Applications to Queueing Theory

Introduction to Stochastic Processes (Erhan Cinlar)
Ch. 5.5, 5.6

Applications to Queueing Theory: M/G/1 Queue

$N_t(\omega)$: number of arrivals during the time interval $[0,t]$.

$Z_1(\omega), Z_2(\omega), \ldots$: service times of customers who depart first, second, ...

$Y_t(\omega)$: number of customers in the system (waiting or being served at time t)

Assumptions:

♣ $N = \{N_t; t \geq 0\} \sim P(a)$

♣ $Z_1, Z_2, \ldots$ i.i.d. $\sim \phi$

- Consider the future of Y from a time T of a departure onward.
- Define X_n as the number of customers in the system just after the instant of the n^{th} departure.

Theorem: X is a MC with the transition matrix

$$P = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 & \cdots \\ q_0 & q_1 & q_2 & q_3 & \cdots \\ & q_0 & q_1 & q_2 & \cdots \\ & & q_0 & q_1 & \cdots \\ & & & q_0 & \cdots \\ & & & & \ddots \end{pmatrix}, \quad q_k = \int_0^\infty \frac{e^{-at}(at)^k}{k!} d\phi(t), \quad k = 0,1,\ldots$$

Proof: We need to show $P\{X_{n+1}=j \mid X_0,\ldots,X_n\}=P\{X_{n+1}=j \mid X_n\}$

$$P\{X_{n+1}=j \mid X_n=i\}=\begin{cases} q_j & i=0, j\ge 0 \\ q_{j+1-i} & i>0, j\ge i-1 \\ 0 & \text{otherwise} \end{cases}$$

- Let T the time of the n^{th} departure.
- Let $Z=Z_{n+1}$ the service time of the $n+1$ customer.

Then, $X_{n+1}=\begin{cases} X_n+(N_{T+Z}-N_T)-1, & X_n>0 \\ N_{S+Z}-N_S, & X_n=0 \end{cases}$ (S:arrival time of the n+1 customer)

Using Poisson properties: $P\{N_{T+Z}-N_T=k \mid X_0,\ldots X_n;T\}=P\{N_Z=k\}$

$$q_k=P\{N_Z=k\}=E\left[P\{N_Z=k \mid Z\}\right]=E\left[\frac{e^{-aZ}(aZ)^k}{k!}\right]=\int_0^\infty \frac{e^{-at}(at)^k}{k!}d\phi(t)$$

- $i=0$ $\quad P\{X_{n+1}=j \mid X_n=0\}=P\{N_{S+Z}-N_S=j\}=P\{N_Z=j\}=q_j$

- $i>0$ $\quad P\{X_{n+1}=j \mid X_n=i\}=P\{N_{T+Z}-N_T=j+1-i\}$

$$=P\{N_Z=j+1-i\}=\begin{cases} q_{j+1-i}, & j\ge i-1 \\ 0, & j<i-1 \end{cases}$$

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The MC X is irreducible and aperiodic. If

$$r = E[N_Z] = aE[Z] = ab$$

then,

- If $r > 1$ all states are transient
- If $r < 1$ all states are recurrent non-null.
- If $r = 1$ all states are recurrent null

Notation:

$r_k = 1 - q_0 - \cdots - q_k$

$r = r_0 + r_1 + \cdots = (q_1 + q_2 + q_3 + \cdots) + (q_2 + q_3 + \cdots) + (q_3 + \cdots) + \cdots$

$= q_1 + 2q_2 + 3q_3 + \cdots$

Proposition: The chain X is recurrent non-null aperiodic if and only if $r < 1$.

Proof: We need to show that

$$\pi = \pi \cdot P, \qquad \pi \cdot 1 = 1$$

$$\left.\begin{array}{rcl} \pi_0 & = & \pi_0 q_0 + \pi_1 q_0 \\ \pi_1 & = & \pi_0 q_1 + \pi_1 q_1 + \pi_2 q_0 \\ \pi_2 & = & \pi_0 q_2 + \pi_1 q_2 + \pi_2 q_1 + \pi_3 q_0 \\ & \vdots & \end{array}\right\} \Rightarrow \begin{array}{rcl} \pi_1 q_0 & = & \pi_0 r_0 \\ \pi_2 q_0 & = & \pi_0 r_1 + \pi_1 r_1 \\ \pi_3 q_0 & = & \pi_0 r_2 \pi_1 r_2 + \pi_2 r_1 \\ & \vdots & \end{array}$$

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Summing all equations ($q_0 = 1 - r_0$, $r = r_0 + r_1 + r_2 + \cdots$)

$$(1-r_0)\cdot\sum_{j=1}^{\infty}\pi_j = \pi_0 r + (r-r_0)\sum_{j=1}^{\infty}\pi_j$$

If $r<1$, then we obtain $\sum_{j=1}^{\infty}\pi_j = \frac{r}{1-r}\pi_0 \quad\Rightarrow\quad \sum_{j=0}^{\infty}\pi_j = \frac{1}{1-r}\pi_0$

The condition $\pi\cdot 1 = 1$ is satisfied with $\pi_0 = 1-r$

Theorem: The limits $\pi(j) = \lim_{n\to\infty} P^n(i,j)$ exist $\forall j\in E$ and are independent of the initial state i.

- If $r\geq 1$, then $\pi(j)=0, \quad \forall j$.
- If $r<1$, then

$$\begin{aligned}
\pi(0) &= 1-r \\
\pi(1) &= (1-r)\frac{r_0}{q_0} \\
&\vdots \\
\pi(j+1) &= (1-r)\sum_{k=1}^{j}\left(\frac{1}{q_0}\right)^{k+1}\sum_{\mathbf{a}\in S_{jk}} r_{a_1} r_{a_2}\cdots r_{a_k}
\end{aligned}$$

where S_{jk} is the set of all k-tuples $\mathbf{a}=(a_1,\ldots,a_k)$ of integers $a_i\geq 1$ with $a_1+\cdots a_k = j$

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More on the recurrent non-null case

Having the limiting distributions , we can compute $E[X_n]$, $Var(X_n)$ etc., in the limit $n \to \infty$.

