

Performance Analysis of a Star Topology of Interconnected Networks Under 2nd-Order Markov Network Output Processes

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Abstract—In this paper, the idea of approximating the output process of slotted multiuser random access communication networks (i.e., the process of the successfully transmitted packets within the networks) by a 2nd-order Markov process is introduced. A method is developed to analytically calculate the parameters of the approximating process for a class of random access algorithms. The method is illustrated by considering a specific random access algorithm from that class. The mean time that a packet spends in the central node of a star topology of interconnected networks (a quantity of practical interest) is incorporated in the evaluation of the accuracy of the proposed approximation. This quantity is calculated under the proposed approximation on the output processes of the interconnected networks, and it is compared to simulation results from the actual system. The previous ideas are applied to networks operating under a specific random access algorithm, and results are obtained for that particular case.

I. INTRODUCTION

CONSIDERABLE work has been done toward the direction of developing communication protocols which determine how a single common resource can be efficiently shared by a large population of users. By now, it is well known that fixed assignment techniques are not appropriate for a system with a large population of bursty users. In this case, random access protocols are more efficient, and many such protocols have been suggested [1], [2]. Usually, the amount of information (in bits) transmitted per time is of fixed length, called a packet. In most of the systems, time is divided into slots of length equal to the time needed for a packet transmission (slotted systems).

The deployment of an ever-increasing number of multiuser random access communication networks brings up the question of how packets, whose destination is in another network, should be handled. Thus, the issue of network interconnection or multihop packet transmission arises [3], [6], [7]. The basic problem in analyzing interconnected systems is that of characterizing the output process of a multiuser random access communication system, i.e., the departure process of the successfully transmitted packets. The output process of a multiuser random access communication system depends on the protocol that has been deployed. Description of that process is a difficult task, and only approximations based on special assumptions have been attempted [4]–[7].

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In a previous work [8], the problem of the characterization of the output process is addressed. There, a Bernoulli and a 1st-order Markov approximation are proposed. It turns out that the 1st-order approximation performs better for moderate and heavy traffic load, with respect to a certain measure of performance which is of practical interest. In this paper, a 2nd-order Markov approximation on the output process of multiuser random access communication networks is proposed. The 2nd-order Markov approximation is intuitively more pleasing than those suggested in [8], since it captures better the dependencies in the output process. These dependencies are introduced by the collision resolution algorithm of the random access protocol.

In the next section, the Markov approximation on the output process is introduced. In Section III, a method to analytically calculate the parameters of the Markov model (steady state and transition probabilities) for a class of random access algorithms is developed. True probabilities, as induced by the actual system, are incorporated for this purpose. The class of random access algorithms for which the method is applicable includes all random access algorithms whose analysis is based on the concept of the session (regenerative cycles). The developed method is illustrated in Section IV by considering a specific random access algorithm from that class. In Section V, the mean time that a packet spends in the central node of a star topology of interconnected networks (which is a quantity of practical interest) is incorporated in the evaluation of the accuracy of the proposed approximation. Finally, some results on the accuracy of the proposed approximation, in the case of the specific random access algorithm considered as an example in this paper, are presented in the last section together with a summary of this work.

II. THE APPROXIMATION OF THE OUTPUT PROCESS

In this section, the proposed approximation on the output process is described. A slotted multiuser random access communication network is considered. It is assumed that the packet input rate to the system is λ packets per slot, and that λ is in the stability region of the system. For such values of λ , all the processes associated with the description of the system are stationary. The communication channel can be in one of the following states: I (idle), if no user is using the channel at that time; S (success), if only one user is transmitting; and C (collision), if more than one user is transmitting at that time. In the previous channel description, we made the assumption that if only one user transmits, then his packet is successfully transmitted. Successfully transmitted packets appear in the output process of the network; I and C states of the channel do not result in a packet output from the network. It is assumed that capture events are not present [18], and that channel errors cannot occur.

We define the output process of a slotted multiuser random-access communication system, $\{a^j\}_{j \geq 0}$, to be a discrete-time binary process associated with the end of the slots. The random variable a^j takes the value 1 if a successful packet transmission occurs in the

j 'th slot, and 0 otherwise. It is clear that the output process can be interpreted as the two-state channel-status process $\{x^j\}_{j \geq 0}$, where $x^j \in \{s, u\}$; by s (success) we denote state S , and by u ("un-success") we denote the union of the states I and C . The purpose of this interpretation of the output process is to relate it to the channel status, which is in interaction with the random access protocol. The latter is true since the evolution of the channel-status process (which determines completely the output process) depends on the current (and possibly the past) channel state and the deployed protocol. This occurs because the state of the channel is fed back to the users, who determine their action based on this feedback and the protocol. A type of feedback information is required by any stable random access algorithm.

