# Markov Processes and Applications

- Discrete-Time Markov Chains
- Continuous-Time Markov Chains
- Applications
  - Queuing theory
  - Performance analysis

# Discrete-Time Markov Chains

Books

- Introduction to Stochastic Processes (Erhan Cinlar), Chap. 5, 6
- Introduction to Probability Models (Sheldon Ross), Chap. 4
- Performance Analysis of Communications Networks and Systems (Piet Van Mieghem), Chap. 9, 11
- Elementary Probability for Applications (Rick Durrett), Chap. 5 (http://www.math.cornell.edu/~durrett/ep4a/bch5.pdf)
- Introduction to Probability, D. Bertsekas & J. Tsitsiklis, Chap. 6

## INTRODUCTION :

$n$th order pdf of some stoc. proc. $\{X_t\}$ is given by

$$f(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = f(x_{t_n} \mid x_{t_n}, x_{t_{n-1}}, \ldots, x_{t_1}) f(x_{t_{n-1}} \mid x_{t_{n-2}}, x_{t_{n-3}}, \ldots, x_{t_1})$$
$$\ldots f(x_{t_2} \mid x_{t_1}) f(x_{t_1})$$

very difficult to have it in general

- If $\{X_t\}$ is an indep. process:

  $$f(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = f(x_{t_n}) f(x_{t_{n-1}}) \ldots f(x_{t_1})$$

- If $\{X_t\}$ is a process with indep. increments:

  $$f(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = f(x_{t_1}) f(x_{t_2} - x_{t_1}) \ldots f(x_{t_n} - x_{t_{n-1}})$$

**Note** : First order pdf's are sufficient for above special cases

- If $\{X_t\}$ is a process whose evolution beyond $t_0$ is (probabilistically) completely determined by $x_{t_0}$ and is indep. of $x_t$ , $t < t_0$, given $x_{t_0}$, then:

  $$f(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = f(x_{t_n} \mid x_{t_{n-1}}) \ldots f(x_{t_2} \mid x_{t_1}) f(x_{t_1})$$

  This is a Markov process ($n$th order pdf simplified)

# Definition of a Markov Process (MP)

A stoch. proc. $\{X_t; t \in I\}$ that takes values from a set $E$ is called a Markov Process (MP) iff :

$$f(x_{t_n} \mid x_{t_{n-1}}, ..., x_{t_1}) = P(x_{t_n} \mid x_{t_{n-1}}) \qquad (E \text{ countable})$$

or

$$f(x_{t_n} \mid x_{t_{n-1}}, ..., x_{t_1}) = f(x_{t_n} \mid x_{t_{n-1}}) \qquad (E \text{ uncountable})$$

for all $x_{t_n}$ and all $t_1 < t_2 < ... < t_n$ and all $n > 0$.

**Notice** : The "next" state $x_{t_n}$ is indep. of the "past" $\{x_{t_1}, ..., x_{t_{n-2}}\}$ provided that the "present" is known.

# Definition of a Markov Chain (MC)

(Discrete - time & discrete - value MP)

If $I$ is countable and $E$ is countable then a MP is called a MC

and is described by the transition probabilities :

$$p(i, j) = P\{X_{n+1} = j \mid X_n = i\} \qquad i, j \in E$$

(indep. of $n$ for a time - homogeneous MC). Assume $E = \{0,1,2,...\}$ (state - space of the MC)

**Transition matrix :**

$$P = \begin{bmatrix} P(0,0) & P(0,1) & \dots & P(0,n) & \dots \\ P(1,0) & P(1,1) & \dots & P(1,n) & \dots \\ \vdots & \vdots & & \vdots & \\ P(n,0) & P(n,1) & \dots & P(n,n) & \dots \\ \vdots & \vdots & & \vdots & \end{bmatrix}$$

$P$ is non - negative, $\sum_j P(i, j) = 1 \quad , \quad \forall i$ (stochastic matrix)

For a given $P$ (stoch. matrix) a MC may be constructed

**Chain rule :**

If $\overline{\pi}$ is a PMF on $E$ s.t. $\pi(i) = P\{X_0 = i\}$ , $i \in E$, then

$$P\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = \pi(i_0) P(i_0, i_1) \ldots P(i_{n-1}, i_n)$$

$$\forall n \in N \quad , \quad i_0, i_1, \ldots, i_n \in E$$

**k - step transitions :**

$\forall\, k \in \mathrm{N}$,

$$P\{X_{n+k} = j \,|\, X_n = i\} = P^k(i, j)$$

$\forall i, j \in E$ , $\forall\, k \in \mathrm{N}$ ; $P^k(i, j)$ is the $(i, j)$ entry of the $k$th power of the transition matrix $P$.

Proof : For $k = 3$ (general $n$ through iterations)

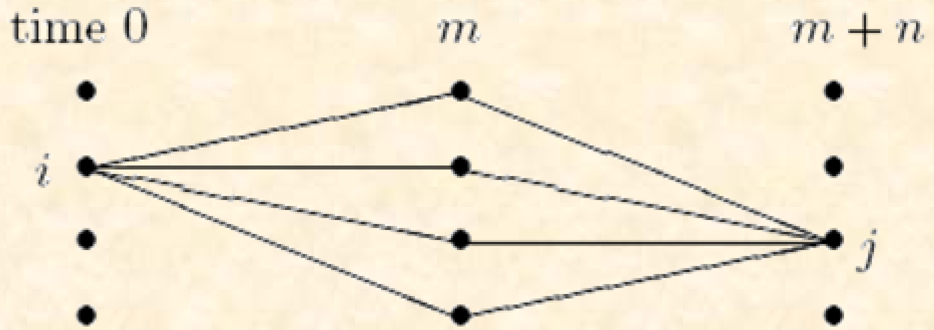
$$P\{X_{n+3} = j \,|\, X_n = i\} = \underbrace{\sum_{l_1 \in E} P(i, l_1) \underbrace{\sum_{l_2 \in E} P(l_1, l_2) P(l_2, j)}_{P^2(l_1, j)}}_{P^3(i, j)}$$

**Chapman Kolmogorov Equations :**

From previous,

$$P^{m+n}(i,j) = \sum_{k \in E} P^m(i,k) P^n(k,j) \qquad i,j \in E$$

In order for $\{X_n\}$ to be in $j$ after $m+n$ steps and starting from $i$ , it will have to be in some $k$ after $m$ steps and move then to $j$ in the remaining $n$ steps.



**Example :** # of successes in Bernoulli process

$\{N_n ; n \geq 0\}$ , $N_n = \#$ of successes in $n$ trials

$N_n = \sum_{i=0}^{n} Y_i$ , $n \geq 0$ , $Y_i$ indep. Bernoulli, $P\{Y_i = 1\} = p$

Notice: $N_{n+1} = N_n + Y_{n+1} \Rightarrow$ evolution of $\{N_n\}$ beyond $n$

does not depend on $\{N_i\}_{i=0}^{n-1}$ (given $N_n$) and thus $\{N_n\}$ is a M.C.

$$P\{N_{n+1} = j \mid N_0, N_1, \ldots, N_n\} = P\{Y_{n+1} = j - N_n \mid N_0, N_1, \ldots, N_n\}$$

$$= \begin{cases} p & \text{if } j = N_n + 1 \\ q = 1 - p & \text{if } j = N_n \\ 0 & \text{otherwise} \end{cases} \quad \text{and } P = \begin{bmatrix} q & p & 0 & \ldots & & \\ 0 & q & p & 0 & \ldots & \\ 0 & 0 & q & p & 0 & \ldots \\ \vdots & & & & & \end{bmatrix}$$

Notice: $\{N_n\}$ is a special M.C. whose increment is indep.

both from present and past (process with indep. increments)

**Example**: Sum of i.i.d. RV's with PMF $\{p_k; k = 0,1,2,...\}$

$$X_n = \begin{cases} 0 & n = 0 \\ Y_1 + Y_2 + ... + Y_n & n \geq 1 \end{cases}$$

$$X_{n+1} = X_n + Y_{n+1}$$

$$P\{X_{n+1} = j \mid X_0, ..., X_n\} = P\{Y_{n+1} = j - X_n \mid X_0, ..., X_n\} = p_{j-X_n}$$

Thus $\{X_n\}$ is a M.C. with $P(i,j) = P\{X_{n+1} = j \mid X_n = i\} = p_{j-i}$

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & p_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

**Example** : Independent trials

$X_0, X_1, \ldots$ i.i.d. with $\pi(k) \quad , \quad k = 0,1,2,\ldots$

$P\{X_{n+1} = j \mid X_0, \ldots, X_n\} = P\{X_{n+1} = j\} = \pi(j)$

$\{X_n\}$ is a M.C.

$$P = \begin{bmatrix} \pi(0) & \pi(1) & \cdots \\ \pi(0) & \pi(1) & \cdots \\ \vdots & \vdots & \\ \pi(0) & \pi(1) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Notice that rows are identical and $P^m = P \quad \forall m \geq 1$

(If $P$ has all rows identical then $X_0, X_1, \ldots$ are i.i.d.)

**Example**: $\{Y_n\}$ are i.i.d. $Y_n \in \{0,1,2,3,4\}$ with $\{p_0, p_1, p_2, p_3, p_4\}$

$X_{n+1} = X_n + Y_{n+1}$ (modulo 5) , $\{X_n\}$ is a M.C.