Instead:

$$X_{n+1} = X_n + M_n - U_n$$

where

$$U_n = 1 - 1_0(X_n)$$

M_n is the number of arrivals during the $n+1$ th service.

$$\lim_{n\to\infty} E[U_n] = 1 - \lim_{n\to\infty} E[1_0(X_n)] = 1 - \lim_{n\to\infty} P\{X_n = 0\} = 1 - \pi(0) = r = a \cdot b$$

$$E[M_n] = \mathbf{r{=}a \cdot b}$$

$$E[M_n^2] = E\Big[E[N_Z^2 \mid Z]\Big] = E\Big[aZ + a^2Z^2\Big] = a \cdot b + a^2c^2$$

$$c^2 = E[Z^2] = \int_0^\infty t^2 d\phi(t)$$

V(X)=σ²=E(X-E(X))²=E(X²)-E(x)² => E(X2) = E(x)² +V(x)

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$$X_{n+1}^2 = X_n^2 + M_n^2 + U_n^2 + 2X_n M_n - 2X_n U_n - 2M_n U_n$$

But

- $U_n^2 = U_n$ (U_n takes values 1, 0)
- $X_n U_n = X_n$ (If $X_n > 0$, then $U_n = 1$, else if $X_n = 0$, then $U_n = 0$)

so that,

$$X_{n+1}^2 = X_n^2 + M_n^2 + U_n + 2X_n M_n - 2X_n - 2M_n U_n$$

Taking expectations of both sides we obtain

$$E[X_{n+1}^2] = E[X_n^2] + E[M_n^2] + E[U_n] + 2E[X_n]E[M_n] - 2E[X_n] - 2E[M_n]E[U_n]$$

and by letting $n \to \infty$

$$0 = ab + a^2c^2 + ab + 2qab - 2q - 2a^2b^2$$

where $$q = \lim_{n\to\infty} E[X_n] = ab + \frac{a^2c^2}{(2-2ab)}$$

Knowing the statistics of X_n we can find the statistics of V_n, (W_n), as $n \to \infty$

$$V_n = W_n + Z_n$$

where

- V_n is the total time spent in the system
- W_n is teh waiting time spent by the n^{th} customer.

What if $r \geq 1$?

Consider $f_k(j)$ the probability starting from state $k+j$, the MC X never enters in the set $\{0,1,\ldots,k\}$

$f_k(j)$ is the maximal solution of the system $h = Q \cdot h, \qquad 0 \leq h \leq 1$

where Q is the matrix obtained from P by deleting all rows and columns corresponding to the states $\{0,1,\ldots,k\}$.

$$Q = \begin{pmatrix} q_1 & q_2 & q_3 & \cdots \\ q_0 & q_1 & q_2 & \cdots \\ & q_0 & q_1 & \cdots \\ & & & \ddots \end{pmatrix}$$

Q does not depend on k, therefore $f_k(j) = f_0(j)$ for all j,k.

Lemma: The probability that X never enters $\{0,1,\ldots,k\}$ starting from $k+j$ is the same as the probability $f(j)$ that X never enters 0 starting from j.

Theorem: Let $f(j)$ be the probability that the queue, starting with j customers never becomes empty. Then,

$$f(j) = 1 - \beta^j, \qquad j = 1,2,\ldots$$

where β is the smallest number in $[0,1]$ satisfying $\beta = q_0 + q_1\beta + q_2\beta^2 + \cdots$

The β is strictly less than one if and only if the traffic intensity $r > 1$. Therefore, X is transient if and only if $r > 1$.

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Exponentially distributed service times $\sim \exp(a)$

i.i.d. interarrival times $\sim \phi$.

In this case $$q_n = \int_0^\infty \frac{e^{-at}(at)^n}{n!} d\phi(t)$$

is the probability that the server completes exactly n services during an interarrival time (provided that there are that many customers).

Define: $r_n = q_{n+1} + q_{n+2} + \cdots$

$$r = \sum_{n=1}^{\infty} nq_n = r_0 + r_1 + r_2 + \cdots$$

r is the expected number of services which the server is capable of completing during an iterarrival time. It can be proved that

- $r \geq 1$ Server can keep up with arrivals (recurrent)
- $r < 1$ Queue size increases to infinity (transient)

If $X_n^\star$ is the number of customers present in the system just before the time T_n of the n^{th} arrival, then

Theorem: $X^\star = \{X_n^\star; n \in N\}$ is a MC with $E = \{0,1,2,\ldots\}$, $P^\star = \begin{pmatrix} r_0 & q_0 & & & \\ r_1 & q_1 & q_0 & & \\ r_2 & q_2 & q_1 & q_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

Applications to Queuing Theory: G/M/1 Queue

Proof: Let M_{n+1} be the number of services completed during the $n+1^{\text{th}}$ interarrival time $[T_n, T_{n+1})$. Then,

$$X^\star_{n+1} = X^\star_n + 1 - M_{n+1}$$

But M_{n+1} is conditionally independent of the past history before T_n given the present number $X^\star_n$. If $Z = T_{n+1} - T_n$

$$P\left\{M_{n+1} = k \mid X^\star_n, Z\right\} = \begin{cases} \dfrac{e^{-aZ}(aZ)^k}{k!} & X^\star_n + 1 > k \\ \displaystyle\sum_{m=k}^{\infty} \dfrac{e^{-aZ}(aZ)^m}{m!} & X^\star_n + 1 = k \\ 0 & \text{otherwise} \end{cases}$$

Taking expectations with respect to Z, which is independent of $X^\star_n$, we obtain

$$P\left\{M_{n+1} = k \mid X^\star_n = i\right\} = \begin{cases} q_k & k \le i \\ r_{k-1} & k = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Equation $X^\star_{n+1} = X^\star_n + 1 - M_{n+1}$ and the previous one provide matrix $P^\star$

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<u>Theorem:</u> $X^\star$ is recurrent non-null if and only if $r > 1$. If $r > 1$,

$$\pi^\star(j) = \lim_{n\to\infty} P^{\star n}(i,j) = \lim_{n\to\infty} P^\star\{X_n^\star = j \mid X_0^\star = i\}$$

and

$$\pi^\star(j) = (1-\beta)\beta^j, \qquad j = 0,1,2,\ldots$$

where β is the unique number satisfying

$$\beta = q_0 + q_1\beta + q_2\beta^2 + \cdots$$

If $r \le 1$ then $\pi^\star(j) = 0$ for all j.