As a consequence of the above discussion, the output process, $\{a^j\}_{j \geq 0}$, and the two-state channel-status process, $\{x^j\}_{j \geq 0}$, are identical. The problem of characterizing the output process of a multiuser random-access communication network is identical to that of characterizing the two-state channel-status process, $\{x^j\}_{j \geq 0}$. The channel-status process $\{x^j\}_{j \geq 0}$ is controlled by the deployed random-access algorithm. In this paper, we approximate this process by a 2nd-order Markov process, $\{\tilde{x}^j\}_{j \geq 0}$, which has the same state space as $\{x^j\}_{j \geq 0}$ and is ergodic within the stability region of the random access algorithm. To study the process $\{\tilde{x}^j\}_{j \geq 0}$, we can equivalently study the underlying 1st-order Markov process $\{y^j\}_{j \geq 0}$ defined as follows: $\{y^j\}_{j \geq 0}$ is a discrete time 1st-order Markov process associated with the end of the slots. The state space of this process is $Z = \{(s, s) = a, (s, u) = b, (u, s) = c, (u, u) = d\}$; the first part of each pair corresponds to the state of the approximating process $\{\tilde{x}^j\}_{j \geq 0}$ at the end of the $(j-1)$ st slot; the second part corresponds to the state of the same process $\{\tilde{x}^j\}_{j \geq 0}$ at the end of the j 'th slot. Having defined the underlying 1st-order Markov chain $\{y^j\}_{j \geq 0}$, we can obtain the binary process $\{\tilde{a}^j\}_{j \geq 0}$ with state space $\{0, 1\}$ from the stationary function $\tilde{a}: S \rightarrow \{0, 1\}$ with

$$\tilde{a}(y^j) = \begin{cases} 1 & \text{if } y = a \text{ or } c \\ 0 & \text{if } y = b \text{ or } d. \end{cases} \quad (1)$$

The process $\{\tilde{a}^j\}_{j \geq 0}$ approximates the output process of the random access communication network. It depends on the 1st-order Markov process $\{y^j\}_{j \geq 0}$, which describes the 2nd-order process $\{\tilde{x}^j\}_{j \geq 0}$; the latter approximates the channel-status process. To completely determine the underlying 1st-order Markov process $\{y^j\}_{j \geq 0}$, we need to estimate its steady-state and transition probabilities. Then the approximating process $\{\tilde{a}^j\}_{j \geq 0}$ is completely determined by (1).

III. PARAMETERS OF THE MARKOV PROCESS

Since the 1st-order Markov process, $\{y^j\}_{j \geq 0}$ is only an approximation, it seems appropriate to estimate its steady state and transition probabilities, by calculating the steady state probabilities that a particular state or state transition occurs in the true process, under stable network operation. The possible state transitions of the 1st-order Markov chain $\{y^j\}_{j \geq 0}$ are shown in Fig. 1. Notice that not all state transitions are possible. Since $\{y^j\}_{j \geq 0}$ is a Markov chain, the following equation must hold:

$$\pi P = \pi \quad (2)$$

where $\pi = (\pi(a), \pi(b), \pi(c), \pi(d))$ is the vector of the steady-state probabilities and

$$P = \begin{bmatrix} p(a, a) & p(a, b) & p(a, c) & p(a, d) \\ p(b, a) & p(b, b) & p(b, c) & p(b, d) \\ p(c, a) & p(c, b) & p(c, c) & p(c, d) \\ p(d, a) & p(d, b) & p(d, c) & p(d, d) \end{bmatrix}$$

is the matrix with the transition probabilities of the Markov chain $\{y^j\}_{j \geq 0}$. From Fig. 1, it is easily concluded that the following

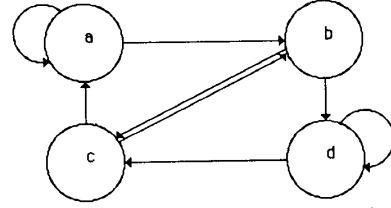


Fig. 1. State transition diagram of the Markov chain $\{y^j\}_{j \geq 0}$.

