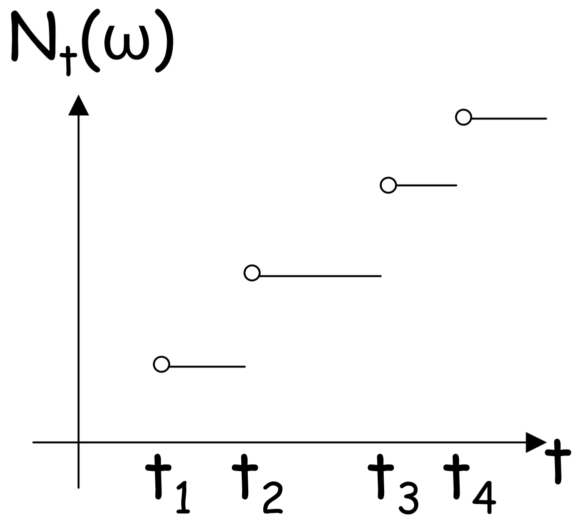
POISSON PROCESSES

COUNTING (ARRIVAL) PROCESS

- $\{N_t(\omega); t>0\}$ defined on some sample space Ω is called a counting process provided that:

  (1) it is non-decreasing
  (2) it increases by jumps only
  (3) it is right continuous
  (4) $N_0(\omega)=0$



POISSON PROCESS

A counting process that satisfies:

1. Each jump is of unit magnitude
2. (independent increments) For any $t,s \geq 0$, $N_{t+s}-N_t$ is independent of $\{N_u(\omega);u\leq t\}$
3. (stationarity) For any $t,s \geq 0$, the distribution of $N_{t+s}-N_t$ is independent of t

Lemma 1 : For every $t \geq 0$, $P\{N_t=0\}=e^{-\lambda t}$ for some $\lambda \geq 0$

Proof : {0 arrivals in [0,t+s]} $\Leftrightarrow$
$\Leftrightarrow$ {0 arrivals in [0,t]} and {0 arrivals in [t,t+s]} , or

$\{N_{t+s}=0\} \Leftrightarrow \{N_t=0\}$ and $\{N_{t+s}-N_t=0\}$
$P\{N_{t+s}=0\}= P\{N_t=0\}\, P\{N_{t+s}-N_t=0\}$ (indep. increments)
$P\{N_{t+s}=0\}= P\{N_t=0\}\, P\{N_s=0\}$ (stationary increments) (*)

Let $N_t=f(t)$, then
(*) $f(t+s)=f(t)f(s)$, $0 \le f(t) \le 1$, $t,s \ge 0$
the only non-zero f(t) satisfying (*) is $e^{-\lambda t}$, $\lambda \ge 0$

Thus $P\{N_t=0\}= e^{-\lambda t}$, $\lambda \ge 0$

Lemma 2: $\lim_{t \to 0} \frac{1}{t} P\{N_t \geq 2\} = 0$

i.e., $\text{Prob}\{\geq 2 \text{ arrivals over a small t}\} \xrightarrow{\text{t} \to 0} 0$ faster that t

Lemma 3: $\lim_{t \to 0} \frac{1}{t} P\{N_t = 1\} = \lambda$ (arrival rate λ)

Proof : $P\{N_t = 1\} = 1 - P\{N_t = 0\} - P\{N_t \geq 2\} \Rightarrow$

$$\lim_{t \to 0} \frac{1}{t} P\{N_t = 1\} = \lim_{t \to 0} \left\{ \frac{1}{t} - \frac{e^{-\lambda t}}{t} - \frac{1}{t} P\{N_t \geq 2\} \right\}$$

$$= \lim_{t \to 0} \frac{1 - e^{-\lambda t}}{t} = \lambda$$

i.e., $\text{Prob}\{1 \text{ arrivals over a small t}\} \xrightarrow{\text{t} \to 0} \lambda t$