<u>Proof:</u> $X^\star$ is recurrent non-null if and only if

$$\nu = \nu \cdot P^\star, \qquad \nu \cdot 1 = 1$$

has a solution.

$$\begin{array}{ccccccccccc}
\nu_0 & = & & & q_1\nu_0 & + & q_2\nu_0 & + & q_3\nu_0 & + & \cdots \\
 & & & & & + & q_2\nu_1 & + & q_3\nu_1 & + & \cdots \\
 & & & & & & & + & q_3\nu_2 & + & \cdots \\
\nu_1 & = & q_0\nu_0 & + & q_1\nu_1 & + & q_2\nu_2 & + & q_3\nu_3 & + & \cdots \\
\nu_2 & = & q_0\nu_1 & + & q_1\nu_2 & + & q_2\nu_3 & + & q_3\nu_4 & + & \cdots
\end{array}$$

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$$\begin{array}{lcccccccccccc}
\nu_0 & = & & & q_1\nu_0 & + & q_2\nu_0 & + & q_3\nu_0 & + & \cdots \\
& & & & & + & q_2\nu_1 & + & q_3\nu_1 & + & \cdots \\
& & & & & & & + & q_3\nu_2 & + & \cdots \\
\nu_1 & = & q_0\nu_0 & + & q_1\nu_1 & + & q_2\nu_2 & + & q_3\nu_3 & + & \cdots \\
\nu_2 & = & q_0\nu_1 & + & q_1\nu_2 & + & q_2\nu_3 & + & q_3\nu_4 & + & \cdots
\end{array}$$

Let $f(j) = \nu_0 + \cdots + \nu_{j-1}, \quad j = 1, 2, \ldots$. Then,

$$\left\{\begin{array}{lcccccccc}
f(1) & = & q_1 f(1) & + & q_2 f(2) & + & q_3 f(3) & + & \cdots \\
f(2) & = & q_0 f(1) & + & q_1 f(2) & + & q_2 f(3) & + & \cdots \\
f(3) & = & & & q_0 f(2) & + & q_1 f(3) & + & \cdots
\end{array}\right\} \Rightarrow \quad f = Q \cdot f$$

We are interested in a solution satisfying

$$\lim_{j\to\infty} f(j) = \sum_{j=0}^{\infty} \nu_j = 1$$

Q was obtained from P by deleting 0^{th} row and column. Such an f exists if and only if X is transient which means that $r > 1$. In this case $f(j) = 1 - \beta^j$. Solving for ν we obtain

$$\nu_0 = f(1) = 1 - \beta, \qquad \nu_1 = f(2) - f(1) = (1-\beta)\beta, \ldots$$

Applications to Queuing Theory: G/M/1 Queue

Theorem: $X^\star$ is transient if and only if $r<1$. If $r<1$, the probability $f^\star(j)$ that the queue starting with j customers never becomes empty is given by

$$f^\star(j) = \pi(0) + \pi(1) + \cdots + \pi(j), \qquad j = 1, 2, \ldots$$

where the $\pi(j)$ are those found in the M/G/1 case.

Proof:

- $f^\star$ is the solution to the system $h = Q^\star h$, $0 \le h \le 1$.
- $Q^\star$ is the matrix obtained from $P^\star$ by deleting the 0^{th} row and column.

The equations for $h = Q^\star h$ are ($f^\star(j) = h_j$)

$$\begin{array}{ccccccccc}
h_1 & = & q_0 h_2 & + & q_1 h_1 & & & & \\
h_2 & = & q_0 h_3 & + & q_1 h_2 & + & q_2 h_1 & & \\
h_3 & = & q_0 h_4 & + & q_1 h_3 & + & q_2 h_2 & + & q_3 h_1 \\
 & \vdots & & & & & & &
\end{array}$$

If we define ….

If we define $\pi_0 = q_0 h_1$, $\pi_1 = (1-q_0)h_1$, and let

$$\pi_j = h_j - h_{j-1}, \qquad j = 2,3,\ldots$$

then the first of the previous equations along with $\pi_0 = q_0 h_1$, implies the equations

$$\begin{aligned} \pi_0 &= q_0\pi_0 + q_0\pi_1 \\ \pi_1 &= q_1\pi_0 + q_1\pi_1 + q_0\pi_2 \end{aligned}$$

and subtracting the equation for h_{j-1} from the one for h_j yields

$$\begin{aligned} \pi_2 &= q_2\pi_0 + q_2\pi_1 + q_1\pi_2 + q_0\pi_3 \\ \pi_3 &= q_3\pi_0 + q_3\pi_1 + q_2\pi_2 + q_1\pi_3 + q_0\pi_4 \end{aligned}$$

In other words, π satisfies $\pi = \pi P$ with P the transition matrix in the M/G/1 case, and we are interested in the solution

$$\pi = \pi P, \qquad \sum_j \pi_j = \lim_j h_j = 1$$

- Such a solution exists if and only if $r < 1$.
- The solution π is connected to h by the relation $h_j = \pi_0 + \cdots + \pi_j$

Special case M/M/1

We can consider this queue as a special case of M/G/1 or G/M/1. In the sequel we use G/M/1. Now the interarrival distribution is given by:

$$\phi(t) = 1 - e^{-\lambda t}, \qquad t \ge 0$$

To compute the limiting distribution of $X^\star$ (queue size just before the n^{th} arrival, we find first β, where

$$\beta = \sum_{k=0}^{\infty} q_k \beta^k = \sum\nolimits_{k=0}^{\infty} \beta^k \int_0^\infty \tfrac{e^{-at}(at)^k}{k!} \lambda e^{-\lambda t} dt = \int_0^\infty e^{-at(1-\beta)} \lambda e^{-\lambda t} dt = \tfrac{\lambda}{\lambda + a - a\beta}$$

The previous equation becomes $\beta = \dfrac{\lambda}{\lambda + a - a\beta}$ or $(1-\beta)(\lambda - ab) = 0$

with solutions $\beta = 1$ and $\beta = \frac{\lambda}{a}$. When $r = \frac{a}{\lambda} > 1$, the smallest solution is $\beta = \dfrac{\lambda}{a}$

So we have $$\lim_{n\to\infty} P\left\{X_n^\star = j\right\} = \left(1 - \frac{\lambda}{a}\right)\left(\frac{\lambda}{a}\right)^j, \qquad j = 0,1,\ldots$$

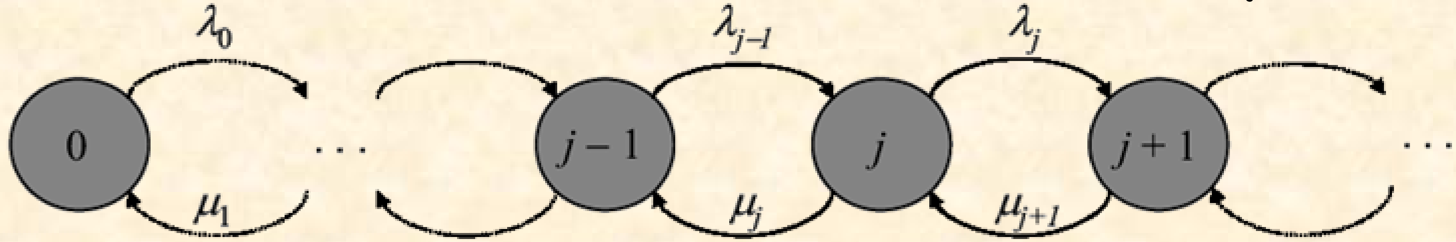
It turns out that $\lim_{t\to\infty} P\left\{Y_t = j\right\} = \left(1 - \dfrac{\lambda}{a}\right)\left(\dfrac{\lambda}{a}\right)^j$, $j = 0,1,\ldots$ for the queue size Y_t at time t.

and $\lim_{n\to\infty} P\left\{X_n = j\right\} = \left(1 - \dfrac{\lambda}{a}\right)\left(\dfrac{\lambda}{a}\right)^j$, $j = 0,1,\ldots$ for the queue size X_n just after the n^{th} departure.