equations hold:

$$p(a, a) + p(a, b) = 1 \quad p(b, c) + p(b, d) = 1 \quad (3a)$$

$$p(c, a) + p(c, b) = 1 \quad p(d, d) + p(d, c) = 1 \quad (3b)$$

$$p(a, c) = p(a, d) = p(b, a) = p(b, b) = p(c, c) \\ = p(c, d) = p(d, a) = p(d, b) = 0. \quad (4)$$

By assuming that the steady-state probabilities and the transition probabilities $p(b, c)$ and $p(a, a)$ are all known, the remaining transition probabilities can be calculated from (2)–(4). If one of the steady state probabilities, e.g., $\pi(c)$, is known, then the rest of them can be calculated from the following equations which relate marginal with joint probabilities:

$$P(u, u) = P(u) - P(u, s) \rightarrow \pi(d) = 1 - \lambda - \pi(c)$$

$$P(s, s) = P(s) - P(u, s) \rightarrow \pi(a) = \lambda - \pi(c) \quad (5)$$

$$P(s, u) = P(s) - P(s, s) \rightarrow \pi(b) = \pi(c).$$

$P(s)$ denotes the probability that a slot is successful, and $P(u)$ is its complement. Under stability, $P(s) = \lambda$. To summarize, all parameters of the Markov process $\{y^j\}_{j \geq 0}$ can be calculated from the probabilities $\pi(c)$, $p(b, c)$, and $p(a, a)$. Notice that $\pi(c) = P(u, s)$ and that

$$p(b, c) = \frac{P(s, u, s)}{P(s, u)} = \frac{P(s, u, s)}{\pi(b)} \quad (6a)$$

$$p(a, a) = \frac{P(s, s, s)}{P(s, s)} = \frac{P(s, s, s)}{\pi(a)} \quad (6)$$

where $P(s, u, s)$ and $P(s, s, s)$ are the joint probabilities of having an (s, u, s) and an (s, s, s) triplet in the output process, respectively. As a consequence, to completely determine the approximating process, it suffices to calculate the probabilities $P(u, s)$, $P(s, u, s)$, and $P(s, s, s)$. All the quantities and the definitions that are used in the rest of the paper appear in Appendix A.

At this point, we concentrate on the class of the random access algorithms whose stability analysis is based on the concept of the session, i.e., the time interval between two consecutive renewal points of the algorithm. This class is quite broad and contains most of the limited and continuous sensing random access algorithms [11]–[17]. For this class of algorithms, the next theorem provides for a method to calculate the desired joint probabilities.

Theorem: Consider a slotted random access algorithm whose operation induces renewal points. Let a session be defined as the time interval between two consecutive renewal points of the algorithm. The joint probabilities $P(u, s)$, $P(s, u, s)$, and $P(s, s, s)$ are given by the following expressions:

$$\pi(c) = P(s, s) = \frac{T^{u, s}}{L} + \lambda e^{-\lambda} \frac{L^u}{L} \quad (7a)$$

$$P(s, u, s) = \frac{T^{s, u, s}}{L} + \frac{L^{s, u}}{L} \lambda e^{-\lambda} + \frac{L^s}{L} I^{u, s} + \frac{L^u}{L} e^{-\lambda} \lambda e^{-\lambda} \quad (7b)$$

$$P(s, s, s) = \frac{T^{s, s, s}}{L} + \frac{L^{s, s}}{L} + \lambda e^{-\lambda} + \frac{L^s}{L} \lambda e^{-\lambda} \lambda e^{-\lambda} \quad (7c)$$

where all the quantities in the above equations are defined in Appendix A.

Proof: We provide a proof of (7b) only. Equations (7a) and (7c) can be proved by following a similar procedure. Let us give the following definitions.

Type 1 (s, u, s) Triplet: It is an (s, u, s) triplet whose three parts belong to the same session.

Type 2 (s, u, s) Triplet: It is an (s, u, s) triplet whose first two parts belong to one session and the third to another. This triplet is assumed to belong to the first of these sessions.

Type 3 (s, u, s) Triplet: It is an (s, u, s) triplet whose first part belongs to one session and the other two to another. This triplet is assumed to belong to the first of these sessions.

Type 4 (s, u, s) Triplet: It is an (s, u, s) triplet whose three parts belong to different sessions. This triplet is assumed to belong to the first of these sessions.