Theorem : If $\{N_t;t\geq 0\}$ is Poisson then

$P\{N_t=k\}=e^{-\lambda t}(\lambda t)^k/k!$, $k=0,1,2,\ldots$ for some $\lambda \geq 0$

Notice on short term behavior : (δ small)

$P\{0 \text{ arrivals in } (t,t+\delta)\} = 1-\lambda\delta+o(\delta^2)$
$P\{1 \text{ arrivals in } (t,t+\delta)\} = \lambda\delta+o(\delta^2)$
$P\{>1 \text{ arrivals in } (t,t+\delta)\} = o(\delta^2)$

Moments of Poisson process

- $E\{N_t\} = \sum_{n=0}^{\infty} nP\{N_t = n\} = \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!} = e^{-\lambda t} \sum_{n=1}^{\infty} n \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda t = \lambda t$

- $E\{N_t^2\} = E\{N_t(N_t - 1) + N_t\} = \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda t}(\lambda t)^n}{n!} + \lambda t$

$$= \sum_{n=0}^{\infty} \underbrace{n(n-1)} \frac{e^{-\lambda t}(\lambda t)^{n-2}(\lambda t)^2}{(n-2)!\underbrace{(n-1)n}} + \lambda t = (\lambda t)^2 + \sum_{n'=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n'}}{n'!} + \lambda t$$

$$= (\lambda t)^2 + \lambda t$$

- $Var\{N_t\} = E\{N_t^2\} - (E\{N_t\})^2 = \lambda t \quad \text{(compare with Wiener's)}$

  $\text{Note}: E\{N_t\} = Var\{N_t\} = \lambda t \ \ \text{(high for large } t\text{)}$

- $R_N(t_1,t_2) = E\{N_{t_1} N_{t_2}\} = E\{N_{t_1}^2 + (N_{t_2} - N_{t_1})N_{t_1}\} =$

$$= \lambda t_1 + (\lambda t_1)^2 + \lambda(t_2 - t_1)\lambda t_1 = \lambda t_1 + \lambda^2 t_1 t_2 \quad (t_1 < t_2)$$

$$= \lambda \min\{t_1,t_2\} + \lambda^2 t_1 t_2 \qquad \text{(Compare with Wiener's)}$$

- $k_N(t_1,t_2) = R_N(t_1,t_2) - E\{N_{t_1}\}E\{N_{t_2}\} = \lambda \min\{t_1,t_2\}$

(Compare with Wiener's)

Note : Above moments similar to those of the Wiener process as a result of the independent increment property, common to both.

Corrolary : if $\{N_t; t \geq 0\}$ is Poisson, then

$$P\{N_{t+s} - N_t = k \mid N_u; u \leq t\} \underset{\text{ind. incr.}}{=} P\{N_{t+s} - N_t = k\}$$

$$\underset{\text{stationary incr.}}{=} P\{N_s = k\} = \frac{e^{-\lambda s}(\lambda s)^k}{k!}$$

Alternative definition of Poisson Process - A

(by checking in samples the validity of the independent increment property)

$\{N_t;t\geq 0\}$ is Poisson with rate λ <u>iff</u>

(a) $N_t(\omega)$ has unit magnitude jumps for almost all ω

(b) $\forall t,s\geq 0 \quad , \quad E\{N_{t+s}-N_t|N_u;u\leq t\}=\lambda s$

Alternative definition of Poisson Process - B

(by checking in samples the validity of the Poisson distribution)

$\{N_t; t\geq 0\}$ is Poisson with rate λ <u>iff</u>

$P\{N_B=k\}=e^{-\lambda b}(\lambda b)^k/k!$, $k=0,1,2,\ldots$

For any subset B of R_t that is the union of a finite number of disjoint intervals whose length sums up to b.