Birth and Death Processes

Introduction to Stochastic Processes (Erhan Cinlar)
Ch. 8.6

Birth Death Process: The steady-state



$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mu_4 & -(\lambda_4+\mu_4) & \lambda_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The steady-state follows from $\pi Q = 0$

$$\pi_0 = \frac{1}{1+\sum_{j=1}^{\infty}\prod_{m=0}^{j-1}\frac{\lambda_m}{\mu_{m+1}}} \qquad \pi_j = \frac{\prod_{m=0}^{j-1}\frac{\lambda_m}{\mu_{m+1}}}{1+\sum_{j=1}^{\infty}\prod_{m=0}^{j-1}\frac{\lambda_m}{\mu_{m+1}}} \quad j \geq 1$$

$\Sigma_1 = \sum_{j=1}^{\infty}\prod_{m=0}^{j-1}\frac{\lambda_m}{\mu_{m+1}}$ must converge to have a steady-state

The steady-state

The process is transient if and only if the embedded MC is transient.
For a recurrent chain, $r_{ij} = \Pr[T_j < \infty | X_0 = i]$ equals 1
(every state j is certainly visited starting from initial state i)
For the embedded MC (gambler's ruin), it holds that

$$\Pr[T_0 < \infty | X_0 = j] = 1 - \frac{\sum_{k=0}^{j-1} \prod_{m=1}^{k} \frac{q_m}{p_m}}{\sum_{k=0}^{N-1} \prod_{m=1}^{k} \frac{q_m}{p_m}}$$

Can be equal to one for $N \to \infty$ only if $\lim_{N\to\infty} \sum_{k=0}^{N-1} \prod_{m=1}^{k} \frac{q_m}{p_m} = \infty$

Transformed to the birth and death rates $\Sigma_2 = \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\mu_m}{\lambda_m} = \infty$

Furthermore, the infinite series $\boxed{\Sigma_1 = \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}}$ must converge to have a steady-state distribution

- If $\Sigma_1 < \infty$ and $\Sigma_2 = \infty$ the BD process is positive recurrent
- If $\Sigma_1 = \infty$ and $\Sigma_2 = \infty$, it is null recurrent
- If $\Sigma_2 < \infty$, it is transient

Birth Death Process: The steady-state

$$\pi_0 = \frac{1}{1 + \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}} \qquad \pi_j = \frac{\prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}}{1 + \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}} \quad j \geq 1$$

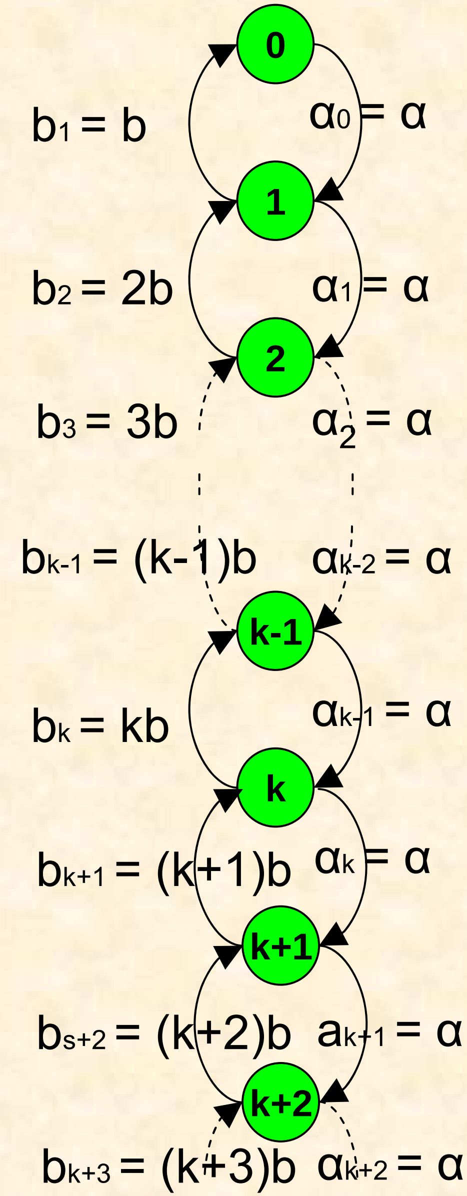
$\Sigma_1 = \sum_{j=1}^{\infty} \prod_{m=0}^{j-1} \frac{\lambda_m}{\mu_{m+1}}$ must converge to have a steady-state

$$\lambda_j \to a_j \qquad \mu_j \to b_j$$

A limiting distribution exists iff $c = \sum_{i=0}^{\infty} \nu_i = 1 + \sum_{i=1} \frac{a_0 \cdot a_1 \cdots a_{i-1}}{b_1 \cdot b_2 \cdots b_i} < \infty$

If $c < \infty$, then the limiting distribution is $\pi(j) = \begin{cases} \dfrac{1}{c} & j = 0 \\ \dfrac{a_0 \cdot a_1 \cdots a_{i-1}}{c \cdot b_1 \cdot b_2 \cdots b_i} & j \geq 1 \end{cases}$

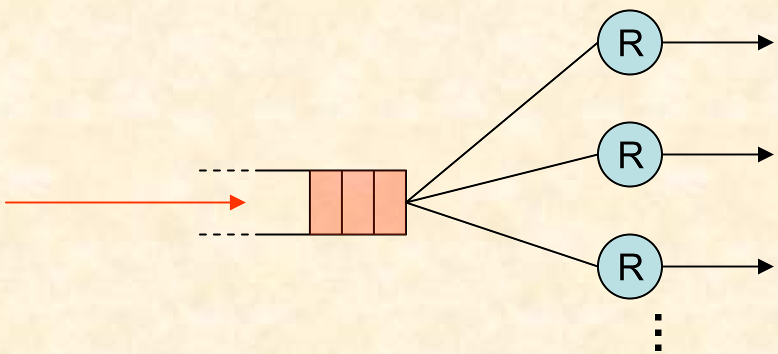
Example: $M/M/\infty$ queue.



There are infinitely many servers so that no customer ever waits.