Let us also define the following quantities:

l_n : the length of the n th session from the time origin (in slots),
 $w_n^{s,u,s}$: the number of (s, u, s) triplets of the n th session,
 $\tau_{i,n}^{s,u,s}$: the number of type i (s, u, s) triplets of the n th session.
 Clearly,

$$w_n^{s,u,s} = \sum_{i=1}^4 \tau_{i,n}^{s,u,s}.$$

The joint probability $P(s, u, s)$ is calculated as the limit of the ratio of the total number of (s, u, s) triplets by the end of the N th session divided by the length of all N sessions, as $N \rightarrow \infty$.

$$P(s, u, s) = \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N w_n^{s,u,s}}{\sum_{n=1}^N l_n}.$$

The above expression can be written as follows:

$$\begin{aligned} P(s, u, s) &= \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N \left\{ \sum_{i=1}^4 \tau_{i,n}^{s,u,s} \right\}}{\sum_{n=1}^N l_n} \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^4 \frac{\frac{1}{N} \sum_{n=1}^N \tau_{i,n}^{s,u,s}}{\frac{1}{N} \sum_{n=1}^N l_n} \\ &= \sum_{i=1}^4 \frac{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \tau_{i,n}^{s,u,s}}{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N l_n}. \end{aligned}$$

Clearly, $\{\tau_{i,n}^{s,u,s}\}_{n \geq 0}$ and $\{l_n\}_{n \geq 0}$ are i.i.d. Since $E\{l_n\} = L < \infty$ and $E\{\tau_{i,n}^{s,u,s}\} = T^{s,u,s} < L < \infty$ within the stability region of the system, the strong law of large numbers asserts that [19]

$$P(s, u, s) = \frac{T^{s,u,s}}{L} + \frac{1}{L} \sum_{i=2}^4 \lim_{N \rightarrow \infty} \frac{1}{N} \left\{ \sum_{n=1}^N \tau_{i,n}^{s,u,s} \right\}. \quad (8)$$

For each i , $i = 2, 3, 4$, the processes $\{\tau_{i,n}^{s,u,s}\}_{n \geq 0}$ are not generally independent, but they are strictly stationary. The expected values of the corresponding random variables are the following ($\tau_{i,n}^{s,u,s}$, $i = 2, 3, 4$, are actually indicator functions; their expectations represent probabilities):

$$\begin{aligned} E\{\tau_{2,n}^{s,u,s}\} &= P\{\text{last two slots of a session are } (s, u), \\ &\quad \text{multiplicity of next session is one}\} \\ &= L^s u \lambda e^{-\lambda} \end{aligned}$$

$$E\{\tau_{3,n}^{s,u,s}\} = P\{\text{last slot of a session is } S, \text{ first two slots of next session are } (u, s)\} = L^s I^{u,s}$$

$$\begin{aligned} E\{\tau_{4,n}^{s,u,s}\} &= P\{\text{last slot of a session is } s, \\ &\quad \text{multiplicity of next session is zero, multiplicity} \\ &\quad \text{of second next session is one}\} = L^s e^{-\lambda} \lambda e^{-\lambda}. \end{aligned}$$

It is easy to observe that the processes $\{\tau_{i,n}^{s,u,s}\}_{n \geq 0}$, $i = 2, 3, 4$, are metrically transitive [20], [21]. From the previous expressions and (8), we obtain (7b) by invoking the ergodic theorem for stationary processes [20], [21]. \square

Notice that the previous theorem holds for any random access algorithm that induces renewal points. For such algorithms, the quantities involved in (7) are meaningful. Equation (7) is useful only if the involved quantities can be computed. A method to compute these quantities is developed for random access algorithms whose stability analysis utilizes the concept of the session. As it is illustrated in the next section, through an example, the quantities of interest can be computed by writing equations similar to those developed for the stability analysis of the algorithm. To write these equations, one should incorporate the steps of the particular algorithm under consideration, and thus a more detailed discussion on the procedure for a general algorithm from this class is not possible beyond this point.