Proposition : (Uniformity of the distribution of the time of Poisson arrival occurrences over an interval)

Let $A_1 \cup A_2 \cup ... \cup A_n = B$, $\{A_i\}_{i=1}^{n}$ disjoint , $|A_i| = a_i$ (length)

$k_1 + k_2 + ... + k_n = k$, all in N, $|B| = b$. Then

$$P\{\underbrace{N_{A_1} = k_1, N_{A_2} = k_2, ..., N_{A_n} = k_n}_{c} \mid \underbrace{N_B = k}_{d}\} = \frac{k!}{k_1!...k_n!}\left(\frac{a_1}{b}\right)^{k_1}...\left(\frac{a_n}{b}\right)^{k_n}$$

$$\text{Proof}: P\{c \mid d\} = \frac{P\{c,d\}}{P\{d\}} = \frac{P\{c\}}{P\{d\}} = \frac{P\{N_{A_1} = k_1\}...P\{N_{A_n} = k_n\}}{P\{N_B = k\}}$$

$\{N_{A_i}\}_i$ independent since $\{A_i\}_i$ disjoint.

On the estimation of λ:

Strong Law of Large Numbers (SLLN) justifies: $\lambda = \lim_{t\to\infty} \frac{N_t(\omega\omega}{t}$ a.e.

Proof: Use discrete unit time axis, n

$$(*)\ N_n = \underbrace{N_1 + (N_2 - N_1) + (N_3 - N_2) + \ldots + (N_n - N_{n-1})}_{\text{are all i.i.d. with mean } \lambda \Rightarrow \text{SLLN holds}}$$

On the limiting behavior of N_t:

From (*) $\Rightarrow N_t$ is the sum of i.i.d. RV's $\Rightarrow$

Central Limit Theorem implies

$$\lim_{t\to\infty} P\left\{\frac{N_t - \lambda t}{\sqrt{\lambda t}} \le x\right\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad \text{(PDF for N(0,1))}$$

(good approximation for $\lambda t \ge 10$ - λt is the variance)

Proposition: (distribution of interarrival times)

$$P\{T_{n+1} - T_n \le s \mid T_0, T_1, \ldots, T_n\} = 1 - e^{-\lambda t} \quad s \ge 0$$

i.e., it is exponentially distributed and independent of past arrival times.

Proof:

$$P\{T_{n+1} - T_n > s \mid T_0, T_1, \ldots, T_n\} = P\{N_{T_{n+s}} - N_{T_n} = 0 \mid T_0, T_1, \ldots, T_n\}$$

$$= P\{N_{T_{n+s}} - N_{T_n} = 0 \mid N_u; u \le T_n\} = \text{(indep. incr. of Poisson)}$$

$$= P\{N_{T_{n+s}} - N_{T_n} = 0\} = e^{-\lambda s}$$

Memorylessness of interarrival times

Since exponential, the distribution of $T_{n+1}-T_n$ is memoryless:

$$P\{T_{n+1}-T_n>t+s|T_{n+1}-T_n>t\}=P\{T_{n+1}-T_n>s\}$$

i.e., knowing that t time units have passed since the last arrival does not affect the time when the next arrival will occur, which remains exponential with the same parameter λ.

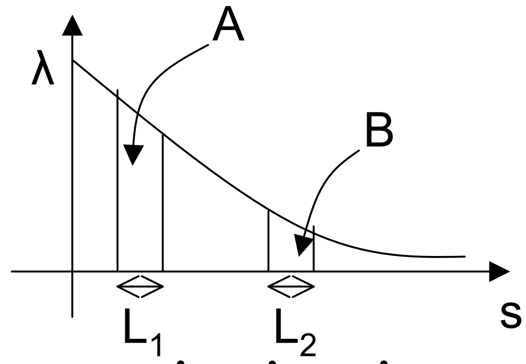
Stated differently: No matter which time instant t I observe the system, the evolution of future arrival times is independent of t and past arrival times.