Arrivals form a Poisson process with rate α

Service times are exponential with mean 1/b



$$c = e^{r} < \infty, \quad r = \frac{a}{b}$$

$$\pi(j) = \frac{e^{-r} r^{j}}{j!}, \qquad j = 0, 1, \ldots \qquad \text{(Poisson with parameter } r\text{)}$$

$$E[Y] = r$$

Example: $M/M/\infty$ queue.

$$Q=\begin{pmatrix} -a_0 & a_0 & & & \\ b_1 & -a_1-b_1 & a_1 & & \\ & b_2 & -a_2-b_2 & a_2 & \\ & & & & \ddots \end{pmatrix}=\begin{pmatrix} -a & a & & & \\ b & -a-b & a & & \\ & 2b & -a-2b & a & \\ & & & & \ddots \end{pmatrix}$$

$$c=1+\sum_{i=1}\frac{a_0\cdot a_1\cdots a_{i-1}}{b_1\cdot b_2\cdots b_i}=1+\sum_{i=1}\frac{a^i}{i!b^i}\overset{r=\frac{a}{b}}{=}\sum_{i=0}^{\infty}\frac{r^i}{i!}\overset{\overbrace{\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x}^{*}}{=}e^r$$

$$\pi(j)=\begin{cases}\dfrac{1}{c}=\dfrac{1}{e^r} & j=0\\[2ex] \dfrac{a_0\cdot a_1\cdots a_{j-1}}{c\cdot b_1\cdot b_2\cdots b_i}=\dfrac{1}{e^r}\dfrac{a^j}{j!b^j}=\dfrac{r^j}{e^r j!} & j\geq 1\end{cases},\qquad \text{or}\qquad \pi(j)=\frac{r^j}{e^r j!}$$

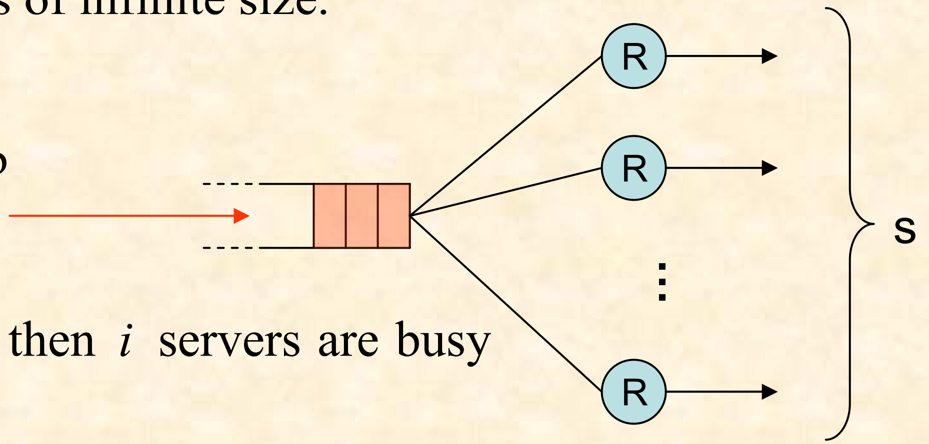
$$E[Y]=\sum_{j=0}^{\infty} j\,\pi_j=\sum_{j=1}^{\infty} j\,\frac{r^j}{e^r j!}=e^{-r}r\sum_{j=1}^{\infty}\frac{r^{j-1}}{(j-1)!}=e^{-r}r\sum_{k=0}^{\infty}\frac{r^k}{k!}\overset{\overbrace{\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x}^{*}}{=}e^{-r}re^r=r$$

Example: $M/M/s/\infty$ queue.

There are s servers, and the waiting room is of infinite size.

Arrivals form a Poisson process with rate α

Service times are exponential with mean 1/b



If there are $i < s$ customers in the system, then i servers are busy working independently of each other

If $i \geq s$, then all s -servers are busy

Therefore,

$$a_0 = a_1 = \cdots = a; \quad b_1 = b, b_2 = 2b, \ldots, b_s = sb, b_{s+1} = sb, \ldots$$

♣ A limiting distribution exists if and only if $r = \frac{a}{sb} < 1$.

Example: $M/M/s/\infty$ queue.

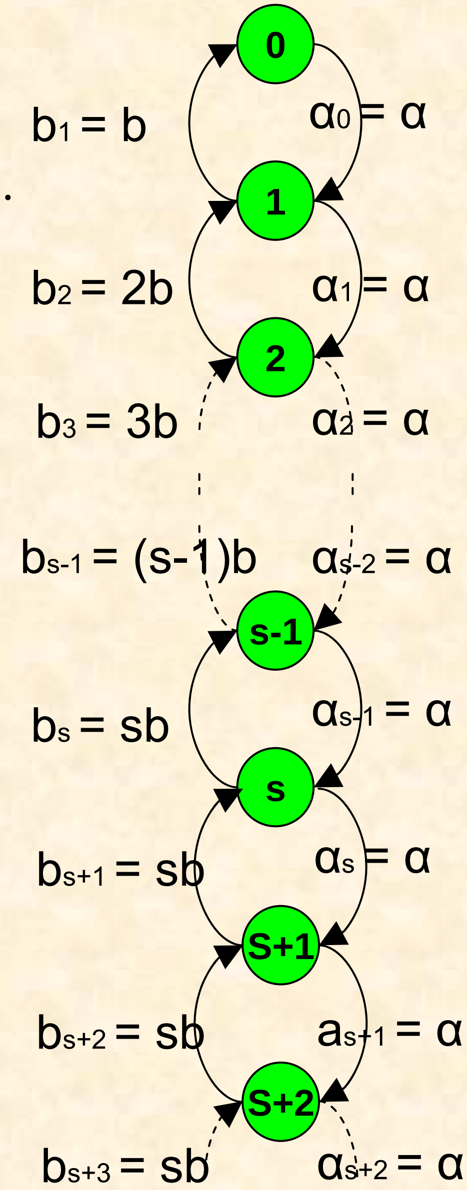
$$a_0 = a_1 = \cdots = a; \quad b_1 = b, b_2 = 2b, \ldots, b_s = sb, b_{s+1} = sb, \ldots$$

$$c = 1 + \sum_{i=1}^{s-1} \frac{s^i r^i}{i!} + \frac{s^s r^s}{s!(1-r)}, \quad r = \frac{a}{sb} < 1$$

$$\pi(j) = \begin{cases} \dfrac{1}{c} & j = 0 \\ \dfrac{1}{c}\dfrac{s^j r^j}{j!} & 0 < j < s \\ \dfrac{1}{c}\dfrac{s^s r^j}{s!} & s \le j \end{cases}$$