IV. CALCULATION OF THE PARAMETERS FOR A SPECIFIC RANDOM ACCESS ALGORITHM

In this section, we illustrate the procedure that leads to the calculation of the quantities that appear in (7), by applying it to a particular random access algorithm. Although the average session length calculations for the stability analysis have been carried out elsewhere, we reproduce the procedure for two reasons. First, it considerably simplifies the calculations of the quantities of interest. Second, it gives the reader some idea as to how the stability equations are related to those used for the calculations of the desired quantities in this paper. The reader can see how the stability equations are modified to give the desired ones. As a matter of fact, the coefficients of the resulting linear systems of equations do not change at all in the case of the algorithm under consideration. That way, the reader can get better insight of the procedure presented here and, as a consequence, be able to apply it in the case of other algorithms with greater ease.

Consider a multiuser random access slotted communication network in which a binary-feedback (collision/noncollision, C/NC), limited-sensing collision-resolution algorithm is deployed. The network input traffic is assumed to be Poisson with intensity λ packets per slot. This algorithm has been developed and analyzed in [10]–[12]. The characterization of the process of the successfully transmitted packets, i.e., the output process of the network, is an open problem. A brief description of the collision-resolution algorithm is provided at this point. Each user is assigned a counter whose initial value is zero (no packet to be transmitted). This counter is updated according to the steps of the algorithm and the feedback from the channel. Upon packet arrival, the counter value increases to one. Users whose values are equal to one at the beginning of a slot, transmit in that slot. If the channel feedback is collision (C), each counter whose value is greater than one increases it by one; each counter whose value equals one, maintains this value with probability p (splitting probability) or increases it to two with probability $1 - p$. If the channel feedback is noncollision (NC), all nonzero counters decrease their values by one. A detailed description of the algorithm can be found in [10] and [11].

An important quantity for the analysis of random access algorithms, which induce regenerative points, is the session. A session is defined as the time interval between two renewal points in the operation of the system. For the particular algorithm under consideration, the renewal points of interest are determined by an imaginary marker. This marker is essentially an imaginary counter whose

value varies according to the channel feedback. Its value is originally set zero ($b = 0$). If $b = 0$ and the feedback is C , then $b = 2$; if $b = 0$ and the feedback is NC , then $b = 0$. If $b > 0$ and the feedback is C , then $b = b + 1$; if $b > 0$ and the feedback is NC , then $b = b - 1$. The slots in which $b = 0$ are renewal points of the system. Due to the statistical splitting among collided users, the system might be empty and the marker still be positive. This implies that the marker determines only a subsequence of the instants when the system is empty. The session is determined as the time interval between successive renewal points, as determined by the marker. The length of such sessions is easy to describe via recursive equations. The multiplicity of a session is defined as the number of packet transmission attempts in the first slot of the session.

An important quantity for the calculation of the quantities in (7) is the mean session length, L . The latter can be calculated by following procedures similar to those that appear in [11] and [13]–[15]. In fact, for the specific algorithm under consideration, L has been calculated in [10] and [11]. We believe that the recursive equations with respect to l_k , which describe the operation of the system, will be very helpful for the better understanding of the procedure for the calculation of the quantities in (7). For this reason, we start by calculating L . From the description of the algorithm, the following equations can be written, with respect to l_k , $k = 1, 2, \dots$:

$$l_0 = 1, \quad l_1 = 1$$

$$l_k = 1 + l_{\phi_1 + f_1} + l_{k - \phi_1 + f_2}, \quad k \geq 2.$$

f_1 and f_2 come from two independent Poisson random variables over $T = 1$ (length of a slot) with probability function $P_f(\cdot)$ and intensity λ ; ϕ_1 comes from a binomial with parameters k and p ($p = 0.5$) and probability function $b_k(\cdot)$. The second of the above equations can be explained as follows. The length of a session of multiplicity $k \geq 2$ consists of the slot wasted in the collision, plus the length of the subsession of multiplicity $\phi_1 + f_1$ (which will be initiated in the next slot), plus the length of the subsession of multiplicity $k - \phi_1 + f_2$ (which will be initiated after the end of the subsession of multiplicity $\phi_1 + f_1$). Subsessions are statistically identical to the sessions of the same multiplicity. ϕ_1 is the number of users whose counter content remained one after the splitting; f_1, f_2 is the number of new users which will be activated (have a packet for transmission) and enter the system in the first slot of the corresponding subsession.