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & p_4 \\ p_4 & p_0 & p_1 & p_2 & p_3 \\ p_3 & p_4 & p_0 & p_1 & p_2 \\ p_2 & p_3 & p_4 & p_0 & p_1 \\ p_1 & p_2 & p_3 & p_4 & p_0 \end{bmatrix}$$

$\sum \text{rows} = 1$ (stoch. matrix)

$\sum \text{columns} = 1$ (here)

(double - stochastic matrix)

**Example**: Remaining lifetime

An equipment is replaced by an identical as soon as it fails

$p_k = \Pr\{\text{a new equip. lasts for } k \text{ time units}\} \qquad k = 1, 2, \ldots$

$X_n$ = remaining lifetime of equip. at time $n$

$$X_{n+1}(\omega) = \begin{cases} X_n(\omega) - 1 & \text{if } X_n(\omega) \geq 1 \\ Z_{n+1}(\omega) - 1 & \text{if } X_n(\omega) = 0 \end{cases}$$

$Z_{n+1}(\omega)$ is the lifetime of equip. installed at time $n$

It is independent of $X_0, X_1, \ldots, X_n$

$X_n$ is a M.C.

- $i \geq 1$:

$$P(i,j) = P\{X_{n+1} = j \mid X_n = i\} = P\{X_n - 1 = j \mid X_n = i\}$$

$$= P\{X_n = j+1 \mid X_n = i\} = \begin{cases} 1 & \text{if } j = i-1 \\ 0 & \text{if } j \neq i-1 \end{cases}$$

- $i = 0$:

$$P(0,j) = P\{X_{n+1} = j \mid X_n = 0\} = P\{Z_{n+1} - 1 = j \mid X_n = 0\}$$

$$= P\{Z_{n+1} = j+1\} = p_{j+1}$$

$$P = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

**Theorem** : (conditional indep. of future from past given present)

Let $Y$ be a bounded function of $X_n, X_{n+1}, \ldots$ . Then

$$\mathrm{E}\{\mathrm{Y}|X_0, X_1, \ldots, X_n\} = \mathrm{E}\{\mathrm{Y}|X_n\}$$

**Proposition :**

$$\mathrm{E}\{f(X_n, X_{n+1}, \ldots)|X_n = i\} = \mathrm{E}\{f(X_0, X_1, \ldots)|X_0 = i\}$$

**Corollary** : $f$ a bounded function on $E \times E \times \ldots$

Let $\quad g(i) = \mathrm{E}\{f(X_0, X_1, \ldots)|X_0 = i\}$.

Then $\forall n \in N \qquad \mathrm{E}\{f(X_n, X_{n+1}, \ldots)|X_0, X_1, \ldots, X_n\} = g(X_n)$

**Stopping Times :**

Previous results derived for fixed time $n \in \mathrm{N}$

What if time is an RV instead?

- If for a RV $T$ , the past $\{X_m ; m \leq T\}$ and the future $\{X_m ; m \geq T\}$ are conditionally indep. given present $X_T$, then the strong Markov property is said to hold at $T$.
- If $T$ is a stopping time, then above hold true ($T$ is a stopping time if the event $\{T \leq n\}$ can be determined by looking at $X_0, X_1, \ldots, X_n$)

**For any stopping time $T$ :**

- $E\{f(X_T, X_{T+1}, \ldots) \mid X_n, n \le T\} = E\{f(X_T, X_{T+1}, \ldots) \mid X_T\}$
- For $g(i) = E\{f(X_0, X_1, \ldots) \mid X_0 = i\}$

$$E\{f(X_T, X_{T+1}, \ldots) \mid X_n; n \le T\} = g(X_T)$$

e.g., if $f(a_0, a_1, \ldots) = \begin{cases} 1 & \text{if} \quad a_m = j \\ 0 & \text{if} \quad a_m \ne j \end{cases} \qquad j \in E, m \in N$

$$E\{f(X_0, X_1, \ldots) \mid X_0 = i\} = P\{X_m = j \mid X_0 = i\} = P^m(i, j)$$

$$E\{f(X_T, X_{T+1}, \ldots) \mid X_n, n \le T\} = P\{X_{T+m} = j \mid X_n; n \le T\}$$

- Strong Markov property at T:

$$P\{X_{T+m} = j \mid X_n; n \le T\} = P^m(X_T, j)$$

## Visits to a state

$X = \{X_n; n \in N\}$ MC, State space $E$, Transition matrix $P$.

Notation: $P_i\{A\} = P\{A \mid X_0 = i\}$ and $E_i[Y] = E[Y \mid X_0 = i]$

Let $j \in E$, $\omega \in \Omega$ and Define:

$N_j(\omega)$ = total number of times state $j$ appears in $X_0(\omega), X_1(\omega), \ldots$.

- ♣ $N_j(\omega) < \infty$, $X$ eventually leaves state $j$ never to return.
- ♣ $N_j(\omega) = \infty$, $X$ visits $j$ again and again.

Let $T_1(\omega), T_2(\omega), \ldots$ the successive indices $n \geq 1$ for which $X_n(\omega) = j$.

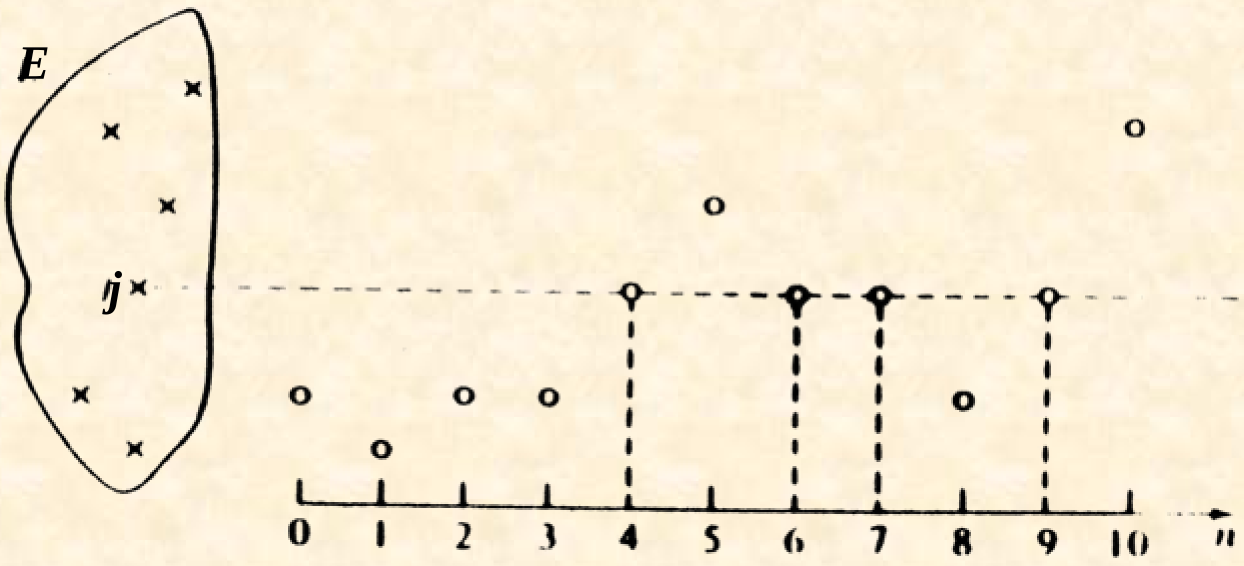
- ♣ If $\nexists n$ then $T_1(\omega) = T_2(\omega) - T_1(\omega) = \cdots = \infty$
- ♣ If $j$ appears a finite number of times $m$, then $T_{m+1}(\omega) - T_m(\omega) = T_{m+2}(\omega) - T_m(\omega) = \cdots = \infty$

$\forall n \in N$, $\{T_m(\omega) \leq n\}$ is equivalent to $j$ appears in $\{X_1(\omega), \cdots, X_n(\omega)\}$ at least $m$ times.

$T_m$ is a stopping time.

**Example**

$$T_1(\omega)=4,\quad T_2(\omega)=6,\quad T_3(\omega)=7,\quad T_4(\omega)=9,\ldots$$



**Proposition:** $\forall i \in E, \quad k, m \geq 1$

$$P_i\left\{T_{m+1} - T_m = k \middle| T_1, \ldots, T_m\right\} = \begin{cases} 0 & \{T_m = \infty\} \\ P_j\{T_1 = k\} & \{T_m < \infty\} \end{cases}$$

**Computation of** $P_j\{T_1 = k\}$. Let $F_k(i,j) = P_i\{T_1 = k\}$

$$k = 1 \Rightarrow F_k(i,j) = P_i\{T_1 = 1\} = P_i\{X_1 = j\} = P(i,j)$$

$$\begin{aligned} k \geq 2 \Rightarrow F_k(i,j) &= P_i\{X_1 \neq j, \cdots, X_{k-1} \neq j, X_k = j\} \\ &= \sum\nolimits_{b \in E - \{j\}} P_i\{X_1 = b\} P_i\{X_2 \neq j, \cdots, X_{k-1} \neq j, X_k = j \mid X_1 = b\} \\ &= \sum\nolimits_{b \in E - \{j\}} P_i\{X_1 = b\} P_b\{X_1 \neq j, \cdots, X_{k-2} \neq j, X_{k-1} = j\} \end{aligned}$$

Thus,

$$F_k(i,j) = \begin{cases} P(i,j) & k = 1 \\ \displaystyle\sum_{b \in E - \{j\}} P(i,b) F_{k-1}(b,j) & k \geq 2 \end{cases}$$

**Example:** Let $j=3$ an the transition matrix $P=\begin{pmatrix}1 & 0 & 0\\ 1/2 & 1/6 & 1/3\\ 1/3 & 3/5 & 1/15\end{pmatrix}$

Find $f_k(i)=F_k(i,j), \qquad i=1,2,3$

- $k=1$. In this case $f_1$ is the $3^{rd}$ column of matrix P.

