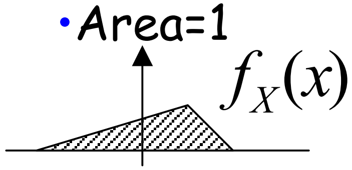
EXAMPLE-PDF



$F_X(x)$
$F_X(\infty)=1$
1
$F_X(x_0)$
A
$F_X(x_0^-)$
$P\{X=x_1\}=F_X(x_1)-F_X(x_1^-)=0$
$F_X(-\infty)=0$
$P\{X=x_0\}=F_X(x_0)-F_X(x_0^-)=A$
0
x_0
x_1
x
$F_X(x)$ non - decreasing

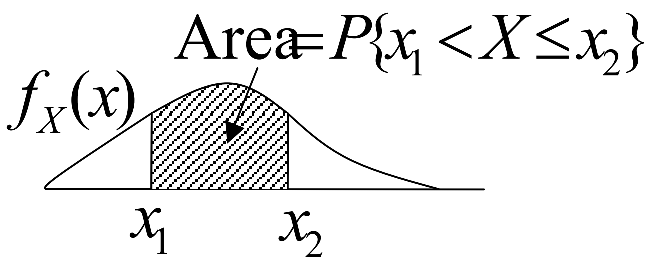
SOME PROPERTIES OF pdf

- $\int_{-\infty}^{\infty} f_X(x)dx = 1 \quad (= F_X(+\infty))$



- $f_X(x) \geq 0 \qquad$ (since $F_X(x)$ is non-decreasing)

- $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(t)dt$



MORE ON RVs

- Conditional & joint PDF/pdf/PMF; expectations; moments, moment generating functions & characteristic functions

EXAMPLE:

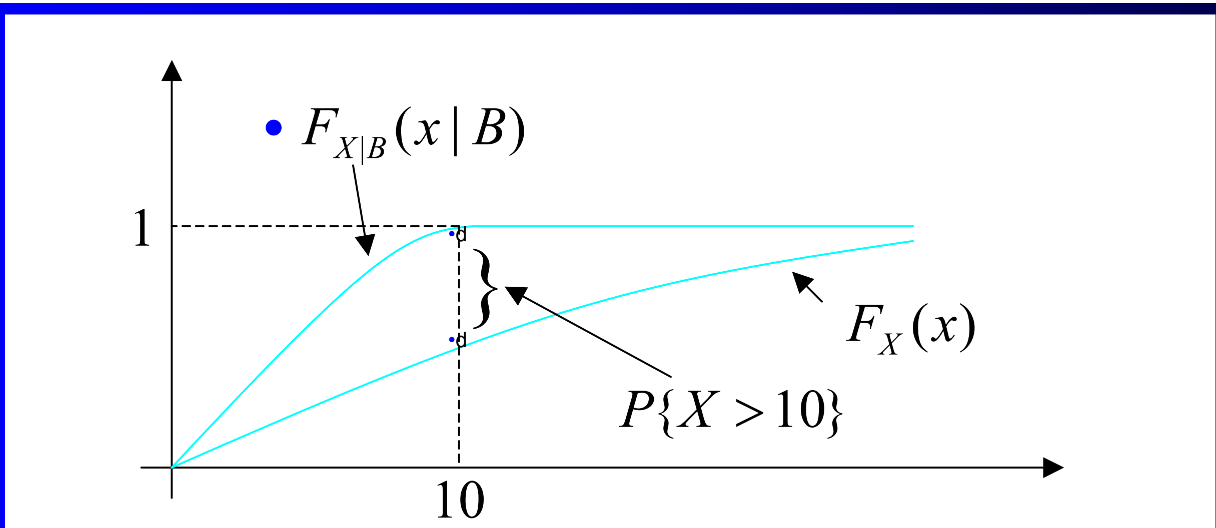
Given $F_X(x)$ and $B=\{X \leq 10\}$ calculate $F_{X/B}(x/B)$

ANSWER:

$$F_{X/B}(x/B) = P\{X \leq x \mid B\} = \frac{P\{X \leq x, B\}}{P\{B\}} = \frac{P\{X \leq x, X \leq 10\}}{P\{X \leq 10\}}$$

$$(i) \quad \text{if } x \geq 10 \quad : \quad F_{X/B}(x/B) = \frac{P\{X \leq 10\}}{P\{X \leq 10\}} = 1$$

$$(ii) \quad \text{if } x < 10 \quad : \quad F_{X/B}(x/B) = \frac{P\{X \leq x\}}{P\{X \leq 10\}} = \frac{1}{P\{X \leq 10\}} F_X(x)$$



<u>Note</u>:

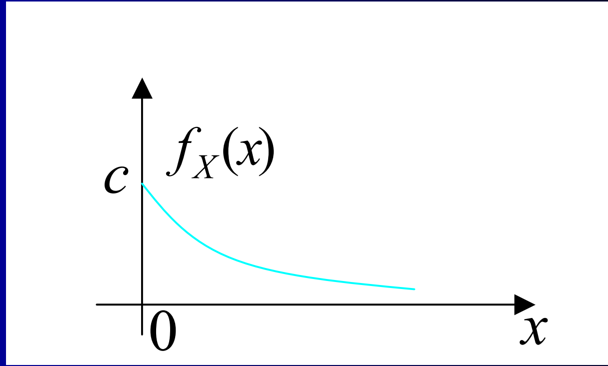
Given X≤10, cond. PDF must reach 1 at x=10

P{X>10} is "distributed" to values of X≤10 by increasing the original mass by 1/P{X≤10}

EXAMPLE: memoryless property of exponential RV

$$F_X(x) = 1 - e^{-cx} \quad , x \geq 0$$

$$f_X(x) = ce^{-cx} \quad , x \geq 0$$



Consider that duration of calls is exp

QUESTION: Given that a call is still on t units after initiation (i.e., X≥t), calculate the probability it will still be on after s time units (i.e.,find A=P{X>t+s|X>t})

ANSWER:

$$A=\frac{P\{X>t+s,X>t\}}{P\{X>t\}}=\frac{P\{X>t+s\}}{P\{X>t\}}=$$

$$=\frac{1-(1-e^{-c(s+t)})}{1-(1-e^{-ct})}=\frac{e^{-cs}e^{-ct}}{e^{-ct}}=e^{-cs}=P\{X>s\}$$

That is :

$$P\{X>t+s\mid X>t\}=P\{X>s\}$$

(the history factor (i.e. already on for t units) does not alter the result)

SEQUENCES OF RVs $\{X_1, X_2, \ldots, X_n, \ldots\}$

SOME KEY QUESTIONS:

- What is the behavior as X_n as $n \to \infty$ or n is very large?
- What is » » as $\Sigma^n_{i=1} X_n$ » » » » ?