By considering the expected values in the previous equations with respect to all random variables involved, we obtain

$$L_0 = 1, \quad L_1 = 1 \quad (9a)$$

$$L_k = 1 + \sum_{f_1=0}^{\infty} \sum_{\phi_1=0}^k P_f(F_1 = f_1) b_k(\Phi_1 = \phi_1) L_{\phi_1 + f_1}$$

$$+ \sum_{f_2=0}^{\infty} \sum_{\phi_1=0}^k P_f(F_2 = f_2) b_k(\Phi_1 = \phi_1) L_{k - \phi_1 + f_2}, \quad k \geq 2. \quad (9b)$$

The infinite dimensional linear system of equations in (9) can be written in the general form

$$L_k = h_k + \sum_{j=0}^{\infty} a_{kj} L_j, \quad k \geq 0. \quad (10)$$

The most widely used definition of stability is the one which relates it with the finiteness of L_k , for $k < \infty$. In [10] and [11], it has been found that the system is stable for Poisson input rates $\lambda < S_{\max} = 0.36$ (packets/packet length). The authors in [10] and [11] were actually able to find a (linear) upper bound on L_k which is finite for $k < \infty$. S_{\max} is then defined as the supremum over all rates λ for which such a bound, L_k^u , was possible to obtain.

The existence of $L_k^u < \infty$, for $k < \infty$, implies that (10) has a nonnegative solution, L_k ; the solution \tilde{L}_k of the finite dimensional

system of equations

$$\tilde{L}_k = h_k + \sum_{j=0}^J a_{kj} \tilde{L}_j, \quad 0 \leq k \leq J, \quad (11)$$

is a lower bound on L_k and $\tilde{L}_k \rightarrow L_k$ as $J \rightarrow \infty$ [11], [14], [15].

It turns out that for sufficiently large J (e.g., 15), \tilde{L}_k is extremely close to L_k ; thus, for all practical purposes, L_k is considered to be equal to \tilde{L}_k , especially for λ outside the neighborhood of S_{\max} . The latter can be shown by calculating a tight upper bound on L_k and observing that it almost coincides with \tilde{L}_k (see [11], [14], and [15] for the procedure).

By solving (11) with a_{kj} and h_k , given by,

$$a_{0j} = a_{1j} = 0, \quad 0 \leq j \leq J \quad (12a)$$

$$a_{kj} = \sum_{\phi_1=0}^{\min\{j, k\}} P_f(F_1 = j - \phi_1) b_k(\Phi_1 = \phi_1)$$

$$+ \sum_{\phi_1=\max\{k-j, 0\}}^k P_f(F_1 = j - k + \phi_1) b_k(\Phi_1 = \phi_1),$$

$$0 \leq j \leq J, \quad 0 \leq k \leq J \quad (12b)$$

and

$$h_k = 1, \quad 0 \leq k \leq J, \quad (13)$$

we calculate the mean session length of multiplicity k . Since the multiplicities of successive sessions are independent and identically distributed random variables, the mean session length L is calculated by averaging L_k over all k ; k is the number of arrivals in a slot from a Poisson process with intensity λ . In fact, the average for $k \leq J$ is sufficient for all practical purposes.

To calculate the quantities in (7), we follow a procedure similar to that developed for the calculation of the mean session length L . The recursive equations with respect to the appropriate random variables and the resulting systems of linear equations are shown in Appendix B. By solving truncated versions of these systems of linear equations, we compute the desired quantities. Actually, as it is the case with the calculation of L , what is calculated is a lower bound on the corresponding quantities. As the number of equations considered increases, the bound converges to the true value.

V. PERFORMANCE OF THE APPROXIMATION

In the previous sections, the 2nd-order Markov process, $\{\tilde{x}^j\}_{j \geq 0}$, which approximates the two-state channel-status process, $\{x^j\}_{j \geq 0}$ (or the output process, $\{a^j\}_{j \geq 0}$), was described, and the parameters of the 1st-order equivalent Markov process, $\{y^j\}_{j \geq 0}$, were analytically calculated. In this section, we present a method to evaluate the accuracy of the proposed approximation.

The characterization of the output process of a multiuser random access communication network is extremely useful in analyzing the performance of systems of interconnected multiuser random access communication networks. The delay induced by the interconnecting links/nodes is a good measure of the performance of the interconnection. In a star topology of interconnected networks, the mean time that a packet spends in the central node is an important performance measure of the interconnection; it is thus desired that this quantity be calculated. This is the reason for the incorporation of the previous mean time in the evaluation of the accuracy of the proposed approximation. By comparing this quantity, calculated under the proposed approximation, to the one from the simulation of the actual system, we can estimate the accuracy of the approximation.