Thus, **there is no need to maintain any record regarding past arrivals to determine future ones (great simplification in system modeling).**

Burstiness of Poisson arrivals

Let τ be the generic RV for $\tau_n = T_{n+1} - T_n$

$P\{\tau \le s\} = 1 - e^{-\lambda s} \Leftrightarrow f_\tau(s) = \lambda e^{-\lambda s}$



Since $f_\tau(s)$ decreases with $s \Rightarrow$

$P\{\tau \approx L_1\} \approx A > B \approx P\{\tau \approx L_2\}$ (for $L_1 < L_2$)

Thus, **short interarrival times occur more frequently than long ones.**

Thus **arrivals appear in bursts (clusters) (Poisson is a fairly bursty arrival process)**

Make the distinction between "burstiness of arrivals" and "uniformity of the times of arrivals over an interval"

Alternative definition of Poisson Process - C

A counting (arrival) process is Poisson <u>iff</u> the associated interarrival times are independent and identically distributed exponential RVs.

Moments of interarrival times:

$$E\{T_{n+1} - T_n\} = \int_0^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda}$$

$$E\{T_{n+1} - T_n\} = \frac{1}{\lambda^2}$$

On the arrival (not interarrival) times Poisson Process

Moments of arrival times , T_n

$$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \dots + (T_n - T_{n-1})$$

$$E\{T_n\} = \frac{n}{\lambda^2} \quad , \quad Var\{T_n\} = \frac{n}{\lambda^2} \quad \text{(since } T_n - T_{n-1} \text{ indep.)}$$

Distribution of arrival times , T_n (Erlang - n)

$$P\{T_n \le t\} = P\{N_t \ge n\} = 1 - P\{N_t < n\} = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda\lambda t^{k}}{k!}$$

$$f(t) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!} \quad , \quad t \ge 0$$

<u>Note:</u>

Erlang - n is the distribution of the interarrival times of groups of n Poisson arrivals

Erlang - n is the distribution of the sum of n exponential and identical RV's

Example :

$\{U_k\}_{k\geq 1}$ are car interarrival times (assumed indep.)

U_k is erlang - 2 distributed, i.e.,

$$P\{U_k \leq t\} = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} \quad , \quad t \geq 0$$

U_k may be viewed as the sum of two interarrival times in a Poisson process $\{N_t\}$ with rate λ. Let $\{T_k\}_{k\geq 1}$ be the arrival times of that Poisson.

Then

$$U_1 = T_2 \quad , \quad U_2 = T_4 \quad , \quad U_3 = T_6 \quad , \ldots$$

If M_t = # cars that passed by before or at t

$$P\{M_t = k\} = P\{2k \leq N_t < 2(k+1)\} = P\{N_t = 2k\} + P\{N_t = 2k+1\}$$

$$= \frac{e^{-\lambda t}(\lambda\lambda t^{2k}}{2k!} + \frac{e^{-\lambda t}(\lambda\lambda t^{2k+1}}{(2k+1)!} \quad , \quad k = 0,1,2,\ldots$$

Example: (hardware lifetime and replacement cost)
Assume exponential lifetime of a piece of hardware which is replaced by an identical upon failure. Replacement cost is β and the discount rate of money is α (i.e., 1$ spent at time t has present value $e^{-\alpha t}$, α is the interest rate). Find the expected cost.

N_t =# of failures in [0, t]
$\{N_t\}$ is Poisson due to the indep. and expon. lifetimes
$T_n(\omega) = n$th failure for realization ω
$\beta \mathrm{e}^{-\alpha T_n(\omega)}$ = present value of the cost of nth replacement
$C(\omega($ = total cost for the (particular) realization ω

$$C(\omega(= \sum_{n=1}^{\infty} \beta \mathrm{e}^{-\alpha T_n(\omega)} \quad , \quad \omega \in \Omega$$

$$E\{C\} = E\{C(\omega(\} = \beta \sum_{n=1}^{\infty} E\{\mathrm{e}^{-\alpha T_n(\omega)}\} = \beta \sum_{n=1}^{\infty} E\{\mathrm{e}^{-\alpha T_n}\}$$

to use the i.i.d. property of inter - failures write:

$$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + ... + (T_n - T_{n-1})$$

$$E\{C\} = \beta \sum_{n=1}^{\infty} E\left\{ e^{-\alpha T_1} e^{-\alpha (T_2 - T_1)} ... e^{-\alpha (T_n - T_{n-1})} \right\}$$

$$= \beta \sum_{n=1}^{\infty} E\{e^{-\alpha T_1}\} E\{e^{-\alpha (T_2 - T_1)}\} ... E\{e^{-\alpha (T_n - T_{n-1})}\} = \beta \sum_{n=1}^{\infty} [E\{e^{-\alpha T_1}\}]^n$$

$$E\{e^{-\alpha T_1}\} = \int_0^{\infty} e^{-\alpha T_1} \lambda e^{-\lambda t} dt = \frac{\lambda}{\lambda + \alpha} \quad \text{, thus}$$

$$E\{C\} = \beta \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda + \alpha} \right)^n = \beta \frac{\dfrac{\lambda}{\lambda + \alpha}}{1 - \dfrac{\lambda}{\lambda + \alpha}} = \frac{\beta \lambda}{\alpha}$$

The result may be derived by setting $f(t) = \beta e^{-\alpha t} \quad , \quad f(T_n) = \beta e^{-\alpha T_n}$
and using the following

Proposition : for a non-negative function $f(\cdot)$ on R_+

$$E\left\{\sum_{n=1}^{\infty} f(T_n)\right\} = \lambda\int_0^{\infty} f(t)dt$$

where T_n is the occurrence time (arrival) of the nth event in a Poisson process with rate λ.

Proof :

$$E\{f(T_n)\} = \int_0^{\infty} f(t)\frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}dt \quad , \quad \text{thus}$$

$$E\left\{\sum_{n=1}^{\infty} f(T_n)\right\} = \sum_{n=1}^{\infty}\int_0^{\infty} f(t)\frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}dt$$

$$= \int_0^{\infty} \lambda f(t)\sum_{n=1}^{\infty}\frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}dt = \lambda\int_0^{\infty} f(t)dt$$

Stopping time of an arrival process

A RV T is a stopping time of an arrival process $\{N_t\}$ if the occurrence of the event $\{T \le t\}$ is determined by $\{N_u; u \le t\}$, i.e., by knowing the history of the arrival process up to t.

Example 1: T_5=the time of the 5^{th} arrival is stopping time since $\{N_u; u \le t\}$ determines if $\{T_5 \le t\}$.

Example 2: T=first time interarrival time exceeds some value C. It is a stopping time.

Poisson arrivals over [T,T+s] , for T a RV

If T=t (a fixed random point in time) we know that

N_{T+s}-N_T is independent from $\{N_u;u\leq s\}$ and Poisson with rate λs.

If T is not fixed but a RV then above holds <u>if T is stopping time</u> and then

$P\{N_{T+s}-N_T=k| N_u;u\leq T\}=e^{-\lambda s}(\lambda s)^k/k!$, k=0,1,…

e.g., $T=T_n$ is a stopping time

T= time of occurrence of the largest interarrival is not a stopping time since $\{T\leq t\}$ cannot be determined by $\{N_u;u\leq t\}$ since the future evolution of the arrival process is needed as well.

Example: Buses arrive as Poisson with λ=0.2 per minute. Inspector arrives at time of the 5th bus arrival after time t_0 and will stay for 60 minutes. Find the distribution of buses to arrive within these 60 minutes.