USEFUL IN:

- Determining a behavior in steady-state (i.e. X_n, $n \to \infty$); also existence
- Dealing with averages of large number of samples

SENSES OF CONVERGENCE OF $\{X_n(\omega)\}\text{-}>X(\omega)$

(a) Everywhere (e.): iff it holds ;ω[Ω

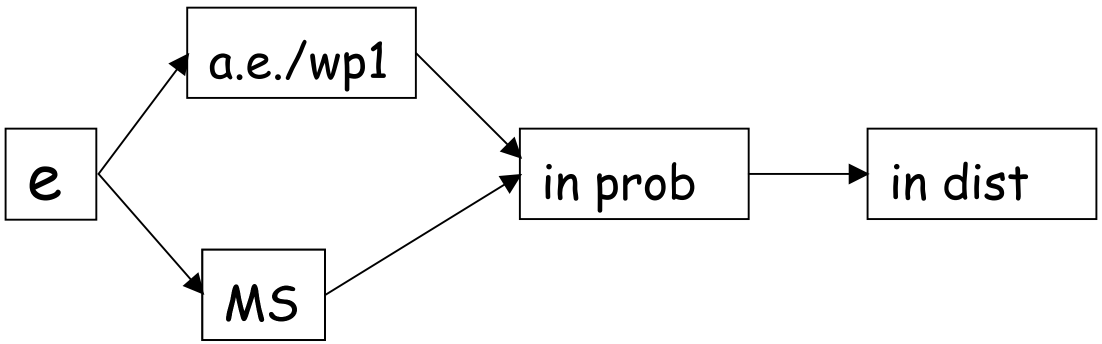
(b) Almost Everywhere or with probability 1 (a.e./wp1): iff it does not hold ONLY for ω[A with P(A)=0

(c) In Probability:
iff $\lim_{n\text{-}>\infty} P\{|X_n(\omega)-X(\omega)|\geq\varepsilon\}=0$;$\varepsilon>0$

(d) Mean Square (M.S.):
iff $\lim_{n\text{-}>\infty} E\{|X_n(\omega)-X(\omega)|^2\}=0$

(e) In distribution: iff $\lim_{n\text{-}>\infty} F_{Xn}(x)=F_X(x)$
; point x of continuity

COMPARISON OF SENSES OF CONVERGENCE



COMPARISON OF SENSES OF CONVERGENCE

EXAMLE/APPLICATION: LAW OF LARGE NUMBERS: (Averaging)

- $\text{WLLN}: \{X_i\}$ indep. RVs, $E\{X_n\} < \infty, \sigma^2 < \infty; S_n = \sum_{i=1}^{n} X_i$; then

  $$Y_n = \frac{S_n - E\{S_n\}}{n} \to Y = 0 \text{ in prob. (weak Law of Large Numbers - WLLN)}$$

  (if $\{X_i\}$ are i.i.d., $\frac{S_n}{n} \to E\{X_n\}$ in prob;

  could be used to estimate the mean of unknown RV)

- $\text{SLLN}: \{X_i\}$ indep. and (a) i.i.d. with $E\{X_i\} < \infty$ or (b) $\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$; then

  $$Y_n = \frac{S_n - E\{S_n\}}{n} \underset{n \to \infty}{\to} \text{a.e. (Strong Law of Large Numbers - SLLN)}$$

EXAMPLE: Proof of WLLN for i.i.d. RVs $\{X_i\}$

Using Chebyshev's inequality for $Y_n = S_n/n$:

$$A = P\{|Y_n - E\{Y_n\}| \geq \delta\} \leq \frac{Var\{Y_n\}}{\delta^2}$$

$$E\{Y_n\} = E\{X_i\}$$

$$Var\{\frac{1}{n}S_n\} = \frac{1}{n^2}Var\{S_n\} \overset{\text{uncorrelated } \{X_i\}}{=} \frac{1}{n}Var\{X_i\}$$

$$\text{Thus} : A \leq \frac{Var\{X_i\}}{n\delta^2} \underset{n\to\infty}{\to} 0$$

$$\text{and } \frac{S_n}{n} \to E\{X_i\} \text{ in prob.}$$

EXAMPLE/APPLICATION: CENTRAL LIMIT THEOREM (normalizing)

$\{X_i\}$ indep. RVs; $\mu_i ; \sigma_i^2$

$$Y_n = \frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i}{\left[\sum_{i=1}^{n} \sigma_i^2\right]^{\frac{1}{2}}} \quad , \quad E\{Y_n\} = 0, Var\{Y_n\} = 1$$

under general conditions $F_{Y_n}(x) \to N(0,1)$ (Normal PDF)

(iff X_i i.i.d. and $\sigma < \infty$, it always holds)

COMMENT ON CLT

To hold look for a lot of RVs with good variability

$$X_i \text{ indep.} \Rightarrow w_i = \frac{X_i - E\{X_i\}}{\left(\sum \sigma_i^2\right)^{\frac{1}{2}}} \quad \text{indep.}$$

$$Y_n = \sum w_i \Rightarrow f_{Y_n} = f_{w_1} \otimes f_{w_2} \otimes \ldots \otimes f_{w_n}$$

$$\Rightarrow \Phi_{Y_n}(\omega) = \Phi_{w_1}(\omega)\Phi_{w_2}(\omega)\ldots\Phi_{w_n}(\omega)$$

$$\Phi_{w_i}(\omega) = E\{e^{j\omega z}\} = \text{characteristic function of } w_i$$