  Hence, $f_1(1)=F_1(1,j)=0$, $f_1(2)=F_1(2,j)=1/3$, $f_1(3)=F_1(3,j)=1/15$

- $k\ge 2$. In this case $f_k=\begin{pmatrix}F_k(1,j)\\ F_k(2,j)\\ F_k(3,j)\end{pmatrix}=\begin{pmatrix}\sum_{b\in E-\{j\}} P(1,b)F_{k-1}(b,j)\\ \sum_{b\in E-\{j\}} P(2,b)F_{k-1}(b,j)\\ \sum_{b\in E-\{j\}} P(3,b)F_{k-1}(b,j)\end{pmatrix}=Q\cdot f_{k-1}$ where $Q=\begin{pmatrix}1 & 0 & 0\\ 1/2 & 1/6 & 0\\ 1/3 & 3/5 & 0\end{pmatrix}$

After some algebra $\quad f_1=\begin{pmatrix}0\\ 1/3\\ 1/15\end{pmatrix} \quad f_2=\begin{pmatrix}0\\ 1/18\\ 1/5\end{pmatrix} \quad f_3=\begin{pmatrix}0\\ 1/108\\ 1/30\end{pmatrix} \quad f_4=\begin{pmatrix}0\\ 1/648\\ 1/180\end{pmatrix} \quad \cdots$

and in general

$$F_k(1,3)=0, \qquad F_k(2,3)=\frac{1}{3}\left(\frac{1}{6}\right)^{k-1}, \qquad F_k(3,3)=\begin{cases}\dfrac{1}{15} & k=1\\ \dfrac{3}{5}\left(\dfrac{1}{6}\right)^{k-2}\dfrac{1}{3} & k\ge 2\end{cases}$$

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Now we can state:

- Starting at state $1$, $X$ never visits $3$ with probability: $P_1\{T_1=+\infty\}=1$
- Starting at state $2$, $X$ first visits $3$ at $k$ with probability: $\frac{1}{3}(\frac{1}{6})^{k-1}$
- Starting at state $2$, $X$ never visits $3$ with probability: $P_2\{T_1=+\infty\}=1-P_2\{T_1<+\infty\}=1-\sum_{k=1}^{\infty}\frac{1}{3}(\frac{1}{6})^{k-1}=\frac{3}{5}$
- Starting at state $3$, $X$ never visits $3$ again with probability: $P_3\{T_1=+\infty\}=1-P_3\{T_1<+\infty\}=\frac{52}{75}$

Now, for every $i, j$ we define

$$F(i,j) = P_i\{T_1 < +\infty\} = \sum_{k=1}^{\infty} F_k(i,j)$$

♣ $F(i,j)$ expresses the probability: starting at $i$ the MC will ever visit state $j$.

$$F(i,j) = P(i,j) + \sum_{b \in E - \{j\}} P(i,b)F(b,j), \qquad i \in E$$

If by $N_j$ we denote the total number of visits to state $j$, then

$$P_j\{N_j = m\} = F(j,j)^{m-1}\left(1 - F(j,j)\right)$$

and for $i \neq j$, $$P_i\{N_j = m\} = \begin{cases} 1 - F(i,j) & m = 0 \\ F(i,j)F(j,j)^{m-1}\left(1 - F(j,j)\right) & m = 1,2,\ldots \end{cases}$$

>From the previous we obtain the Corollary:

$$P_j\{N_j < +\infty\} = \begin{cases} 1 & F(j,j) < 1 \\ 0 & F(j,j) = 1 \end{cases}$$

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Let $R(i,j) = E_i[N_j]$ ( $R$ is called the **potential** matrix of $X$ )

Then,

$$R(i,j) = F(i,j)\,R(j,j) + (1 - F(i,j))\,0$$

$$R(j,j) = \tfrac{1}{1-F(j,j)}$$

$$R(i,j) = F(i,j)\,R(j,j)\ , \quad (i{\neq}j)$$

**<u>Computation of</u>** $R(i,j)$ **<u>first and then</u>** $F(i,j)$

Define:

$$1_j(k) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \Rightarrow 1_j(X_n(\omega)) = \begin{cases} 1, & X_n(\omega) = j \\ 0, & X_n(\omega) \neq j \end{cases}$$

Then,

$$N_j(\omega) = \sum_{n=0}^{\infty} 1_j(X_n(\omega))$$

$$R(i,j) = E_i\left[\sum_{n=0}^{\infty} 1_j(X_n)\right] = \sum_{n=0}^{\infty} E_i\left[1_j(X_n)\right] = \sum_{n=0}^{\infty} P_i\{X_n = j\} = \sum_{n=0}^{\infty} P^n(i,j)$$

In matrix notation:

$$R = I + P + P^2 + \cdots \Rightarrow RP = PR = P + P^2 + \cdots = R - I$$

from which we obtain

$$R(I-P) = (I-P)R = I$$

## Classification of states

X: MC, with state space $E$, transition matrix $P$
$T$ : The time of first visit to state $j$
$N_j$ : The total number of visits to state $j$

**Definition**

♣ State $j$ is called **recurrent** if $P_j\{T<\infty\}=1$

♣ State $j$ is called **transient** if $P_j\{T=\infty\}>0$

♣ A recurrent state $j$ is called **null** if $E_j[T]=\infty$

♣ A recurrent state $j$ is called **non-null** if $E_j[T]<\infty$

♣ A recurrent state $j$ is called **periodic** with period $\delta$, if $\delta\geq 2$ is the greatest integer for which

$$P_j\{T=n\delta \text{ for some } n\geq 1\}=1$$

- If $j$ is recurrent then starting at $j$ the probability of returning to $j$ is 1.

$$F(j,j)=1 \Rightarrow R(j,j)=E_j[N_j]=+\infty \Longleftrightarrow P_j\{N_j=+\infty\}=1$$

- If $j$ is transient then there exists a positive probability $1-F(j,j)$ of never returning to $j$.

$$F(j,j)<1 \Rightarrow R(j,j)=E_j[N_j]<\infty \Longleftrightarrow P_j\{N_j<\infty\}=1$$

In this case $R(i,j)=F(i,j)R(j,j)<R(j,j)<\infty$ and since $R(i,j)=\sum_n P^n(i,j)$ we conclude that

$$\lim_{n\to\infty} P^n(i,j)\to 0$$

**Theorem:**

♣ If $j$ transient or recurrent null then

$$\forall i \in E, \qquad \lim_{n\to\infty} P^n(i,j) \to 0$$

♣ If $j$ recurrent non-null then

$$\pi(j) = \lim_{n\to\infty} P^n(j,j) > 0 \quad \text{and} \quad \forall i \in E, \qquad \lim_{n\to\infty} P^n(i,j) = F(i,j)\pi(j)$$

♣ If $j$ periodic with period $\delta$, then a retutrn to $j$ is possible only at steps numbered $\delta$, $2\delta$, $3\delta$, ...

$$P^n(j,j) = P_j\{X_n = j\} > 0 \text{ only if } n \in \{0, \delta, 2\delta, ..\}$$

| Recurrent non-null | Recurrent null | Transient |
|---|---|---|
| $P_j\{T<\infty\}=1$ | | $P_j\{T=\infty\}>0$ |
| $E_j[T]<\infty$ | $E_j[T]=\infty$ | |
| $F(j,j)=1 \Rightarrow R(j,j)=E_j[N_j]=+\infty \Longleftrightarrow P_j\{N_j=+\infty\}=1$ | | $F(j,j)<1 \Rightarrow R(j,j)=E_j[N_j]<\infty$ $\Longleftrightarrow P_j\{N_j<\infty\}=1$ |
| $\pi(j)=\lim\limits_{n\to\infty} P^n(j,j)>0$ and $\forall i \in E,$ $\lim\limits_{n\to\infty} P^n(i,j)=F(i,j)\pi(j)$ | $\forall i \in E, \quad \lim\limits_{n\to\infty} P^n(i,j) \to 0$ | |

♣ A recurrent state $j$ is called **periodic** with period $\delta$, if $\delta \geq 2$ is the greatest integer for which

$$P_j\{T=n\delta \text{ for some } n\geq 1\}=1$$

♣ If $j$ periodic with period $\delta$, then a return to $j$ is possible only at steps numbered $\delta$, $2\delta$, $3\delta$, ...

$$P^n(j,j)=P_j\{X_n=j\}>0 \text{ only if } n\in\{0,\delta,2\delta,..\}$$

We say that state $j$ can be reached from state $i$ $i \to j$, if $\exists n \geq 0 : P^n(i,j) > 0$

$i \to j$, iff $F(i,j) > 0$

**Definition:**

- A set of states is **closed** if no state outside it can be reached from any state in it.
- A state forming a closed set by itself is called an **absorbing** state
- A closed set is called **irreducible** if no proper subset of it is closed.
- A MC is called irreducible if its only closed set is the set of all states

**Comments:**

♣ If $j$ is absorbing then $P(j,j) = 1$.

♣ If MC is irreducible then all states can be reached from each other.

♣ If $C = \{c_1, c_2, \cdots\} \in E$ is a closed set and $Q(i,j) = P(c_i, c_j)$, $c_i, c_j \in C$, then $Q$ is a Markov matrix.

♣ If $i \to j$ and $j \to k$ then $i \to k$.