A star topology of N interconnected networks is shown in Fig. 2. Each input stream represents the output process from a multiuser random-access slotted communication system. Let λ_i be the output rate (in packets per slot) of the i th network. A packet arrival in the central node is declared at the end of the slot in which the packet was successfully transmitted. Thus, the arrival process associated

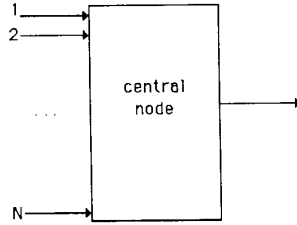


Fig. 2. A star topology of interconnected networks.

with each input line is a discrete process. The arrival points in all streams coincide; that is, the networks are assumed to be synchronized and all slots are of the same length. The service time in the central node is constant and equal to one, which is assumed to be the length of a slot. This implies that arriving and departing packets have the same length. The first in-first out (FIFO) service policy is adopted. More than one arrival (from different input streams) that occurs at the same arrival point is served in a randomly chosen order. The buffer capacity of the central node is assumed to be infinite.

A discrete time, single server queueing system, with a finite number of independent input streams and per stream arrivals governed by an underlying finite-state Markov chain, has been analyzed in [9]. The queueing system that is considered in this section is a special case of the general system in [9]. If λ_i is within the stability region of the corresponding network and if [18]

$$\sum_{i=1}^N \lambda_i < 1,$$

then the queueing system is stable. The average number of packets Q in the central node can be calculated as the sum of the solutions of 4^N linear equations [9]. Then, the mean time that a packet spends in the central node D is given in conjunction with Little's formula by the following expression:

$$D = \frac{Q}{\sum_{i=1}^N \lambda_i}. \quad (14)$$

Note that under stable operation of the networks, the adopted mapping rule in (1) implies that the input rate of the i th stream to the central node is equal to the input rate to the corresponding network.

Let us denote by \bar{x} and \bar{y} the N -dimensional vectors that describe the states of the N Markov chains in two consecutive time slots, $\bar{x}, \bar{y} \in Z = Z^1 \times Z^2 \times \cdots \times Z^N$; $p(\bar{x}, \bar{y})$ denotes the transition probability from state \bar{x} to state \bar{y} . Let $p(j; \bar{y})$ denote the probability that there are j packets in the central node, and that the N -dimensional Markov chain is in state \bar{y} , and let $P(z; \bar{y})$ be the corresponding generating function. Then, the average number of packets in the system Q is given by the sum of the solutions of 4^N linear equations [9]. These equations are given by

$$\sum_{\nu=0}^N \sum_{\bar{x} \in F_\nu} [2(\nu-1)P'(1; \bar{x}) + (\nu-1)(\nu-2)P(1; \bar{x}) + 2(\nu-1)p(0; \bar{x})] = 0 \quad (15a)$$

and any $4^N - 1$ from the following:

$$P'(1; \bar{y}) = \sum_{\nu=0}^N \sum_{\bar{x} \in F_\nu} [(\nu-1)P(1; \bar{x}) + P'(1; \bar{x}) + p(0; \bar{x})] \cdot p(\bar{x}, \bar{y}), \quad \bar{y} \in \bar{S}. \quad (15b)$$

The unknown quantities in (15) are $P'(1; \bar{y})$, $\bar{y} \in \bar{S}$; $P'(1; \bar{y})$ denotes the value of the derivative of $P(z; \bar{y})$ at $z = 1$. The set F_ν

is given by

$$F_\nu = \left\{ \bar{x} = (x_1, \dots, x_N) \in Z : \sum_{i=1}^N a_i(x_i) = \nu \right\},$$

where $a_i(\cdot)$ is the mapping associated with the i th network. Since the input streams to the central node are independent, we have that

$$p(\bar{x}, \bar{y}) = \prod_{i=1}^N p_i(x_i, y_i), \quad P(1; \bar{x}) = \pi(\bar{x}) = \prod_{i=1}^N \pi_i(x_i)$$

and $p(0; \bar{x})$ is estimated from

$$p(0; \bar{x}) = \frac{\sum_{\bar{y} \in F_0} p(\bar{y}/\bar{x}) \pi(\bar{y})}{\sum_{\bar{y} \in F_0} \pi(\bar{y})}$$

where $\pi_i(x_i)$ and $p_i(x_i, y_i)$ are the steady state and state transition probabilities of the Markov chain associated with the i th input stream; p_0 is the probability that the central node is empty. The latter is given by [9]

$$p_0 = 1 - \sum_{i=1}^N \lambda_i.$$

By solving the 4^N linear equations that are given by (15), and summing up the solutions, the average number of packets in the central node Q is obtained. Then, the mean time that a packet spends in the system is calculated from (14).