$$(\text{for normal } Y : \Phi_Y = e^{-\frac{\omega^2}{z}})$$

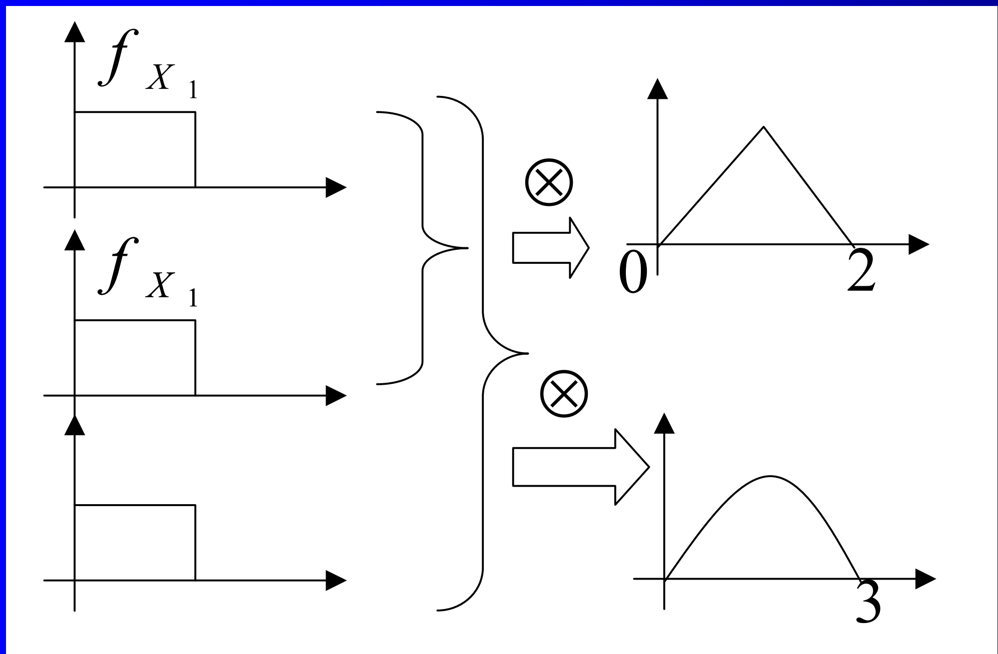
- if $f_{w_i}(\cdot)$ are delta fns (0 variance) $\Rightarrow \Phi_{w_i}(\omega) = 1$

$$\Rightarrow \Phi_{Y_n}(\omega) = 1 \neq e^{-\frac{\omega^z}{z}} \quad \text{(CLT cannot hold)}$$

- if only m RVs have variance and rest have almost zero,

$$\Phi_{Y_n}(\omega) \approx \Phi_{w_1}(\omega) \ldots \Phi_{w_m}(\omega) \cdot 1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \neq e^{-\frac{\omega^z}{z}}$$

- Same conclusions in time domain as well :



Alternatively, the above variance requirement can be stated as:

$$\sigma_i^2 \ll \sum_{i=1}^{n} \sigma_i^2$$

STOCHASTIC PROCESSES

Prob space $\{\Omega, \mathsf{F}, P\}$

A mapping $X(\mathrm{t}, \omega), \omega \in \Omega, t \in I$

(index set e.g. time (discrete or continuous))

is a stochastic process iff $X(t_0, \omega)$ is a RV for any fixed t_0

Interpretation 1 :