To find the closed set $C$ that contains $i$ we work as follows:

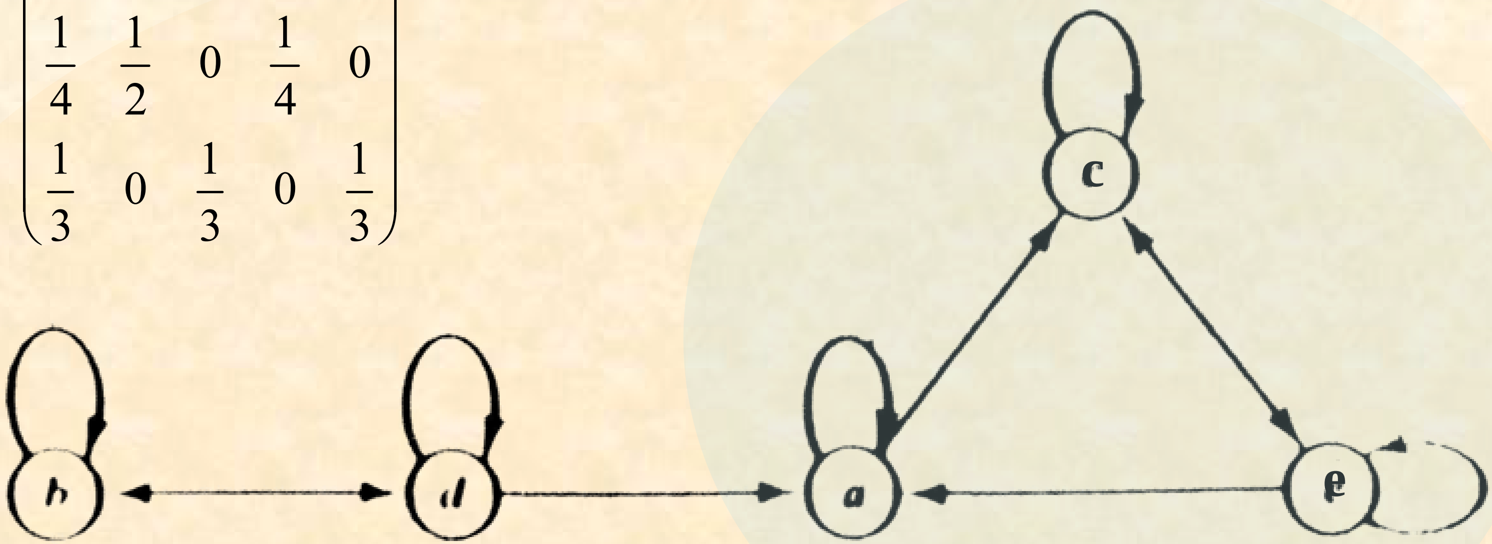
- Starting with $i$ we include in $C$ all states $j$ that can be reached from $i$: $P(i,j) > 0$.
- We next include in $C$ all states $k$ that can be reached from $j$: $P(j,k) > 0$.
- We repeat the previous step

**Example:** MC with state space $E = \{a,b,c,d,e\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

**Comments:**

- Closed sets: $\{a,c,e\}$ and $\{a,b,c,d,e\}$
- There are two closed sets. Thus, the MC is not irreducible.



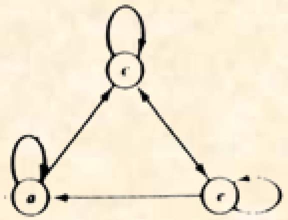
**Example:** MC with state space $E=\{a,b,c,d,e\}$ and transition matrix

$$P=\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

**Comments:**

- Closed sets: $\{a,c,e\}$ and $\{a,b,c,d,e\}$
- There are two closed sets. Thus, the MC is not irreducible.
- If we delete the $2^{nd}$ and $4^{th}$ rows we obtain the Markov matrix:

$$Q=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$



If we relabel the states $1=a$, $2=c$, $3=e$, $4=b$ and $5=d$ we get

$$\overline{P}=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

**<u>Lemma</u>** If $j$ recurrent and $j \to k \Rightarrow k \to j$. Thus, $F(k, j) = 1$.

<u>Proof:</u> If $j \to k$ then $k$ is reached without returning to $j$ with probability **a**. Once $k$ is reached, the probability that $j$ is never visited again is $1 - F(k, j)$. Hence,

$$1 - F(j, j) \geq \mathrm{a}(1 - F(k, j)) \geq 0$$

But $j$ is recurrent, so that $F(j, j) = 1 \Rightarrow F(k, j) = 1$

♠ As a result: If $j \to k$ but $k \not\to j$, then $j$ **must** be **transient**.

**Theorem:** From recurrent states only recurrent states can be reached.
**Theorem:** In a Marcov chain the recurrent states can be divided in a unique manner, into irreducible closed sets $C_1$, $C_2$, …, and after an appropriate arrangement:

$$P = \begin{pmatrix} P_1 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ 0 & 0 & P_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ Q_1 & Q_2 & Q_3 & \cdots & Q \end{pmatrix}$$

**Theorem:** Let $X$ an irreducible MC. Then, one of the following holds:

- All states are transient.
- All states are recurrent null
- All states are recurrent non-null
- Either all aperiodic or if one is periodic with period $\delta$, all are periodic with the same period.

Proof: Since $X$ is irreducible then $j \to k$ and $k \to j$, which means that $\exists r, s$: $P^r(j,k) > 0$ and $P^s(k,j) > 0$. Pick the smallest $r, s$ and let $\beta = P^r(j,k) P^s(k,j)$.

- If $k$ recurrent $\Rightarrow j$ recurrent.
- If $k$ transient $\Rightarrow j$ transient. (If it was recurrent then $k$ would be recurrent)
- If $k$ recurrent null then $P^m(k,k) \to 0$ as $m \to \infty$. But $P^{n+r+s}(k,k) \geq \beta P^n(j,j) \Rightarrow P^n(j,j) \to 0$

**Corollary:** If C irreducible closed set of **finitely** many states, then $\nexists$ recurrent null states.

Proof: If one is recurrent null then all states are recurrent null.
Thus, $\lim_{n\to\infty} P^n(i,j)=0, \qquad \forall i,j\in C$. But,

$$\forall i\in C, n\geq 0, \sum_{j\in C} P^n(i,j)=1 \Rightarrow \lim_{n\to\infty}\sum_{j\in C} P^n(i,j)=1$$

Because, we have finite number of states

$$\lim_{n\to\infty}\sum_{j\in C} P^n(i,j)=\sum_{j\in C}\lim_{n\to\infty} P^n(i,j)=0$$

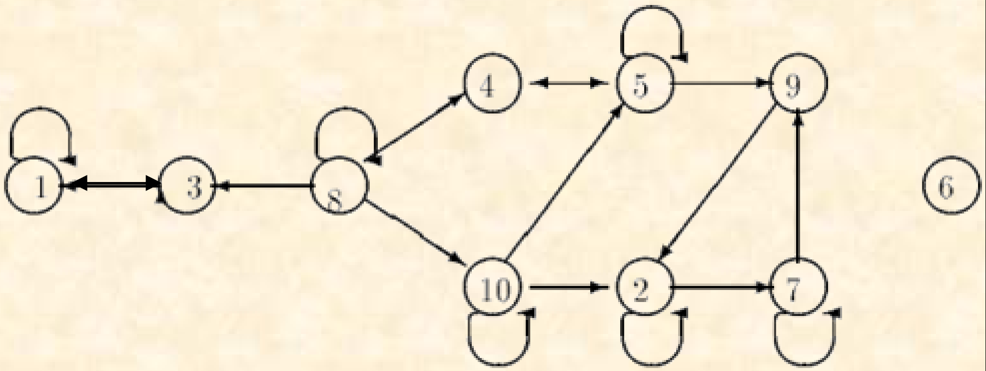
**Corollary:** If $C$ is an irreducible closed set with finitely many states then there are **no** transient states

**Algorithm - Finite number of states**

- Identify irreducible closed sets.
- All states belonging to an irreducible closed set are recurrent positive
- The rest of the states are transient
- Periodicity is checked to each irreducible set

**Example:**

The irreducible closed sets are $\{1,3\}$, $\{2,7,9\}$ and $\{6\}$. The states $\{4,5,8,10\}$ are transient. If we relabel the states we obtain




$$P = \begin{pmatrix}
1 & & & & & & & & & \\
 & \frac{1}{2} & \frac{1}{2} & & & & & & & \\
 & 1 & 0 & & & & & & & \\
 & & & \frac{1}{3} & \frac{2}{3} & 0 & & & & \\
 & & & 0 & \frac{1}{4} & \frac{3}{4} & & & & \\
 & & & 1 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3}
\end{pmatrix}$$

**Example:** Let $N_n$ the number of successes in the first $n$ Bernoulli trials. As we have seen

$$P(i,j) = P\{N_{n+1} = j \mid N_n = i\} = \begin{cases} p & j = i+1 \\ q & j = i \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$P = \begin{pmatrix} q & p & 0 & \cdots \\ 0 & q & p & \cdots \\ 0 & 0 & q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\forall j$ we have $j \to j+1$ but $j+1 \not\to j$. This means that $j$ is **not** recurrent. Since the MC is irreducible all states are **transient**.

**Example:** Remaining lifetime

Remember: $$X_{n+1}(\omega) = \begin{cases} X_n(\omega) - 1 & X_n(\omega) \geq 1 \\ Z_{n+1}(\omega) - 1 & X_n(\omega) = 0 \end{cases}$$

from which we obtain:

$i \geq 1$ $$P(i, j) = P\{X_{n+1} = j \mid X_n = j\} = P\{X_n - 1 = j \mid X_n = j\} = \begin{cases} 1 & j = i - 1 \\ 0 & j \neq i - 1 \end{cases}$$

$i = 0$ $$P(0, j) = P\{X_{n+1} = j \mid X_n = 0\} = P\{Z_{n+1} - 1 = j \mid X_n = 0\}$$
$$= P\{Z_{n+1} = j + 1\} = p_{j+1}$$

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

>From state 0 we reach state $j$ in one step. From $j$ we can reach $j-1$, $j-2$, ..., 1, $0$. Thus, all states can be reached from each other, which means that the MC is irreducible. Since, $P(0,0) > 0$ the MC is aperiodic. Return to state $0$ occurs if the lifetime is finite:

$$\sum_j p_j = 1 \Rightarrow F(0,0) = \sum_j p_j = 1$$

Since state $0$ is recurrent, all states are recurrent.
If the expected lifetime:

$$\sum_j j p_j = +\infty$$

then state $0$ is null and all states are recurrent null.
If the expected lifetime:

$$\sum_j j p_j < \infty$$

then state $0$ is non-null and all states are recurrent non-null.