$$E[Y] = sr + \frac{s^s r^s}{s!} \frac{r}{(1-r)^2} \frac{1}{c}$$

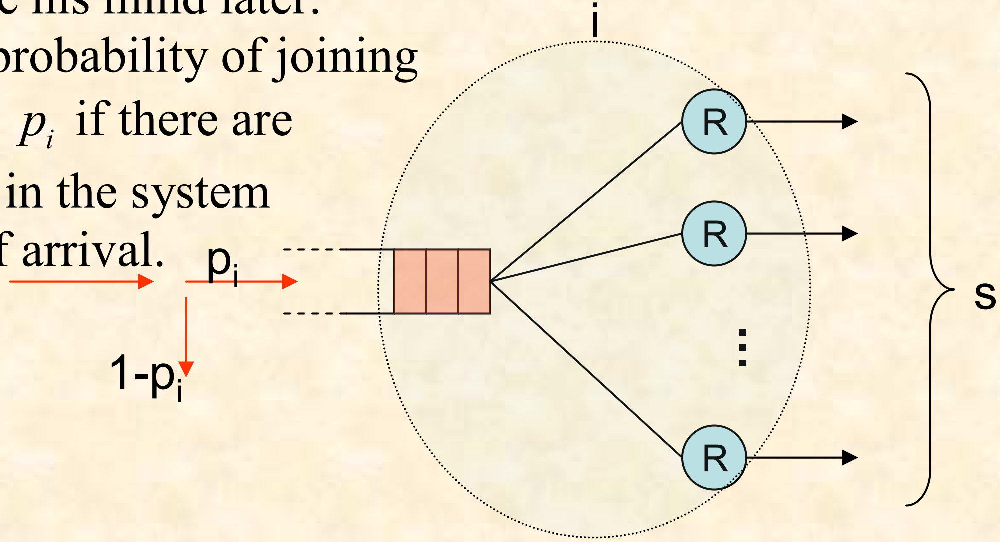
$$E[Y_q] = \frac{1}{c} \frac{s^s}{s!} r^s \frac{r}{(1-r)^2}$$

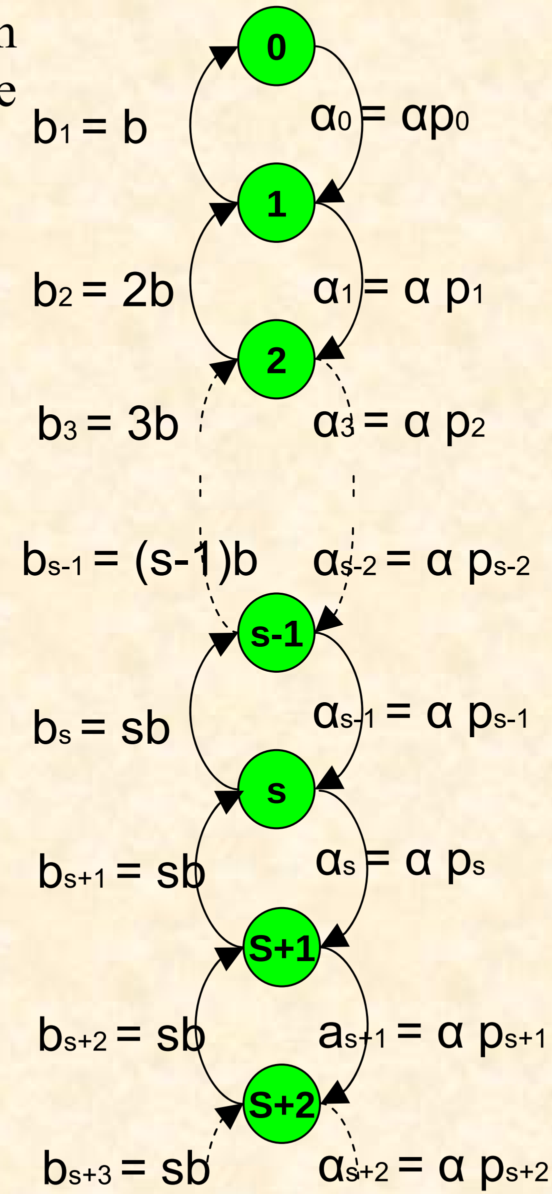


Example: $M/M/s/\infty$ queue.

An arriving customer is permitted to balk: if he finds the system too crowded, he may leave, but once he joins the system he cannot change his mind later.

Suppose the probability of joining the queue is p_i if there are i-customers in the system at the time of arrival.





If the queue size at time t is $Y_t = i$, and if there were no service completions during $[t, t+u]$, then the probability that there are no additions to the queue during $[t, t+u]$ is

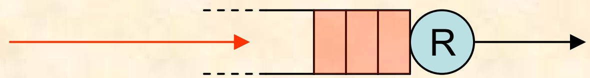
$$\sum_{n=0}^{\infty} \frac{e^{-au}(au)^n}{n!}(1-p_i)^n = e^{-au}\sum_{n=0}^{\infty}\frac{(au(1-p_i))^n}{n!} \underbrace{=}_{\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x} e^{-au}e^{au(1-p_i)} = e^{-ap_i u}$$

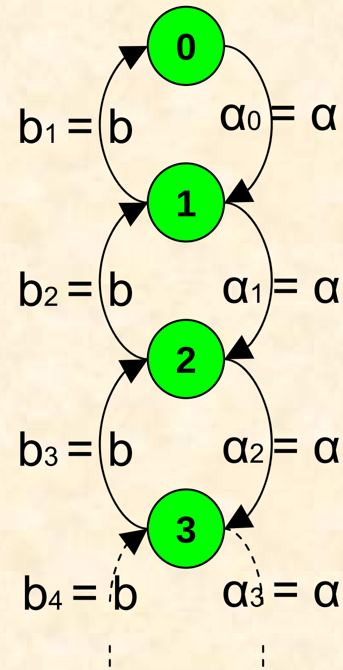
Hence, $a_0 = ap_0, a_1 = ap_1, \ldots a_i = ap_i$

$b_1 = b, b_2 = 2b, \ldots, b_s = sb, b_{s+1} = b_{s+2} = \cdots = sb$

Example: $M/M/1/\infty$ queue.

Customers arrive according to a Poisson process with rate a, service times are exponential with mean $\frac{1}{b}$, there is a single server and infinite queues are permissible.





Y_t is the number of customers in the system at time t.

Y_t is a special birth-death process, where

$$a_0 = a_1 = a_2 = \cdots = a; \qquad b_1 = b_2 = \cdots = b$$

The parameter $r = \frac{a}{b}$ is called traffic intensity.