It is a collection of RVs $\{X_t(\omega); t \in I\}$

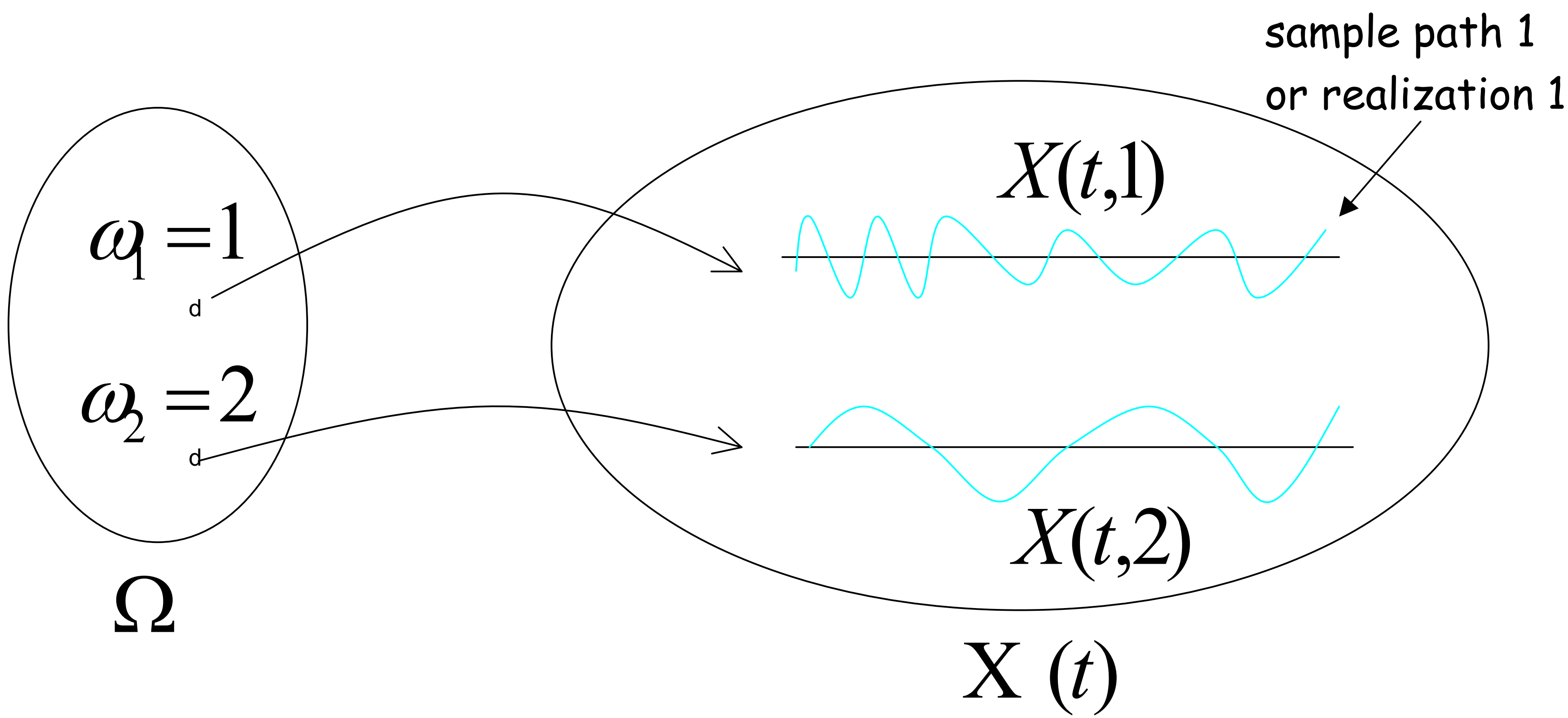
defined on common prob space

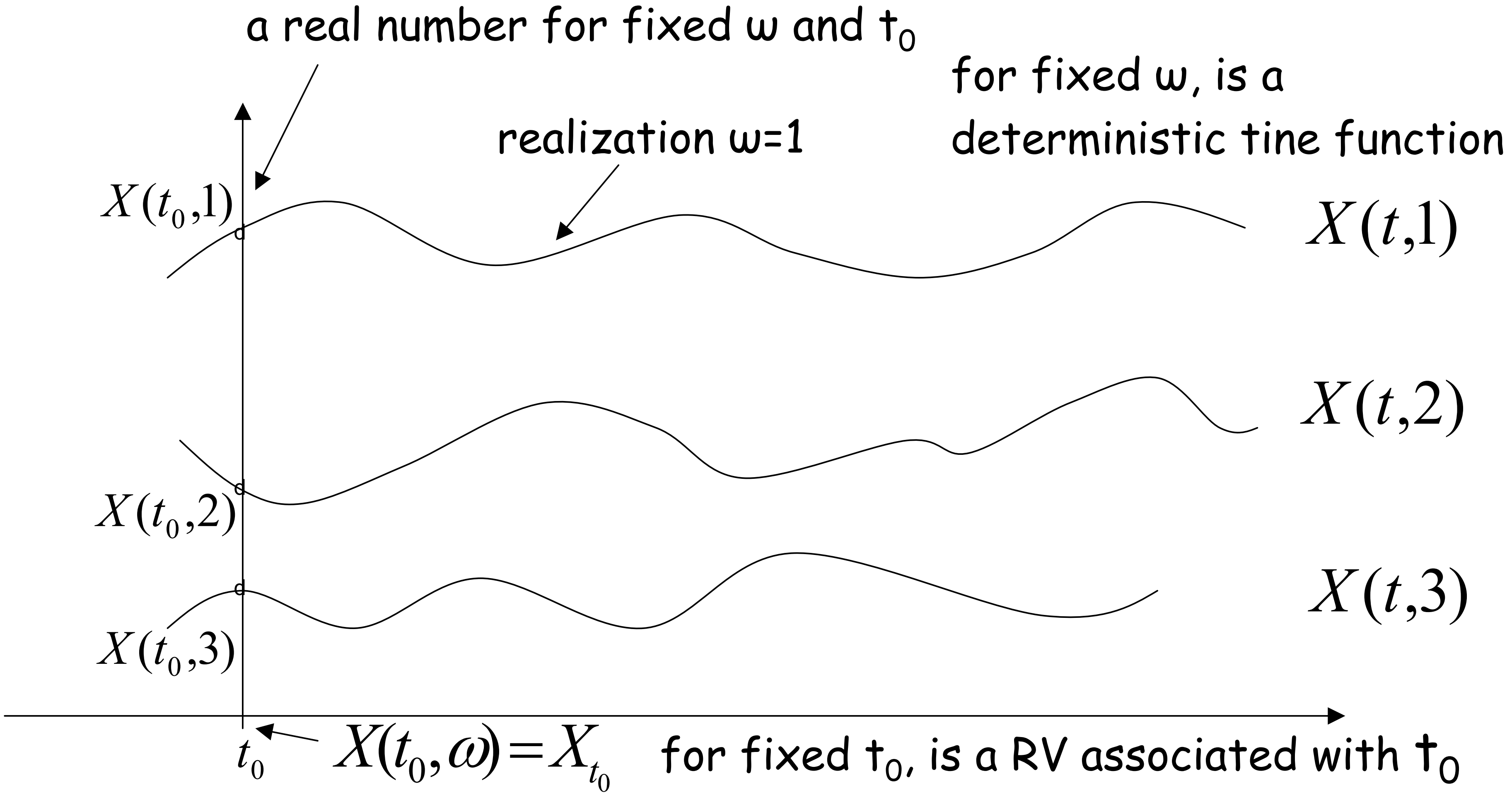
Interpretation 2 :

It is a mapping $\Omega \to X(t)$

$X(t)$ contain time functions instead of real numbers (case of RV)

For fixed t_0 , $X(t_0)$ is a RV

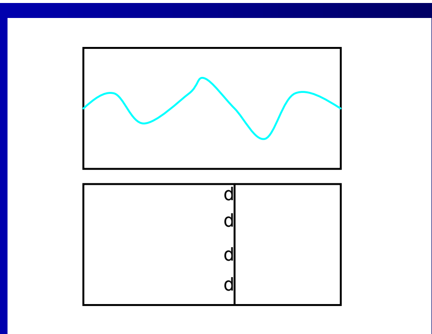




Oscilloscope "view"

Freeze image -> Sample path

Block screen except a slit



DESCRIPTION OF A STOCH PROCESS

Joint PDF : $F_{\overline{X}}(X_1, X_2, \ldots, X_n, \ldots)$

Mean : $\mu_X(n) = E\{X_n\}$

Autocorrelation fn : $R_X(k,l) = E\{X_k X_l^*\}$

Autocovariance fn : $K_X(k,l) = E\{(X_k - \mu_X(k))(X_l - \mu_X(l)^*)$

$$= R_X(k,l) - \mu_X(k)\mu_X(l)$$

Can be facilitated by:

- **Taking advantage of special structure of some processes (indep. increments, Markov, etc.)**
- **Stationarity**
- **Ergodicity**

PROCESS WITH INDEPENDENT INCREMENTS

$\forall\ N > 1$, all $n_1 < n_2 < \ldots < n_N$

$Y_{n_1}, Y_{n_2} - Y_{n_1}, \ldots, Y_{n_N} - Y_{n_{N-1}}$ are indep.

then for $\{Y_n\}_n$ with indep. increments:

$$F_{\overline{Y}}(y_1, y_2, \ldots, y_N) = F_{Y_1}(y_1) F_{Y_2 - Y_1}(y_2 - y_1) \ldots F_{Y_1}(y_1) F_{Y_N - Y_{N-1}}(y_N - y_{N-1})$$