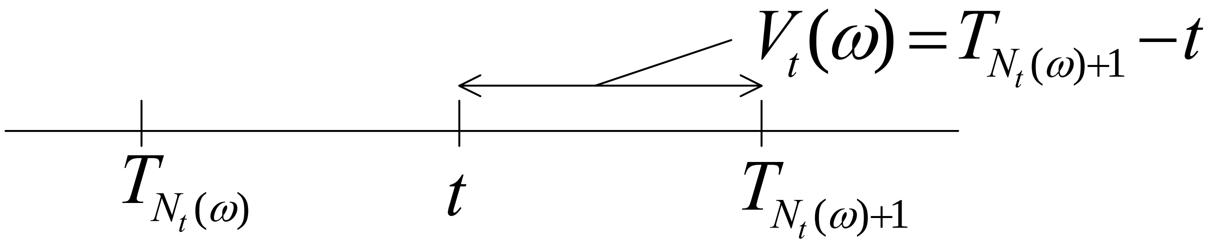
Answer:

Time of arrival of inspector $T = T_{N_{t_0}(\omega)+5} \quad \omega \in \Omega$

T is a stopping time. Thus,

$$P\{N_{T+60} - N_T = k\} = \frac{e^{-12} 12^k}{k!} \quad , \quad k = 0,1,...$$

Forward Recurrence Times



V_t = (remaining) time between current time t and the next arrival

Theorem: $P\{V_t \le u \mid N_s ; s \le t\} = 1 - e^{-\lambda u} \quad , \quad u \ge 0$

(i.e., the distribution of V_t is the same as that of an interarrival)

Proof : $\{V_t \le u\} = \{T_{N_t+1} - t \le u\} = \{T_{N_t+1} - t > u\}^c$

$$= \{T_{N_t+1} > t + u\}^c = \{N_{t+u} - N_t = 0\}^c$$

Thus, $\mathrm{P}\{V_t \le u \mid N_s ; s \le t\} = P\{\{N_{t+u} - N_t = 0\}^c \mid N_s ; s \le t\}$

$$= 1 - P\{N_{t+u} - N_t = 0 \mid N_s ; s \le t\} = 1 - P\{N_{t+u} - N_t = 0\}$$

$$= 1 - P\{N_u = 0\} = 1 - e^{-\lambda u}$$

$$\text{Note:}\quad \mathrm{E}\{V_t\} = \frac{1}{\lambda} \quad \text{and} \quad E\{T_{n+1} - T_n\} = \frac{1}{\lambda}$$

$$E\{T_{N_t+1} - T_{N_t}\} = E\{T_{N_t+1} - -t + t - T_{N_t}\} = E\{V_t\} + E\{t - T_{N_t}\}$$

$$= \frac{1}{\lambda} + E\{t - T_{N_t}\}$$

That is, the interarrival interval that we happen to observe (that covers t) is larger, on average, than an ordinary such interval between two arrivals!!!

Example :

The time between the two consecutive arrivals containing our time of arrival to the bus stop is almost twice as large, on the average, as the typical bus interarrival time, assuming Poisson bus arrivals.

(Reason why the bus is always more late than usual
- or claimed by the company -
when we arrive at the bus stop)

Uniqueness of Poisson superposition property: If L & M are renewal processes and their superposition N is renewal, then all 3 are Poisson (renewal process: i.i.d. but not necessarily exponential interarrival times).

Decomposition of a Poisson process: $N=\{N_t; t\geq 0\}$ Poisson with rate λ, $\{X_n; n=1,2,\ldots\}$ Bernoulli with param. p

$\{S_n; n=1,2,\ldots\}$=# of successes in n trials

$N_t(\omega)$ trials (i.e., arrivals that are split based on p) are carried out in [0,t]

$M_t(\omega)=S_{N_t(\omega)}(\omega)$ is the number of successes over [0,t]

$L_t(\omega)= N_t(\omega)-M_t(\omega)$ is the number of failures over [0,t]

Theorem : $M = \{M_t; t \geq 0\}$ & $L = \{L_t; t \geq 0\}$ are Poisson with rate λp and $\lambda(1-p)$, respectively and M & L are independent.