## Algorithm - Infinite number of states

**Theorem:** Let $X$ an irreducible MC, and consider the system of linear equations:

$$\nu(j) = \sum_{i \in E} \nu(i) P(i, j), \qquad j \in E$$

Then all states are **recurrent non-null** iff there exists a solution $\nu$ with

$$\sum_{j \in E} \nu(j) = 1$$

**Theorem:** Let $X$ an irreducible MC with transition matrix $P$, and let $Q$ be the matrix obtained from $P$ by deleting the $k$-row and $k$-column for some $k \in E$. Then all states are **recurrent** if and only if the only solution of

$$h(i) = \sum_{j \in E_0} Q(i, j) h(j), \qquad 0 \le h(i) \le 1, \qquad i \in E_0$$

is $h(i) = 0$ for all $i \in E_0$. $E_0 = E - \{k\}$.

- Use first theorem to determine whether all states are recurrent non-null or not.
- In the latter case, use the second theorem to determine whether the states are transient or not.

**Example:** Random walks.

$$P = \begin{pmatrix} 0 & 1 & & & \cdots \\ q & 0 & p & & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- All states can be reached from each other, and thus the chain is irreducible.
- A return to state $0$ can occur only at steps numbered 2,4,6,... Therefore, state $0$ is periodic with period $\delta = 2$.
- Since $X$ is irreducible all states are periodic with period 2.
- Either all states are recurrent null, or all are recurrent non-null, or all the states are transient.

Check for a solution of $\nu = \nu P$.

$$\begin{aligned} \nu_0 &= q\nu_1 \\ \nu_1 &= \nu_0 + q\nu_2 \\ \nu_2 &= p\nu_1 + q\nu_3 \\ \nu_3 &= p\nu_2 + q\nu_4 \end{aligned}$$

Hence,

$$\nu_1 = \tfrac{1}{q}\nu_0$$

$$\nu_2 = \tfrac{1}{q}\left(\tfrac{1}{q}\nu_0 - \nu_0\right) = \tfrac{p}{q^2}\nu_0$$

$$\nu_3 = \tfrac{1}{q}\left(\tfrac{p}{q^2} - \tfrac{p}{q}\right)\nu_0 = \tfrac{p^2}{q^3}\nu_0$$

Any solution is of the form

$$\nu_j = \frac{1}{q}\left(\frac{p}{q}\right)^{j-1}\nu_0, \qquad j = 1, 2, \ldots$$

If $p < q$, then $p/q < 1$ and

$$\sum_{j=0}^{\infty}\nu_j = \left(1 + \frac{1}{q}\sum_{j=1}^{\infty}\left(\frac{p}{q}\right)^{j-1}\right)\nu_0 = \frac{2q}{q-p}\nu_0$$

If we choose $\nu_0 = \frac{q-p}{2q}$ then $\sum \nu_j = 1$ and

$$\nu(j) = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{p}{q}\right), & j = 0 \\ \dfrac{1}{2q}\left(1 - \dfrac{p}{q}\right)\left(\dfrac{p}{q}\right)^{j-1}, & j \geq 1 \end{cases}$$

In this case all states are **recurrent non null**

If $p > q$ either all states are recurrent null or all states are transient. Consider the matrix

$$Q = \begin{pmatrix} 0 & p & & & \cdots \\ q & 0 & p & & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The equation $h = Qh$ gives ( $h_i = h(i)$ )

$$h_{i+1} = \left[ \left(\frac{q}{p}\right)^i + \left(\frac{q}{p}\right)^{i-1} + \cdots + \frac{q}{p} + 1 \right] h_1$$

- If $p = q$ then $h_i = ih_1$ for all $i \geq 1$ and the only way to have $0 \leq h_i \leq 1$ for all $i$ is by choosing $h_1 = 0$ which implies $h_i = 0$ that is all states are **recurrent null**.
- If $p > q$, then choosing $h_1 = 1 - (q/p)$, we get

$$h_i = 1 - \left(\frac{q}{p}\right)^i$$

which also satisfies $0 \leq h_i \leq 1$. In this case all states are **transient**.

## Calculation of $R$ and $F$

- ♣ $R(i,j) = E_i[N_j]$ Expected number of visits to state $j$.
- ♣ $F(i,j) =$ The probability of **ever** reaching state $j$ starting at $i$.

$j$ **Recurrent state:** $F(j,j) = 1 \Rightarrow R(j,j) = \infty$

$$R(i,j) = F(i,j)R(j,j) \qquad R(i,j) = \begin{cases} 0 & F(i,j) = 0 \\ +\infty & F(i,j) > 0 \end{cases}$$

$j$ **Transient /** $i$ **Recurrent state:** $F(i,j) = 0 \Rightarrow R(i,j) = 0$

$i, j$ **Transient**

Let $D = \{$ the transient states $\}$, $Q(i,j) = P(i,j)$, $S(i,j) = R(i,j)$, $i, j \in D$.

Then $$P = \begin{pmatrix} K & 0 \\ L & Q \end{pmatrix} \Rightarrow P^m = \begin{pmatrix} K^m & 0 \\ L_m & Q^m \end{pmatrix}$$

Hence, $$R = \sum_{m=0}^{\infty} P^m = \begin{pmatrix} \sum K^m & 0 \\ \sum L_m & \sum Q^m \end{pmatrix} \Rightarrow S = \sum_{m=0}^{\infty} Q^m = I + Q + Q^2 + \cdots$$

**Computation of** $S$

$$S = I + Q + Q^2 + \cdots \Rightarrow$$
$$SQ = QS = Q + Q^2 + \cdots = S - I \Rightarrow$$
$$(I - Q)S = I\,, \quad S(I - Q) = I$$

**Proposition:** If there are finitely many transient states $S = (I - Q)^{-1}$

♣ When the set $D$ of transient states is infinite, it is possible to have more than one solution to the system.

**Theorem:** $S$ is the minimal solution of $(I - Q)Y = I$, $Y \geq 0$

**Theorem:** $S$ is the unique solution of $(I - Q)Y = I$ if and only if the only bounded solution of $h = Qh$ is $h = 0$, or equivalently

$$h = Qh,\, 0 \leq h \leq 1 \Longleftrightarrow h = 0$$

**Example:** Let $X$ a MC with state space $E = \{1,2,3,4,5,6,7,8\}$

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 & | & & & | & & & \\ 0. & 0.6 & 0.4 & | & & & | & & & \\ 0.5 & 0.5 & 0. & | & & & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ & & & | & 0. & 1. & | & & & \\ & & & | & 0.8 & 0.2 & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ 0. & 0. & 0. & | & & & | & 0.4 & 0.6 & 0. \\ 0.4 & 0.4 & 0. & | & & & | & 0. & 0. & 0.2 \\ 0.1 & 0. & 0.3 & | & & & | & 0.6 & 0. & 0. \end{pmatrix}$$

- $\{1,2,3\}$ are recurrent positive aperiodic.
- $\{4,5\}$ are recurrent positive aperiodic.
- $\{6,7,8\}$ are transient

$$Q = \begin{pmatrix} 0.4 & 0.6 & 0. \\ 0. & 0. & 0.2 \\ 0.6 & 0. & 0. \end{pmatrix} \Rightarrow S = (I-Q)^{-1} = \begin{pmatrix} 0.6 & -0.6 & 0. \\ 0. & 1. & -0.2 \\ -0.6 & 0. & 1. \end{pmatrix}^{-1}$$

$j$ recurrent, can be reached from $i$

$j$ recurrent, cannot be reached from $i$

$j$ transient, $i$ recurrent

$j$, $i$ transient

$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\hphantom{\infty\ \infty\ \infty\ \ 0\ \ 0}}^{j \text{ recurrent}} & \overbrace{\hphantom{0\ \ 0\ \ 0}}^{j \text{ transient}} \end{array} \\
\begin{array}{c} i \text{ recurrent} \\ \\ i \text{ transient} \end{array} &
R = \left(\begin{array}{ccc|cc|ccc}
\infty & \infty & \infty & 0 & 0 & 0 & 0 & 0 \\
\infty & \infty & \infty & 0 & 0 & 0 & 0 & 0 \\
\infty & \infty & \infty & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \infty & \infty & 0 & 0 & 0 \\
0 & 0 & 0 & \infty & \infty & 0 & 0 & 0 \\
\hline
\infty & \infty & \infty & 0 & 0 & \frac{125}{66} & \frac{75}{66} & \frac{15}{66} \\
\infty & \infty & \infty & 0 & 0 & \frac{15}{66} & \frac{75}{66} & \frac{15}{66} \\
\infty & \infty & \infty & 0 & 0 & \frac{75}{66} & \frac{45}{66} & \frac{75}{66}
\end{array}\right)
\end{array}
$$

S

**Computation of** $F(i, j)$

♣ $i, j$ **recurrent** belonging to the same irreducible closed set

$$F(i, j) = 1$$

♣ $i, j$ **recurrent** belonging to different irreducible closed sets

$$F(i, j) = 0$$

♣ $i, j$ **transient** Then $R(i, j) < \infty$ and

$$F(j, j) = 1 - \frac{1}{R(j, j)}, \qquad F(i, j) = \frac{R(i, j)}{R(j, j)}$$

♣ $i$ **transient**, $j$ **recurrent** ????

**Lemma:** If $C$ is irreducible closed set of recurrent states, then for any transient state $i$:

$$F(i, j) = F(i, k)$$

for all $j$, $k \in C$.

Proof: For $j, k \in C \Rightarrow F(j, k) = F(k, j) = 1$. Thus, once the chain reaches any one of the states of $C$, it also visits all the other states. Hence, $F(i, j) = F(i, k)$ is the probability of entering the set $C$ from $i$.