Example: BINOMIAL COUNING PROCESS

$S_n = \sum X_i \quad , \quad X_i$ i.i.d. wp p

$E\{S_n\} = np \quad (S_n$ not WSS)

Let $I_{k_n} = S_n - S_k = X_{k+1} + \ldots + X_n \quad , n > k$

I_{k_n} indep. of S_n

Stationarity

- Strict Sense Stationarity (SSS) :

$$f_X(x_0,\ldots,x_{n-1}) = f_X(x_k,\ldots,x_{k+(n-1)}) \quad \forall\, k\,, \forall n \geq 1$$

(Neither prob behavior of each x_k

nor interdependencies change with k)

- Wide Sense Stationarity (WSS):

$$\mu_X(n) = \text{constant} \quad \forall n \geq 1$$

$$K_X(k,l) = K_X(k+\rho, l+\rho) \quad \forall \rho \geq 1$$

(both mean and covariance are time-shift invariant)

Note: if $\{x_n\}$ is not WSS it is also not SSS

THE WIENER PROCESS

$$\Omega = \{s, f\} \quad , \quad p_f = 1 - p_s$$

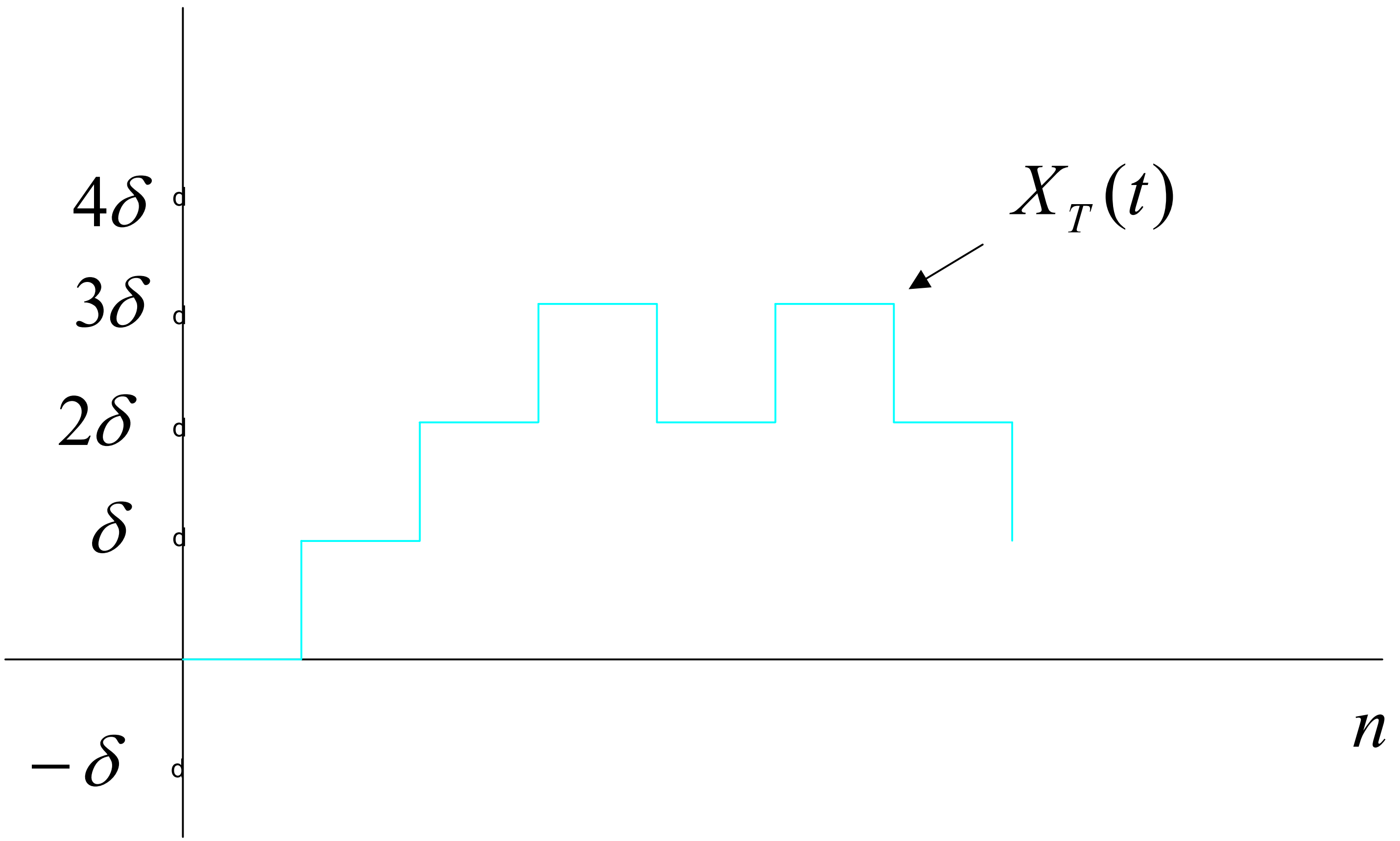
Let $\quad w : \Omega \to \{+\delta, -\delta\}$

where $\quad w = \begin{cases} +\delta & \text{if} \quad w = s \\ -\delta & \text{if} \quad w = f \end{cases}$

Let $\; X_n = w_1 + w_2 + \ldots + w_1$

$\{X_n(\omega)\}$ is a random sequence which increases by $+\delta$ or decreases by δ in each step. (Random walk)

THE WIENER PROCESS



For fixed n, $X_n(\omega)$ is a RV with prob.:

$$P\{X_n = r\delta\} = \Pr\{\#\,successes \; - \#\,failures \; = r\} =$$

$$= P\{k - (n-k) = r\} = P\{k = \frac{n+r}{2}\}$$

$$= \binom{n}{(n+r)/2} p_s^{(n+r)/2} p_f^{\,n-(n+r)/2} \underset{p_f = p_s = 1/2}{=}$$

$$= \binom{n}{(n+r)/2} 2^{-n} \qquad (\frac{n+r}{2} \text{ integer})$$

$$E\{X_n\} = \sum_{k=1}^{n} E\{W_k\} = 0$$

$$E\{X_n^2\} = \sum_{k=1}^{n} E\{W_k^2\} = \sum_{k=1}^{n} \left\{ \frac{1}{2}\delta^2 + \frac{1}{2}\delta^2 \right\} = n\delta^2$$

Let $X_T(t)$ be the piecewise constant version of X_n. That is,

$$X_T(t) = \sum_{k=1}^{\infty} W_k \cdot u(t - kT) \quad , \quad W_k = \begin{cases} +\delta & wp \quad 1/2 \\ -\delta & wp \quad 1/2 \end{cases}$$

Clearly

Let $X_T(t)$ be the piecewise constant version of X_n. That is,

$$X_T(t)\Big|_{t = kT} = X_k = \sum_{i=1}^{k} w_i$$

The PMF and moments of $X_T(t)$ can be obtained from those of X_k

Definition of Wiener Process

- The Wiener process (or Wiener-Levy or Brownian motion) is the process whose distribution is the limiting distribution of $X_T(t)$ as T->0; the jump size δ *goes* to zero as well and X(t) is a continuous state continuous time stochastic process.