Proof : Suffices to show that

$$P\{M_{t+s} - M_t = m, L_{t+s} - L_t = k \mid M_u, L_u; u \leq t\} =$$

$$= \frac{e^{-\lambda ps}(\lambda ps)^m}{m!} \cdot \frac{e^{-\lambda(1-p)s}(\lambda(1-p)s)^k}{k!} \quad , \quad k, m = 0,1,2...$$

$\forall \quad t, s \geq 0.$

$\{M_{t+s} - M_t = m, L_{t+s} - L_t = k\} \Leftrightarrow$ (bring in $\mathrm{N_t}$)

$\{N_{t+s} - N_t = m + k, M_{t+s} - M_t = m\} \Leftrightarrow$ (bring in $S_{\mathrm{N_t}}$)

$\{N_{t+s} - N_t = m + k, S_{\mathrm{N_{t+s}}} - S_{\mathrm{N_t}} = m\} = \mathbf{A}$

Now, $\{M_u, L_u; u \le t\} \Leftrightarrow \{N_u; u \le t, X_1, X_2, ..., X_{N_t}\} = \mathbf{B}$

Notice that $N_{t+s} - N_t$ & $S_{N_{t+s}} - S_{N_t}$ (and thus $\mathbf{A}$) are indep. of $\mathbf{B}$

$$
\begin{aligned}
P(\mathbf{A}) &= \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} - N_t = m+k, S_{N_{t+s}} - S_{N_t} = m\} \\
&= \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} = m+k+n, S_{m+k+n} - S_n = m\} \\
&= \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} = n+m+k\} P\{S_{m+k+n} - S_n = m\} \\
&= \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} - N_t = m+k\} P\{S_{m+k} = m\} \\
&= P\{N_{t+s} - N_t = m+k\} P\{S_{m+k} = m\} \\
&= \frac{e^{-\lambda s} (\lambda s)^{m+k}}{(m+k)!} \cdot \frac{(m+k)!}{m!k!} p^m (1-p)^k
\end{aligned}
$$

Example: N is the Poisson arrival process of cars (rate λ). $N^1, N^2, \ldots, N^5$ are the arrival processes of cars with 1,2,…,5 passengers (0.3,0.3,0.2,0.1,0.1 are the passenger occupancy probabilities for 1,2,…,5).

$N^1, N^2, \ldots, N^5$ are Poisson with rates $0.3\lambda, 0.3\lambda, 0.2\lambda, 0.1\lambda, 0.1\lambda$.

Expected # of passengers per unit time:

$E\{N^1+2N^2+3N^3+4N^4+5N^5\}=$
$0.3\lambda+2*0.3\lambda+3*0.2\lambda+4*0.1\lambda+5*0.1\lambda$

Compound Poisson process

(allowing jumps of <u>any</u> size in a Poisson process)

Definition A: $Z=\{Z_t;t\geq 0\}$ is a compound Poisson provided that:

(a) $Z_t(\omega)$ has only finitely many jumps in any finite interval (a.e.)

(b) for all $t,s\geq 0$, $Z_{t+s}-Z_t$ is indep. of $\{Z_n;u\leq t\}$

(c) for all $t,s\geq 0$, the distribution of $Z_{t+s}-Z_t$ depends on s (indep. of t)

Note:

- if $N=\{N_t;t\geq 0\}$ is the process that counts the number of jumps in $(0,t]$, then (b) & (c) ==> N is Poisson.

- Z & N differ in the fact that jumps in Z are not all equal to one (1) but are RV's $\{X_1,X_2,\ldots\}$.

  (b) & (c) ==> $\{X_1,X_2,\ldots\}$ are i.i.d. and, thus, indep. of $\{T_1,T_2,\ldots\}$.

- If $\{T_1,T_2,\ldots\}$ are Poisson arrivals times & $\{X_1,X_2,\ldots\}$ are i.i.d. RV's indep. of $\{T_1,T_2,\ldots\}$ then the sum of all X_i such that $T_i\leq t$, Z_t, forms a compound Poisson process.

Definition: Z is a compound Poisson <u>iff</u> its jump times form a Poisson process & the magnitudes of its jumps are i.i.d RV's independent of the jump times.