Let

$$P = \begin{pmatrix} P_1 & & & & \\ & P_2 & & & \\ & & P_3 & & \\ & & & \ddots & \\ Q_1 & Q_2 & Q_3 & & Q \end{pmatrix}$$

Lump all states of $C_j$ together to make one absorbing state:

$$\hat{P} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ b_1 & b_2 & b_3 & \cdots & b_m & Q \end{pmatrix}, \quad b_j(i) = \sum_{k \in C_j} P(i,k), i \in D$$

The probability of ever reaching the absorbing state $j$ from the transient state $i$ by the chain with the transition matrix $\hat{P}$ is the same as that of ever reaching $C_j$ from $i$.

$$\hat{P} = \begin{pmatrix} I & 0 \\ B & Q \end{pmatrix}, \qquad B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}, \qquad B(i,j) = \sum_{k \in C_j} P(i,k), \quad \imath \in D$$

$$\hat{P}^n = \begin{pmatrix} I & 0 \\ B_n & Q^n \end{pmatrix}, \qquad B_n = (I + Q + Q^2 + \cdots + Q^{n-1})B$$

$B_n(i,j)$ is the probability that starting from $i$, the chain enters the recurrent class $C_j$

Define:

$$G = \lim_{n\to\infty} B_n = \left( \sum_{k=0}^{\infty} Q^k \right) B = SB$$

♣ $G(i,j)$ is the probability of ever reaching the set $C_j$ from the transient state $i$: ($F(i,j)$)

**<u>Proposition:</u>** Let $Q$ the matrix obtained from $P$ by deleting all the rows and columns corresponding to the recurrent states, and let $B$ be defined as previously, for each transient $i$ and recurrent class $C_j$.

- Compute $S$
- Compute $G = SB$
- $G(i,j) = F(i,k)$, $\forall k \in C_j$.

• If there is only one recurrent class and finitely many transient states, then things are different.
In this case, it can be proved that:

$$G = 1 \Rightarrow F(i,j) = 1, \qquad \forall j \in C$$

**Example:** Let $X$ a MC with state space $E = \{1,2,3,4,5,6,7,8\}$

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 & | & & & | & & & \\ 0. & 0.6 & 0.4 & | & & & | & & & \\ 0.5 & 0.5 & 0. & | & & & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ & & & | & 0. & 1. & | & & & \\ & & & | & 0.8 & 0.2 & | & & & \\ - & - & - & | & - & - & | & - & - & - \\ 0. & 0. & 0. & | & & & | & 0.4 & 0.6 & 0. \\ 0.4 & 0.4 & 0. & | & & & | & 0. & 0. & 0.2 \\ 0.1 & 0. & 0.3 & | & & & | & 0.6 & 0. & 0. \end{pmatrix}$$

$i, j$ recurrent belonging to the same irreducible closed set

$i, j$ recurrent belonging to different irreducible closed sets

$$
\begin{array}{ll}
 & \begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{j \text{ recurrent}} \qquad \overbrace{\qquad\qquad\qquad\qquad}^{j \text{ transient}} \end{array} \\
\begin{array}{l} i \text{ recurrent} \\ \\ i \text{ transient} \end{array} &
F = \left(\begin{array}{ccc|cc|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & - & - \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
- & - & - & - & - & - & - & - \\
1 & 1 & 1 & 0 & 0 & 0.472 & 1. & 0.20 \\
1 & 1 & 1 & 0 & 0 & 0.12 & 0.12 & 0.20 \\
1 & 1 & 1 & 0 & 0 & 0.60 & 0.60 & 0.12
\end{array}\right)
\end{array}
$$

$j$ transient, $i$ recurrent

$j, i$ transient

$$F(j, j) = 1 - \frac{1}{R(j, j)},$$

$$F(i, j) = \frac{R(i, j)}{R(j, j)}$$

one (reachable) recurrent class and finitely many transient states

**Example:**

$$P=\begin{pmatrix} 0.5 & 0.5 & & & & & \\ 0.8 & 0.2 & & & & & \\ & & 0. & 0.4 & 0.6 & & \\ & & 1. & 0. & 0. & & \\ & & 1 & 0. & 0. & & \\ 0.1 & 0. & 0.2 & 0.2 & 0.1 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0. & 0.1 & 0.2 & 0.4 \end{pmatrix} \Rightarrow \hat{P}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{pmatrix}$$

Thus,

$$S=(I-Q)^{-1}=\begin{pmatrix} 0.7 & -0.1 \\ -0.2 & 0.6 \end{pmatrix}^{-1}=\begin{pmatrix} 1.50 & 0.25 \\ 0.50 & 1.75 \end{pmatrix}$$

$$G=S\cdot B=\begin{pmatrix} 1.50 & 0.25 \\ 0.50 & 1.75 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 \\ 0.2 & 0.2 \end{pmatrix}=\begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix}$$

and $F=\begin{pmatrix} 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ & & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & & \\ 0.2 & 0.2 & 0.8 & 0.8 & 0.8 & \frac{1}{3} & \frac{1}{7} \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.6 & \frac{1}{3} & \frac{3}{7} \end{pmatrix}$

## Recurrent states and Limiting probabilities

♣ Consider only an irreducible set of states.

**Theorem:** Suppose $X$ is irreducible and aperiodic. Then all states are recurrent non-null if and only if

$$\pi(j) = \sum_{i \in E} \pi(i) P(i, j),\ j \in E, \quad \sum_{j \in E} \pi(j) = 1$$

has a solution $\pi$. If there exists a solution $\pi$, then it is strictly positive, there are no other solutions, and we have

$$\pi(j) = \lim_{n \to \infty} P^n(i, j), \forall i, j \in E$$

**Corollary:** If $X$ in an irreducible aperiodic MC with finitely many states (no-null states, no transient states), then

$$\pi \cdot P = \pi, \quad \pi \cdot 1 = 1$$

has a unique solution. The solution $\pi$ is strictly positive, and $\pi(j) = \lim_{n \to \infty} P^n(i, j)$, $\forall i, j$.

♣ A **probability** distribution $\pi$ which satisfies $\pi = \pi \cdot P$, is called an **invariant** distribution for $X$.

♣ If $\pi$ is the initial distribution of $X$, that is, $P\{X_0 = j\} = \pi(j), \quad j \in E$

then $P\{X_n = j\} = \sum_i \pi(i) P^n(i, j) = \pi(j)$, for any $n \in E$

Proof: $\pi = \pi \cdot P = \pi \cdot P^2 = \cdots = \pi \cdot P^n$

**Algorithm:** for finding $\lim_{n\to\infty} P^n(i, j)$

- Consider the irreducible closed set containing $j$
- Solve for $\pi(j)$. Thus, we find $\lim_{n\to\infty} P^n(j, j)$
- For every $i$ (not necessarily in E)

$$\lim_{n\to\infty} P^n(i, j) = F(i, j) \lim_{n\to\infty} P^n(j, j)$$

Compute $F(i, j)$ first. Then, find $\lim_{n\to\infty} P^n(i, j)$

**Example:**

$$E=\{1, 2, 3\},\ P=\begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0. & 0.4 \\ 0. & 0.4 & 0. \end{pmatrix}$$

$$\pi P=\pi \quad \Rightarrow \quad \begin{array}{ccccccc} \pi(1) & = & \pi(1)0.3 & + & \pi(2)0.6 & & \\ \pi(2) & = & \pi(1)0.5 & + & & + & \pi(3)0.4 \\ \pi(3) & = & \pi(1)0.2 & + & \pi(2)0.4 & + & \pi(3)0.6 \end{array}$$

$$\pi\,1=1$$

System's Solution:

$$\pi=\begin{pmatrix} \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \end{pmatrix} \Rightarrow P^{\infty}=\lim_{n\to\infty} P^{n}(i, j)=\begin{pmatrix} \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \\ \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \\ \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \end{pmatrix}$$

**Example:**

E={1, 2, 3, 4, 5, 6, 7},
$$P=\begin{pmatrix} 0.2 & 0.8 & & & & & \\ 0.7 & 0.3 & & & & & \\ & & 0.3 & 0.5 & 0.2 & & \\ & & 0.6 & 0. & 0.4 & & \\ & & 0. & 0.4 & 0.6 & & \\ 0. & 0.1 & 0.1 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0. & 0.1 & 0.2 & 0.4 \end{pmatrix}$$

$$P_1=\begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix} \Rightarrow \pi_1=\begin{pmatrix} \frac{7}{15} & \frac{8}{15} \end{pmatrix}, \quad P_2=\begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0. & 0.4 \\ 0. & 0.4 & 0.6 \end{pmatrix} \Rightarrow \pi_2=\begin{pmatrix} \frac{6}{23} & \frac{7}{23} & \frac{10}{23} \end{pmatrix}$$

$$\begin{bmatrix} F(6,1) & \cdots & F(6,5) \\ F(7,1) & \cdots & F(7,5) \end{bmatrix}=\begin{bmatrix} 0.2 & 0.2 & 0.8 & 0.8 & 0.8 \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.6 \end{bmatrix}$$

Thus,

$$P^{\infty} = \lim_{n\to\infty} P^n = \begin{pmatrix} \frac{7}{15} & \frac{8}{15} & & & & & \\ \frac{7}{15} & \frac{8}{15} & & & & & \\ & & \frac{6}{23} & \frac{7}{23} & \frac{10}{23} & & \\ & & \frac{6}{23} & \frac{7}{23} & \frac{10}{23} & & \\ & & \frac{6}{23} & \frac{7}{23} & \frac{10}{23} & & \\ \frac{1.4}{15} & \frac{1.6}{15} & \frac{4.8}{23} & \frac{5.6}{23} & \frac{8}{23} & 0. & 0. \\ \frac{2.8}{15} & \frac{3.2}{15} & \frac{3.6}{23} & \frac{4.2}{23} & \frac{6}{23} & 0. & 0. \end{pmatrix}$$