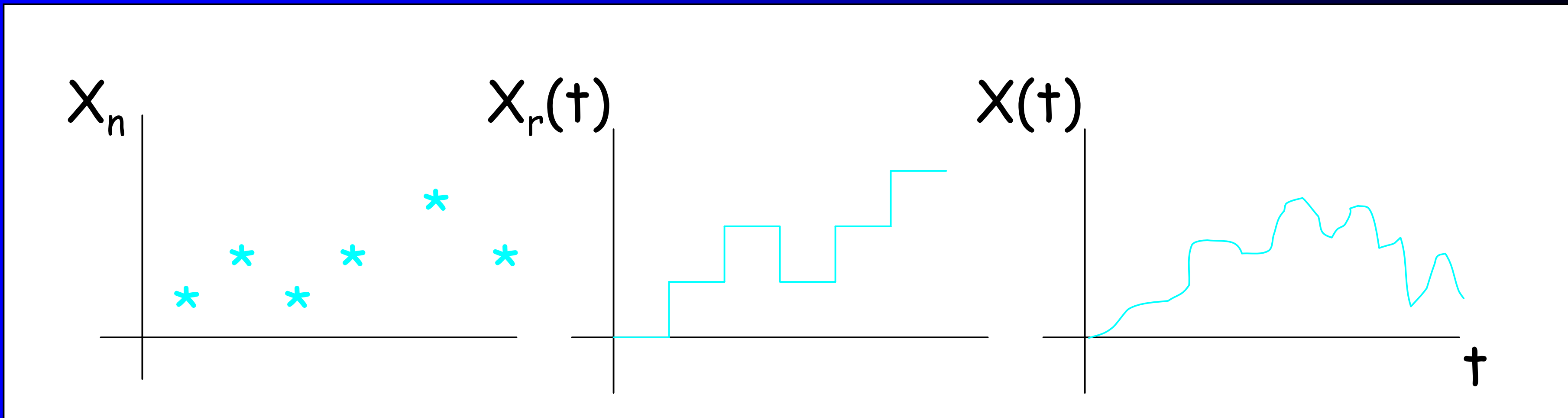
$$E\{X_T(nT)\} = E\{X_n\} = 0$$

$$E\{X_T^2(nT)\} = E\{X_n^2\} = n\delta^2 \Rightarrow Var\{X_T(nT)\} = n\delta^2$$

and $Var\{X_T(t)\} = \dfrac{t}{T}\delta^2 \quad$ for $t = nT$

Suppose that $\delta^2 = aT$ so that the variance does not vanish as T- > 0. Then

$$X(t) = \lim_{\substack{T->0 \\ S^2=aT}} X_T(t) \quad \text{and} \quad Var\{X(t)\} = at$$



Wiener process models the chaotic motion of gas molecules.

Note:

$\{X_n\}_n$ is the sum of n i.i.d. RVs w_i

$$\Downarrow$$

(Central Limit Theorem) limiting process (as n increases) would be Gaussian, and the same would be expected for X(t), the Wiener process (as t->0).

$$E\{X(t)\} = \mu_X(t) = 0 \quad , \quad Var\{X(t)\} = \sigma_x^2(t) = at$$

$$f_{X_t}(x_t) = \frac{1}{\sqrt{2\pi at}} e^{-\frac{x_t}{2at}} \quad t > 0$$

(notice that $X(t)$ is not SSS since $f_{X_t}(x_t)$ changes with time)

- **The Wiener process is an example of Gaussian process since all n^th-order pdf's are Gaussian. This can be seen intuitively as follows:**

Let $\Delta = X(t) - X(s)$, $t > s$. By considering the infinite number of i.i.d. RVs between $X(s)$ and $X(t)$ and applying the CLT we have:

$$f_{\Delta}(y; t-s) = \frac{1}{\sqrt{2\pi a(t-s)}} e^{-\frac{y^2}{2a(t-s)}}$$

since $\quad E\{X(t) - X(s)\} = E\{\Delta\} = 0$

and $\quad E\{(X(t) - X(s))^2\} = a(t-s) \quad t > s$

- Using the pdf for the increment Δ and the independent increment property of the Wiener process we can derive the n^{th}-order pdfs and show they are Gaussian.

$$(f_{X_{t_1}X_{t_2}}(x_{t_1}, x_{t_2}) = f_{X_{t_1}}(x_{t_1}) \cdot f_{X_{t_2}-X_{t_1}}(x_{t_2} - x_{t_1}))$$

➢Covariance or autocorrelation function:

$$K_X(t,s) = E\{X(t)X^*(s)\} \overset{t>s}{=} E\{[X(t)-X(s)+X(s)]X^*(s)\}$$

$$= E\{[X(t)-X(s)]X^*(s)\} + E\{X^2(s)\} = as$$

Thus

$$K_X(t,s) = a \cdot \min(t,s)$$

MEAN SQUARE CALCULUS & ERGODICITY

- CALCULUS: Define limits, integrals & derivatives for SPs
- ERGODICITY: Condition under which time averages=ensemble (over Ω) averages

Mean Square Continuity

$$\text{A SP } \; X(t) \; \text{ is continuous at } t \Leftrightarrow E\{|X(t+\varepsilon)-X(t)|^2\} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

$$\text{N \& S condition} : R_X(t,t) \quad \textit{is continuous at } t$$

$$\left(E\{|X(t+\varepsilon)-X(t)|^2\} = R_X(t+\varepsilon,t+\varepsilon)+R_X(t,t)-R_X(t,t+\varepsilon)-R_X(t+\varepsilon,t) \underset{\varepsilon\to 0}{\longrightarrow} 0 \right)$$

$$\text{Note} : \text{N \& S condition for WSS SP} : R_X(\tau) \; \text{ continuous at } \tau = 0$$

Mean Square Derivative at t

Condition :

$$E\left\{\left|\frac{X(t+\varepsilon_1)-X(t)}{\varepsilon_1}-\frac{X(t+\varepsilon_2)-X(t)}{\varepsilon_2}\right|^2\right\}\underset{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}{\longrightarrow} 0$$

N & S condition : $\dfrac{d^2R_X(t_1,t_2)}{dt_1dt_2}$ exists at $t = t_1 = t_2$

Note : N & S condition for WSS SP : $\dfrac{d^2R_X(\tau)}{d\tau^2}$ exists at $\tau = 0$.