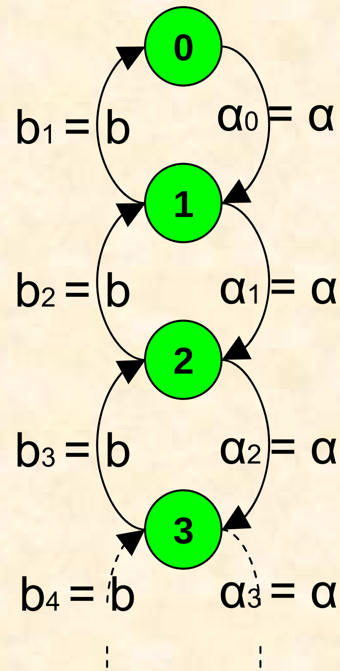
If $r = \frac{a}{b} < 1$, then there is a limiting distribution, which is

$$\pi(j) = \begin{cases} \dfrac{1}{c} & j = 0 \\ \dfrac{a^j}{cb^j} & j \ge 1 \end{cases}, \qquad c = \frac{1}{1-r}$$

Thus, $\pi(j) = (1-r)r^j$, j=0, 1, …

$$E[Y] = \frac{r}{(1-r)}, \qquad E[Y_q] = \frac{r^2}{(1-r)}$$

Example: $M/M/1/\infty$ **queue.**



$$Q = \begin{pmatrix} -a_0 & a_0 & & & \\ b_1 & -a_1-b_1 & a_1 & & \\ & b_2 & -a_2-b_2 & a_2 & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} -a & a & & & \\ b & -a-b & a & & \\ & b & -a-b & a & \\ & & & & \ddots \end{pmatrix}$$

$$c = 1 + \sum_{i=1} \frac{a_0 \cdot a_1 \cdots a_{i-1}}{b_1 \cdot b_2 \cdots b_i} = 1 + \sum_{i=1} \frac{a^i}{b^i} = \sum_{i=0}^{\infty} \frac{a^i}{b^i} = \sum_{i=0}^{\infty} r^i \overset{r<1}{=} \frac{1}{1-r}$$

$$\pi(j) = \begin{cases} \dfrac{1}{c} = (1-r) & j = 0 \\ \dfrac{a_0 \cdot a_1 \cdots a_{j-1}}{c \cdot b_1 \cdot b_2 \cdots b_i} = (1-r)\dfrac{a^j}{b^j} = (1-r)\, r^j & j \geq 1 \end{cases}, \qquad \text{or} \qquad \pi(j) = (1-r)r^j$$

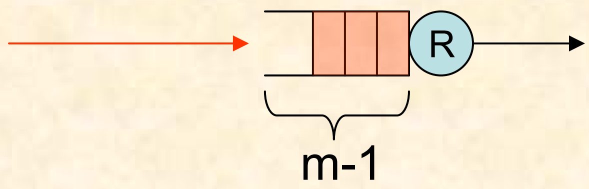
$$E[Y] = \sum_{j=0}^{\infty} j\, \pi_j = \sum_{j=1}^{\infty} j\, (1-r)r^j = (1-r)r \sum_{j=1}^{\infty} j\, r^{j-1} = (1-r)r \frac{1}{(1-r)^2} = \frac{r}{(1-r)}$$

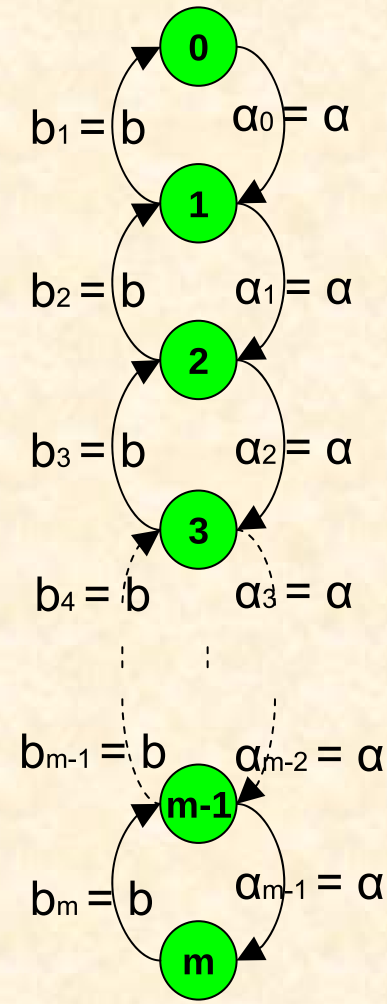
$$E[Y_q] = \sum_{j=2}^{\infty} (j-1)\, \pi_j = \sum_{j=2}^{\infty} (j-1)\, (1-r)r^j = (1-r)r^2 \sum_{j=2}^{\infty} (j-1)\, r^{j-2}$$

$$= (1-r)r^2 \sum_{k=1}^{\infty} k\, r^{k-1} = (1-r)r^2 \frac{1}{(1-r)^2} = \frac{r^2}{(1-r)}$$

Example: $M/M/1/m$ queue.

Exactly as the previous case with the difference that the "waiting" room has finite capacity $m-1$.





Y_t is the number of customers in the system at time t.

Starting with less than m customers, the total number in the system cannot exceed m. A customer arriving to find m or more customers in the system leaves and never returns.

$$a_0 = a_1 = \cdots = a_{m-1} = a; \quad a_m = a_{m+1} = \cdots = 0; \quad b_1 = b_2 = \cdots = b$$

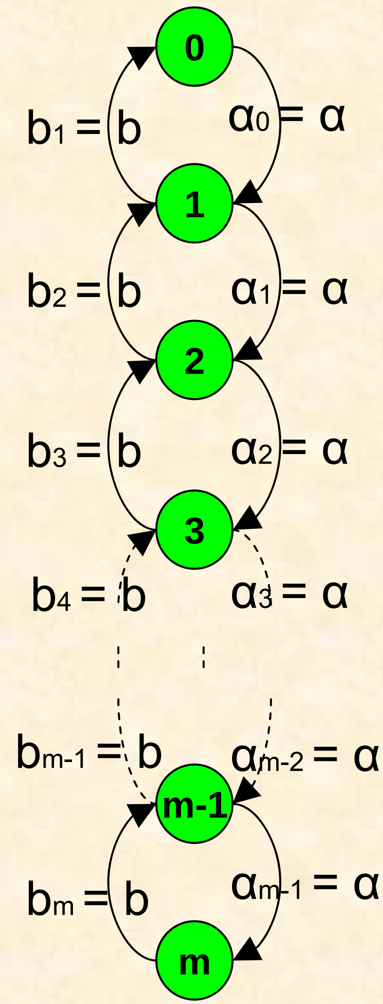
States $m+1, m+2, \ldots$ are all transient.

The $C = \{0,1,\ldots,m\}$ is a recurrent irreducible set.