**Example:**

Random walks: $P = \begin{pmatrix} q & p & & & \cdots \\ q & 0 & p & & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ ( $X$ irreducible aperiodic (since state 0 is aperiodic))

$$\left.\begin{array}{rcl} \pi_0 & = & \pi_0 q + \pi_1 q \\ \pi_1 & = & \pi_0 p + \pi_2 q \\ \pi_2 & = & \pi_1 p + \pi_3 q \\ \vdots & & \vdots \end{array}\right\} \overset{\pi_0=1}{\Longrightarrow} \left.\begin{array}{rcl} \pi_1 & = & \frac{p}{q} \\ \pi_2 & = & \left(\frac{p}{q} - p\right)/q = \frac{p^2}{q^2} \\ \pi_3 & = & \left(\frac{p^2}{q^2} - \frac{p^2}{q}\right)/q = \frac{p^3}{q^3} \\ \vdots & & \vdots \end{array}\right\} \overset{\pi_0=1}{\Longrightarrow} \pi = \begin{pmatrix} 1 & \frac{p}{q} & \frac{p^2}{q^2} & \cdots \end{pmatrix}$$

- ♣ If $p \geq q$: no solution of $\pi = \pi \cdot P$, $\pi \cdot 1 = 1$
- ♣ If $p < q$: $\lim_{n\to\infty} P^n(i,j) = \left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^j$

**Example:** Remaining lifetime

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\left.\begin{array}{rcl} \pi_0 & = & \pi_0 p_1 + \pi_1 \\ \pi_1 & = & \pi_0 p_2 + \quad + \pi_2 \\ \pi_2 & = & \pi_0 p_3 + \quad + \pi_3 \\ \vdots & & \vdots \end{array}\right\} \overset{\pi_0 = 1}{\Longrightarrow} \begin{array}{rcl} \nu_0 & = & 1 \\ \nu_1 & = & 1 - p_1 \\ \nu_2 & = & 1 - p_1 - p_2 \\ \vdots & & \vdots \end{array}$$

Thus,

$$\sum_{j=0}^{\infty} \nu_j = (p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + \cdots) + (p_3 + \cdots) + \cdots$$

$$= p_1 + 2p_2 + 3p_3 + \cdots = m$$

- ♣ $m = E[Z_n]$ is the expected lifetime.
- ♣ If $m = \infty$ then all states are recurrent null and $\lim_{n\to\infty} P^n(i,j) = 0$

**Interpretation of Limiting Probabilities**

**Proposition:** Let $j$ be an aperiodic recurrent non-null state, and let $m(j)$ be the expected time between two returns to $j$. Then,

$$\pi(j) = \lim_{n\to\infty} P^n(j,j) = \frac{1}{m(j)}$$

The limiting probability $\pi(j)$ of being in state $j$ is equal to the **rate** at which $j$ is visited.

**Proposition:** Let $j$ be an aperiodic recurrent non-null and let $\pi(j)$ defined as previously. Then, for almost all $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} 1_j(X_m(\omega)) = \pi(j)$$

♣. If $f$ is a bounded function on $E$, then

$$\sum_{m=0}^{n} f(X_m) = \sum_{j\in E} f(j) \sum_{m=0}^{n} 1_j(X_m)$$

**Corollary:** $X$ irreducible recurrent MC, with limiting probability $\pi$. Then, for any bounded function $f$ on $E$:

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} f(X_m) = \pi \cdot f, \qquad \pi \cdot f = \sum_{j\in E} \pi(j) f(j)$$

Similar results hold for **expectations**

**Corollary:** Suppose $X$ is an irreducible recurrent MC with limiting distribution $\pi$. Then for any bounded function $f$ on $E$

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^{n} E_i[f(X_m)] = \pi\cdot f$$

independent of $i$.

- If $f(j)$ is the reward received whenever $X$ is in $j$, then both the expected average reward in the long run and the actual average reward in the long run converge to the constant $\pi\cdot f$.

The **ratio** of the total reward received during the steps $0,1,\ldots,n$ by using function $f$ to the corresponding amount by using function $g$ is

$$\lim_{n\to\infty}\frac{\sum_{m=0}^{n} f(X_m)}{\sum_{m=0}^{n} g(X_m)} = \frac{\pi\cdot f}{\pi\cdot g}$$

♣ The same holds even in the case that $X$ is only recurrent (can be null or periodic or both)

**Theorem:** Let $X$ be an irreducible recurrent chain with transition matrix $P$. Then, the system

$$\nu = \nu \cdot P$$

has a strictly positive solution; any other solution is a constant multiple of that one.

**Theorem:** Suppose $X$ is irreducible recurrent, and let $\nu$ be a solution of $\nu = \nu \cdot P$. Then for any two functions $f$ and $g$ on $E$ for which the two sums

$$\nu \cdot f = \sum_{i \in E} \nu(i) f(i), \qquad \nu \cdot g = \sum_{i \in E} \nu(i) g(i)$$

converge absolutely and at least one is not zero we have

$$\lim_{n \to \infty} \frac{\sum_{m=0}^{n} E_i[f(X_m)]}{\sum_{m=0}^{n} E_i[g(X_m)]} = \frac{\nu \cdot f}{\nu \cdot g}$$

independently of $i, j \in E$. Moreover we also have

$$\lim_{n \to \infty} \frac{\sum_{m=0}^{n} f(X_m(\omega))}{\sum_{m=0}^{n} g(X_m(\omega))} = \frac{\nu \cdot f}{\nu \cdot g}$$

for almost all $\omega \in \Omega$

Any non-negative solution of $\nu = \nu \cdot P$ is called an **invariant measure** of $X$ .

**Comments:**

- Any irreducible recurrent chain $X$ has an invariant measure, and this is unique up to a multiplication by a constant.
- Furthermore, if $X$ is also non-null, then $\nu \cdot 1 = \sum_j \nu(j)$ is finite, and $\nu$ is a constant multiple of the limiting distribution $\pi$ satisfying $\pi P = \pi$ , $\pi \cdot 1 = 1$
- The existence of an invariant measure $\nu$ for $X$ **does not** imply that $X$ is recurrent.
- For $f = 1_k$, $g = 1_j$ and $i = j$
- $$\lim_{n\to\infty} \frac{E_j\left[\sum_{m=0}^{n} 1_k(X_m)\right]}{E_j\left[\sum_{m=0}^{n} 1_j(X_m)\right]} = \frac{\nu(k)}{\nu(j)}$$
- $\frac{\nu(k)}{\nu(j)}$ is the ratio of the expected number of visits to $k$ during the first $n$ steps to the expected number of returns to $j$ during the same period as $n \to \infty$
- $\frac{\nu(k)}{\nu(j)}$ is the expected number of visits to $k$ between two visits to state $j$

# Periodic States

It is sufficient to consider only an irreducible MC with periodic recurrent states.

**Lemma:** Let $X$ be an irreducible MC with recurrent periodic states with period $\delta$. Then, the states can be divided into $\delta$ disjoint sets $B_1,\ B_2,\ldots,\ B_\delta$ such that $P(i, j) = 0$ unless

$$i \in B_1, j \in B_2, \quad \text{or } i \in B_2, j \in B_3, \quad \text{or } \cdots i \in B_\delta, j \in B_1.$$

**Example:** $X$ MC with $E=\{1,2,3,4,5,6,7\}$

$$P=\begin{pmatrix} & & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & & \\ & & \frac{1}{3} & 0 & \frac{2}{3} & & \\ & & & & & \frac{1}{3} & \frac{2}{3} \\ & & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & & & & & \\ \frac{1}{4} & \frac{3}{4} & & & & & \end{pmatrix}$$

All states are periodic with period $3$. The sets are $B_1=\{1,2\}$, $B_2=\{3,4,5\}$ and $B_3=\{6,7\}$.

>From $B_1$ in one step the MC reaches $B_2$, in two steps $B_3$ and in three steps $B_1$.

$$P^2 = \begin{pmatrix} & & & & & \frac{23}{48} & \frac{25}{48} \\ & & & & & \frac{11}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{2}{3} & & & & & \\ \frac{3}{8} & \frac{5}{8} & & & & & \\ \frac{7}{16} & \frac{9}{16} & & & & & \\ & & \frac{5}{12} & \frac{1}{8} & \frac{11}{24} & & \\ & & \frac{3}{8} & \frac{1}{16} & \frac{9}{16} & & \end{pmatrix}, \quad P^3 = \begin{pmatrix} \frac{71}{192} & \frac{121}{192} & & & & & \\ \frac{29}{72} & \frac{43}{72} & & & & & \\ & & \frac{14}{36} & \frac{3}{36} & \frac{19}{36} & & \\ & & \frac{19}{48} & \frac{3}{32} & \frac{49}{96} & & \\ & & \frac{13}{32} & \frac{7}{64} & \frac{31}{64} & & \\ & & & & & \frac{157}{288} & \frac{131}{288} \\ & & & & & \frac{111}{192} & \frac{81}{192} \end{pmatrix}$$

Note: $\bar{P} = P^3, \qquad \bar{P} = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{pmatrix}$

- Chain corresponding to $\bar{P}$ has three closed sets $B_1$, $B_2$, $B_3$ and each one of these is irreducible, recurrent and aperiodic.
- The previous limiting theory applies to compute $\lim_m P_1^m$, $\lim_m P_2^m$, $\lim_m P_3^m$ separately.

**Theorem:** Let $P$ the transition matrix of an irreducible MC with recurrent periodic states of period $\delta$, and let $B_1,\ B_2,\ldots\ B_\delta$ be as previously. Then, in the MC with transition matrix $\overline{P} = P^\delta$, the classes $B_1,\ B_2,\ldots\ B_\delta$ are irreducible closed sets of aperiodic states.

$$\overline{P} = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_\delta \end{pmatrix}, \qquad P_a(i,j) = P^\delta(i,j), \quad i,j \in B_a$$

**Comments:**

- If $i \in B_a$, then $P_i\{X_m \in B_b\} = 1, \quad b = a + m(\mathrm{mod}\,\delta)$
- $P^n(i,j)$ does not have a limit as $n \to \infty$ except when all the states are null (in which case $P^n(i,j) \to 0, \quad \forall i,j, \quad n \to \infty$ )
- The limits $P^{n\delta+m}(i,j)$ exist as $n \to \infty$, but are dependent on the initial state $i$.