Moments of the Derivative SP

$$E\{X'(t)\} = \frac{d\mu_X(t)}{dt} \qquad , \quad R_{X'}(t_1,t_2) = \frac{d^2 R_X(t_1,t_2)}{dt_1 dt_2}$$

Example - Wiener SP

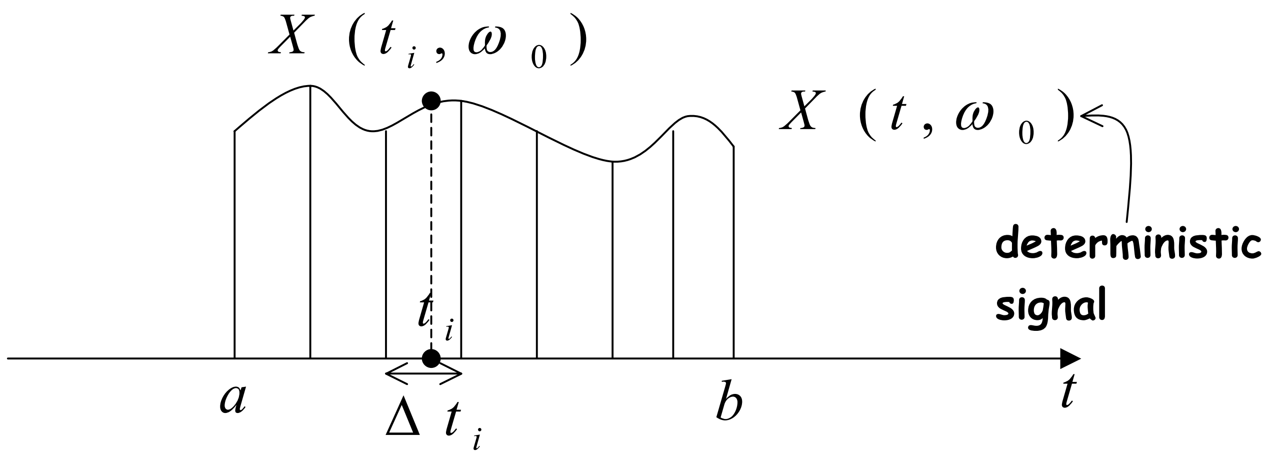
$$R_X(t_1,t_2) = a\min(t_1,t_2) = \begin{cases} at_1 & \text{if} \quad t_1 < t_2 \\ at_2 & \text{if} \quad t_2 < t_1 \end{cases}$$

$$\frac{dR_X(t_1,t_2)}{dt_2} = \begin{cases} 0 & \text{if} \quad t_1 < t_2 \\ a & \text{if} \quad t_2 < t_1 \end{cases} = a \cdot u(t_1 - t_2)$$

$$\frac{d^2 R_X(t_1,t_2)}{dt_1 dt_2} = \frac{d\, a \cdot u(t_1 - t_2)}{dt_1} a\delta(t_1 - t_2) \quad \text{i.e. it is white (uncorrelated) and since Gaussian} \Rightarrow \text{indep.}$$

(as expected for a SP with indep. increments)

Mean Square Integral



$$\text{Note}: \int_a^b X(t)dt = Y \quad \text{(a R.V.)}$$

$$\int_a^b X(t,\omega_0)dt = Y(\omega) = \text{value of } Y \text{ at } \omega_0 = \text{a number}$$

$$\text{N \& S condition}: \quad \int_a^b \int_a^b R_X(t_1,t_2)dt_1 dt_2 \quad \text{exists}$$

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$$\text{For } \ Y = \int_a^b X(t)dt$$

$$\mu_Y = \int_a^b \mu_X(t)dt \quad , \quad E\{Y^2\} = \int_a^b\int_a^b R_X(t_1,t_2)dt_1dt_2$$

$$\text{Example for } X(t) \ \text{ WSS SP} \left(Y = \int_{-T}^{T} X(t)dt \right):$$

$$\sigma_Y^2 = \int_{-T}^{T}\int_{-T}^{T} K_X(t_1-t_2)dt_1dt_2 = \ldots = \int_{-2T}^{2T} K_X(\tau)[2T-|\tau|]d\tau$$

Example for $n(t)$ zero mean, white SP $\left(X(t) = \int_0^t n(\tau) d\tau \right)$:

$$K_n(t_1, t_2) = A\delta(t_1 - t_2)$$

$$\sigma_X^2 = \int_0^t \int_0^t A\delta(t_1 - t_2) dt_1 dt_2 = A\int_0^t dt_1 = At$$

(if $n(t)$ is also Gaussian, then $X(t)$ is Wiener)

ERGODICITY

$X(t)$ is a SP, $X(t,\omega_0)$ = a sample path/realiztion

Time Averages ($A\{\cdot\}$) of (functions of) $X(t,\omega_0)$ are possible

Ensemble Averages ($E\{\cdot\}$) are typically not possible but needed

Question :

$$\mathrm{A}\{\cdot\} \overset{?}{=} \mathrm{E}\{\cdot\}$$

Answer :

Yes if $X(t)$ ergodic in some sense.