Example: $M/M/1/m$ queue.

$r \neq 1$

$$Q = \begin{pmatrix} -a_0 & a_0 & & & \\ b_1 & -a_1-b_1 & a_1 & & \\ & b_2 & -a_2-b_2 & a_2 & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} -a & a & & & \\ b & -a-b & a & & \\ & & \ddots & & \\ & & b & -a-b & a \\ & & & b & -b \end{pmatrix}$$



$$c = \sum_{i=0}^{\infty} v_i = \sum_{i=0}^{m} \frac{a^i}{b^i} = \sum_{i=0}^{m} r^i \overset{r\neq 1}{=} \frac{1-r^{m+1}}{1-r}$$

$$\pi(j) = \begin{cases} \dfrac{1}{c} = \dfrac{1-r}{1-r^{m+1}} & j = 0 \\ \dfrac{a_0 \cdot a_1 \cdots a_{j-1}}{c \cdot b_1 \cdot b_2 \cdots b_i} = \dfrac{1-r}{1-r^{m+1}} \dfrac{a^j}{b^j} = \dfrac{1-r}{1-r^{m+1}} r^j & m \geq j \geq 1 \end{cases},$$

or $\qquad \pi(j) = \dfrac{1-r}{1-r^{m+1}} r^j, m \geq j \geq 0$

$$E[Y] = \sum_{j=0}^{\infty} j \pi_j = \frac{r}{1-r} - r^{m+1} \frac{m+1}{1-r^{m+1}}$$

Example: $M/M/1/m$ queue.

$r=1$

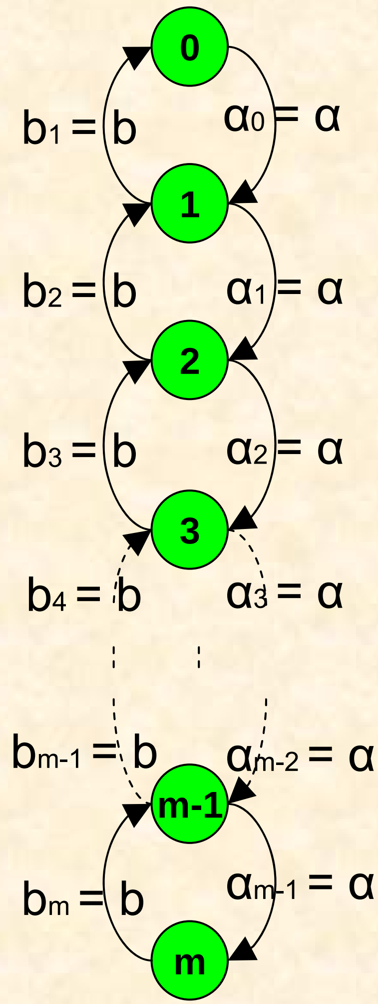
$$A=\begin{pmatrix} -a_0 & a_0 & & & \\ b_1 & -a_1-b_1 & a_1 & & \\ & b_2 & -a_2-b_2 & a_2 & \\ & & & & \ddots \end{pmatrix}=\begin{pmatrix} -a & a & & & \\ b & -a-b & a & & \\ & & \ddots & & \\ & & b & -a-b & a \\ & & & b & -b \end{pmatrix}$$

$$c=\sum_{i=0}^{\infty} v_i=\sum_{i=0}^{m}\frac{a^i}{b^i}=\sum_{i=0}^{m} r^i \overset{r=1}{=} m+1$$

$$\pi(j)=\begin{cases} \dfrac{1}{c}=\dfrac{1}{m+1} & j=0 \\ \dfrac{a_0\cdot a_1\cdots a_{j-1}}{c\cdot b_1\cdot b_2\cdots b_i}=\dfrac{1}{m+1}\dfrac{a^j}{b^j}=\dfrac{1}{m+1} & m\geq j\geq 1 \end{cases},$$

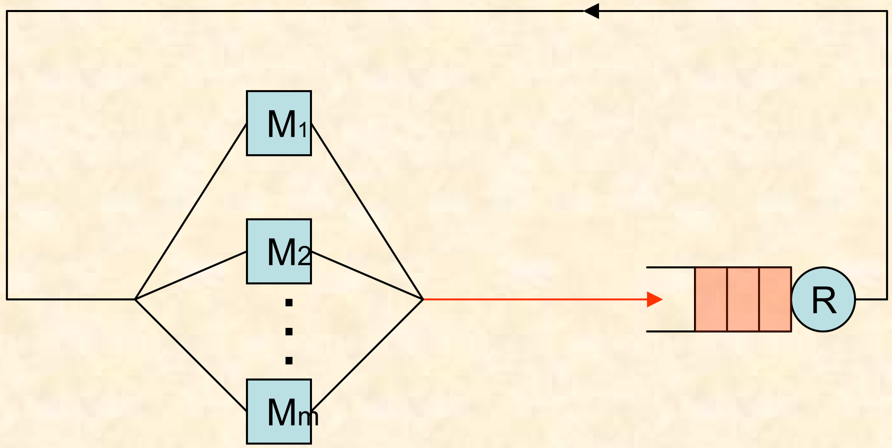
or $\quad \pi(j)=\dfrac{1}{m+1},\ m\geq j\geq 0$

$$E[Y]=\sum_{j=0}^{\infty} j\,\pi_j=\sum_{j=0}^{m} j\frac{1}{m+1}=\frac{1}{m+1}\sum_{j=0}^{m} j=\frac{1}{m+1}\frac{m(m+1)}{2}=\frac{m}{2}$$



Example: Machine repair problem.

- Suppose there are m machines serviced by one repairman.
- Each machine runs without failure, independent of all others, an exponential time with mean $\frac{1}{a}$
- When it fails, it waits until the repairman can come to repair it, and the repair itself takes an exponential distributed amount of time with mean $\frac{1}{b}$.
- Once repaired, the machine is as good as new.



Example: Machine repair problem.

- Let Y_t be the number of failed machines at time t
- If $Y_t = i$, then there are $m-i$ machines working, and the time until the next failure is exponential with parameter $(m-i)a$ (if no machines are repaired in the meantime).

Hence,

- $a_0 = ma, a_1 = (m-1)a, \ldots, a_{m-1} = a; \quad (a_m = a_{m+1} = \cdots = 0)$
- $b_1 = b_2 = \cdots = b$

Limiting distribution:

$$\pi(j) = pm(m-1)\cdots(m-j)\left(\frac{a}{b}\right)^j, \quad j = 0, 1, \ldots, m$$

where $\dfrac{1}{p} = \sum_{i=0}^{m} m(m-1)\cdots(m-i)\left(\dfrac{a}{b}\right)^i$