**Theorem:** Let $P$ and $B_a$ as previously and suppose that the chain is non-null. Then, for any $m \in \{0,1,\ldots,\delta-1\}$

$$\lim_{n\to\infty} P^{n\delta+m}(i,j) = \begin{cases} \pi(j) & i \in B_a, \quad j \in B_b, \quad b = a + m(\mathrm{mod}\,\delta) \\ 0 & \text{otherwise} \end{cases}$$

The probabilities $\pi(j)$, $j \in E$ form the unique solution of

$$\pi(j) = \sum_{i\in E} \pi(i)P(i,j), \qquad \sum_{i\in E} \pi(i) = \delta$$

**Example:** Let $X$ be a MC with state space $E=\{1,2,3,4,5\}$, $P=\begin{pmatrix} & & & 0.5 & 0.5 \\ & & & 0.4 & 0.6 \\ & & & 0. & 1. \\ 0.8 & 0. & 0.2 & & \\ 0. & 1. & 0. & & \end{pmatrix}$

The chain is irreducible, recurrent non-null periodic with period $\delta=2$.

$$P^2=\overline{P}=\begin{pmatrix} 0.4 & 0.5 & 0.1 & & \\ 0.32 & 0.6 & 0.08 & & \\ 0. & 1. & 0. & & \\ & & & 0.4 & 0.6 \\ & & & 0.4 & 0.6 \end{pmatrix} \qquad \pi_1=\begin{pmatrix}0.32 & 0.60 & 0.08\end{pmatrix},\quad \pi_2=\begin{pmatrix}0.4 & 0.6\end{pmatrix}$$

$$\lim_{n\to\infty}P^{2n}=\begin{pmatrix} 0.32 & 0.60 & 0.08 & & \\ 0.32 & 0.60 & 0.08 & & \\ 0.32 & 0.60 & 0.08 & & \\ & & & 0.4 & 0.6 \\ & & & 0.4 & 0.6 \end{pmatrix} \qquad \lim_{n\to\infty}P^{2n+1}=\begin{pmatrix} & & & 0.4 & 0.6 \\ & & & 0.4 & 0.6 \\ & & & 0.4 & 0.6 \\ 0.32 & 0.60 & 0.08 & & \\ 0.32 & 0.60 & 0.08 & & \end{pmatrix}$$

**Example:** Random Walks ( $p < q$ ) $\quad P = \begin{pmatrix} 0 & 1 & & & \\ q & 0 & p & & \\ & q & 0 & p & \\ & & \ddots & \ddots & \end{pmatrix}$

- Cyclic Classes $B_1 = \{0, 2, 4, \ldots\}$ , $B_2 = \{1, 3, 5, \ldots\}$
- Invariant solution $\nu = \nu \cdot P$

$$\nu_0 = 1, \quad \nu_2 = \frac{p}{q^2}, \quad \nu_4 = \frac{p}{q^2}, \quad \ldots$$

$$\nu_1 = \frac{1}{q}, \quad \nu_3 = \frac{p^2}{q^3}, \quad \nu_5 = \frac{p^4}{q^5}, \quad \ldots$$

Normalize: $$\sum \nu_i = 1 + \frac{1}{q}\left[1 + \frac{p}{q} + \frac{p^2}{q^2} + \cdots\right] = 1 + \frac{1}{q}\frac{1}{1 - \frac{p}{q}} = \frac{2}{1 - \frac{p}{q}}$$

Multiply each term by $1 - \frac{p}{q}$. ( $\sum \nu_i = 2$ )

$$(\pi_0, \quad \pi_2, \quad \pi_4, \quad \ldots) = \left(1 - \tfrac{p}{q}\right)\left(1, \quad \tfrac{p}{q^2}, \quad \tfrac{p^3}{q^4}, \quad \ldots\right)$$

$$(\pi_1, \quad \pi_3, \quad \pi_5, \quad \ldots) = \left(1 - \tfrac{p}{q}\right)\left(\tfrac{1}{q}, \quad \tfrac{p^2}{q^3}, \quad \tfrac{p^4}{q^5}, \quad \ldots\right)$$

Hence,

$$\lim_{n\to\infty} P^{2n} = \left(1-\frac{p}{q}\right)\begin{pmatrix} 1 & 0 & \frac{p}{q^2} & 0 & \frac{p^3}{q^4} & \cdots \\ 0 & \frac{1}{q} & 0 & \frac{p^2}{q^3} & 0 & \cdots \\ 1 & 0 & \frac{p}{q^2} & 0 & \frac{p^3}{q^4} & \cdots \\ 0 & \frac{1}{q} & 0 & \frac{p^2}{q^3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \lim_{n\to\infty} P^{2n+1} = \left(1-\frac{p}{q}\right)\begin{pmatrix} 0 & \frac{1}{q} & 0 & \frac{p^2}{q^3} & 0 & \cdots \\ 1 & 0 & \frac{p}{q^2} & 0 & \frac{p^3}{q^4} & \cdots \\ 0 & \frac{1}{q} & 0 & \frac{p^2}{q^3} & 0 & \cdots \\ 1 & 0 & \frac{p}{q^2} & 0 & \frac{p^3}{q^4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

## Transient States

- If a MC has only **finitely** many transient states, then it will eventually leave the set of transient states never to return.
- If there are **infinitely** many transient states, it is possible for the chain to remain in the set of transient states forever.

**Example:**

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & p_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- All states are transient
- If initial state is $i$, then the chain stays forever in the set $\{i, i+1, i+2, ..\}$.

As $n \to \infty$, $X_n(\omega) \to \infty$

Let $A \subset E$, $Q$ the matrix obtained from $P$ by deleting all the rows and columns corresponding to states which are **not** in $A$. Then, for $i, j \in A$

$$Q^n(i,j) = \sum_{i_1 \in A} \cdots \sum_{i_{n-1} \in A} Q(i,i_1)Q(i_1,i_2)\cdots Q(i_{n-1},j) = P_i\left\{X_1 \in A, \cdots, X_{n-1} \in A, X_n = j\right\}$$

$$\sum_{j \in A} Q^n(i,j) = P_i\left\{X_1 \in A, \cdots, X_{n-1} \in A, X_n \in A\right\}$$

The event $\{X_1 \in A, \ldots, X_{n+1} \in A\}$ is a subset of $\{X_1 \in A, \ldots, X_n \in A\}$, therefore

$$\sum_{j \in A} Q^n(i,j) \geq \sum_{j \in A} Q^{n+1}(i,j)$$

Let

$$f(i) = \lim_{n \to \infty} \sum_{j \in A} Q^n(i,j), \qquad i \in A$$

♣ $f(i)$ is the probability that starting at $i \in A$, the chain stays in the set $A$ **forever**.

**Proposition:** The function $f$ is the maximal solution of the system

$$h = Qh, \qquad 0 \le h \le 1$$

Either $f = 0$ or $\sup_{i \in A} f(i) = 1$

♣ An application of the previous proposition was given in a theorem on the classification of states:

**Theorem:** Let $X$ an irreducible MC with transition matrix $P$, and let $Q$ be the matrix obtained from $P$ by deleting the $k$-row and $k$-column for some $k \in E$. Then all states are **recurrent** if and only if the only solution of

$$h(i) = \sum_{j \in E_0} Q(i,j)h(j), \qquad 0 \le h(i) \le 1, \qquad i \in E_0$$

is $h(i) = 0$ for all $i \in E_0$. $E_0 = E - \{k\}$.

Proof:

- Fix a perticular state and name it $0$.
- Since $X$ is irreducible it is possible to go from $0$ to some $i \in A = E - \{0\}$.
- If the probability $f(i)$ of remaining in $A$ forever is $f(i) = 0$ for all $i \in A$, then with probability 1, the chain will leave $A$ and enter $0$ again.
- Hence, if the only solution of the system is $h = 0$, then state $0$ is recurrent , and that in turn implies that all states are recurrent.
- Conversely, if all states are recurrent, then the probability of remaining in the set $A$ forever must be zero, since $0$ will be reached with probability one from any state $i \in A$

**Example:** (Random Walk)

$$Q = \begin{pmatrix} 0 & p & & & \\ q & 0 & p & & \\ & q & 0 & p & \\ & & & & \ddots \end{pmatrix}$$

- If $p > q$ all states are transient.

$$f(i) = 1 - \left(\frac{q}{p}\right)^i, \qquad i = 1, 2, 3, \ldots$$

This is the maximal solution since $\sup_i f(i) = 1$.

<u>Interpretation:</u>

- ➢ Starting at a state $k$ (e.g. $k = 7$) the probability of staying forever within the set $\{1, 2, 3, \ldots\}$ is equal to $1 - \left(\frac{q}{p}\right)^7$.
- ➢ If $k' > k$, the probability of remaining in $\{1, 2, 3, \ldots\}$ is greater.
- ➢ From the shape of $P$: the restriction of $P$ to the set $\{k, k+1, \ldots\}$ is the same as the matrix $Q$. Hence, for all $k \in \{1, 2, 3, \ldots\}$

$$P_{k+i}\{X_1 \geq k, X_2 \geq k, \ldots\} = 1 - \left(\frac{q}{p}\right)^{i+1}$$

For any subset $A$ of $E$, let $f_A(i)$ the probability of remaining forever in $A$ given the initial state $i \in A$. Then,

- If $A$ is an **irreducible recurrent** class, $f_A = 1$.
- If $A$ is a proper subset of an irreducible recurrent class, $f_A = 0$.
- If $A$ is a **finite set of transient** states, $f_A = 0$.
- If $A$ is an **infinite set of transient** states, then either $f_A = 0$ or $f_A \neq 0$.

In the latter case the chain travels through a sequence of sets ($A_1 \supset A_2 \supset A_3 \cdots$) to "infinite".