Definitions :

$$A\{\cdot\} = \lim_{T\to\infty} A_T\{\cdot\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \bullet \, dt$$

$E\{\cdot\} =$ an average of $\bullet$ over all ω

with the corresponding probability as weights

Examples :

For $\bullet = X(t, \omega_0)$

$$A\{X(t,\omega_0)\} \stackrel{?}{=} E\{X(t,\omega_0)\} = E_{\omega_0}\{X(t,\omega_0)\} = \mu_X$$

For $\bullet = X(t,\omega_0) X(t+\tau,\omega_0)$

$$A\{X(t,\omega_0)X(t+\tau,\omega_0)\} \stackrel{?}{=} E\{X(t,\omega_0)X(t+\tau,\omega_0)\} = R_X(\tau)$$

IMPORTANTOBSERVATION A (need for some stationarity)

For $X(t)$ to be POSSIBLE to be ergodic it must be stationary (time shift invariant) to a certain extent

Argument : Since $A\{\cdot\}$ is indep. of time,

if $E\{\cdot\} = A\{\cdot\}$ then

$E\{\cdot\}$ must also be indep. of time (constant E)

IMPORTANTOBSERVATION B (need for some regularity)

For $X(t)$ to be POSSIBLE to be ergodic it must be regular
(i.e. time averages be indep. of ω)

Argument : Since $E\{\cdot\}$ is indep. of ω,
if $A\{\cdot\} = E\{\cdot\}$ then
$A\{\cdot\}$ must also be indep. of ω (constant A)

Comment on need for regularity :

$\triangleright$ For any ω_0 , $X(t, \omega_0)$ should behave over time like any other $X(t, \omega_i)$ so that ω is NOT identifiable by a time average

$\triangleright$ For any ω_0 , $X(t, \omega_0)$ should exhibit over time all kinds of behavior and with the proper "frequency & duration" as for any ω, so that the time average becomes equivalent to ensemble average

DEFINITION : $X(t)$ is ergodic if :

(a) It is stationary to some extent ($E\{\cdot\} = E$ (constant))

(b) It is regular ($A\{\cdot\} = A$ (constant))

(c) Constants E and A are equal

((c) is met if $E\{\}$ & $A\{\}$ operators are interchangeable - Fubinni's theorem)

EXAMPLE : $\Omega = \{-2,-1,1,2\}$, $X(t,\omega) = \omega$ for $-\infty < t < \infty$

, $P(\omega) = \frac{1}{4}$ for $\omega \in \Omega$

Ensemble Average : $E\{X(t,\omega)\} = E\{\omega\} = 0$ (time shift - invariant)

Ensemble Average : $A\{X(t,\omega)\} = \omega$ (not regular)

$X(t,\omega)$ not ergodic.

M.S. ERGODICITY IN THE MEAN

Def : $$A_T\{X(t)\} = \frac{1}{2T}\int_{-T}^{T} X(t)dt \underset{T\to\infty}{\longrightarrow} \mu_X \quad \text{(m.s.)} \quad (*)$$

N & S condition :

$$\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})K_X(\tau)d\tau\right\} = 0 \qquad (**)$$

Proof : $$(*) \Leftrightarrow E\{|A_T\{X(t)\} - \mu_X|^2\} \underset{T\to\infty}{\longrightarrow} 0 \Leftrightarrow \sigma^2_{A_T} \underset{T\to\infty}{\longrightarrow} 0$$

$$\sigma^2_{A_T} = \frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T} K_X(t_1 - t_2)dt_1 dt_2 = (**)$$

Examples:

(a) if $\int_{-\infty}^{\infty} |K_X(\tau)|\,d\tau < \infty$ then $(**) < \frac{1}{2T}\int_{-\infty}^{\infty} |K_X(\tau)|\,d\tau \underset{T\to\infty}{\to} 0$

and $X(t)$ is ergodic in the mean

(b) if $K_X(0) < \infty$ & $K_X(\tau) \underset{\tau\to\infty}{\to} 0$ then $X(t)$ is ergodic in the mean

$$\text{Proof}: (**) < \frac{1}{2T}\left\{\int_{-\alpha}^{\alpha} |K_X(\tau)|\,d\tau + \int_{a<|\tau|<2T} |K_X(\tau)|\,d\tau\right\} \quad (a : |K_X(\tau)| < \varepsilon, |\tau| > a)$$

$$< \frac{1}{2T}\{2a\cdot K_X(0)d\tau + 4T\cdot\varepsilon\} = \frac{2aK_X(0)}{2T} + 2\varepsilon$$

$$= \text{arbitrarily small as } T \to \infty$$

M.S. ERGODICITY IN THE AUTOCORRELATION

Def : $A_T\{X(t)X^*(t+\lambda)\} \underset{T\to\infty}{\longrightarrow} R_X(\lambda) \qquad (m.s.)$

$\Big[$Equivalent : $A_T\{\Phi_\lambda(t)\} \underset{T\to\infty}{\longrightarrow} E\{\Phi_\lambda(t)\} \qquad (m.s.)$

for SP $\Phi_\lambda(t) = X(t)X^*(t+\lambda)$ $\Big]$

N & S condition :

$$(**) \Rightarrow \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)K_{\Phi_\lambda}(\tau)\,d\tau \underset{T\to\infty}{\longrightarrow} 0$$

M.S. ERGODICITY IN DISTRIBUTION

Index function : $I_X(x,t) = \begin{cases} 1 & \text{if} \quad X(t) \le X \\ 0 & \text{otherwise} \end{cases}$

Def : $A_T\{I_X(x,t)\} = \frac{1}{2T}\int_{-T}^{T} I_X(x,t)dt \underset{T\to\infty}{\to} F_X(x)$

N & S condition :

$$(**) \Rightarrow \frac{1}{2T}\int_{-2T}^{2T}(1-\frac{|\tau|}{2\tau})K_{I_X}(x,\tau)d\tau \underset{T\to\infty}{\to} 0$$

where $K_{I_X}(x,\tau) = E\{I_X(x,t)I_X^*(x,t+\tau)\} - [E\{I_X(x,t)\}]^2$

$$= F_{X_t X_{t+\tau}}(x,x) - [F_{X_t}(x)]^2 \qquad (***)$$

Note : $(**)$ implies that $K_{I_X}(x,\tau)$ should vanish to 0 as $T\to\infty$,

i.e. $F_{X_t X_{t+\tau}}(x,x) \underset{T\to\infty}{\to} F_{X_t}(x)\cdot F_{X_t}(x)$

or X_t , $X_{t+\tau}$ should be asymptotically independent

intuitively expexted for $A\{I_X\}$ to be equal to $E\{I_X\}$