VI. SUMMARY AND RESULTS

In this paper, we introduced the idea of approximating the output process of multiuser random access communication networks by a 2nd-order Markov process. The motivation behind the proposed approximation is to capture the dependencies in the true output process which are introduced by the random access algorithm. In Section III, a method to analytically calculate the parameters of the approximating process is developed for networks operating under a random access algorithm from a general class. Equation (7) of the theorem that is proved in that section is applicable to any random access algorithm that induces renewal points. The quantities which are involved in (7) can be calculated in the case of random access algorithms whose stability analysis is based on the concept of the session. The procedure to be followed in that case is illustrated through an example in Section IV. In Section V, we present a method to evaluate the accuracy of the proposed approximation. For that purpose, we chose a star topology of interconnected networks, and we incorporated the mean time that a packet spends in the central node in the evaluation of the accuracy of the approximation. By comparing the latter quantity (which is of practical interest), calculated under the proposed approximation on the output process of the interconnected networks, to that from the simulation of the actual system, we estimate the accuracy of the approximation.

As N (the number of the interconnected networks) increases, the dimensionality of the system of linear equations which need to be solved (15) increases rapidly. For large N , simulation results have shown that the Bernoulli approximation on the output process performs well; its performance improves as N increases. The latter can be explained by the fact that the increased number of independent input streams reduces the dependencies in the total input traffic to the central node. The per-network output traffic must also decrease for the queueing system to be stable. The latter implies that either each network operates away from its stability region, and thus the dependencies in its output process are not strong, or that not all of the successful packets are forwarded to the central node; the packet selection introduced in the latter case results in increased independence in the output process.

As an example, the mean time that a packet spends in the central node of the star topology was calculated in the case of the random access algorithm presented in Section IV. The results (in slots) are shown in Table I, together with the results obtained from the

TABLE I

RESULTS FOR THE MEAN PACKET DELAY IN THE CENTRAL NODE OF A STAR TOPOLOGY OF 3 INTERCONNECTED NETWORKS; λ IS THE PER-NETWORK INPUT (OUTPUT) RATE. THE RESULTS ARE OBTAINED UNDER THE MARKOV APPROXIMATION AND FROM THE SIMULATION OF THE ACTUAL SYSTEM

λ	$\pi(c)$	P(a,a)	P(b,c)	Markov	Sim.
.01	.0098	.0136	.0100	1.01	1.00
.10	.0862	.1271	.0999	1.15	1.01
.20	.1488	.2381	.2006	1.58	1.21
.25	.1728	.2896	.2520	2.19	1.70
.30	.1922	.3403	.3050	4.69	4.28
.31	.1955	.3506	.3160	6.49	6.25
.32	.1985	.3609	.3271	10.99	11.37
.33	.2014	.3714	.3385	42.55	48.89

simulation of the actual system. The maximum per-network output rate under stable operation of the particular algorithm is 0.36 packets per slot. On the other hand, the queueing system of the star topology is stable for total input rates less than 1 packet per slot [18]. By comparing the analytical results, obtained under the approximation of the output process by a 2nd-order Markov chain, to the simulations of the actual system, we conclude that the approximation performs well for the whole range of per-network input rates. The proposed approximation seems to perform better than the Bernoulli or the 1st-order Markov approximations discussed in [8], under heavy traffic. Under heavy traffic, the dependencies introduced by the random access algorithms are strong, and it seems that they are best captured by the proposed approximation. Of course, the performance of the proposed approximation depends on the random access algorithm deployed within the network. In this paper, we developed the approximating model and the methods to compute its parameters for a class of random access algorithms, and this is the main contribution of this paper. Results are presented only for a special case, and the conclusions can be extended to other cases only intuitively.

APPENDIX A

In this appendix, we define the quantities and the variables that are used in this paper.

- (*u, s*) pair: a pair of consecutive slots with the first slot being in state *u* and the second in state *s*.
- Internal (*u, s*) pair: an (*u, s*) pair is internal if both slots belong to the same session.
- (*s, u, s*) triplet: a triplet of consecutive slots that are in states *s, u, s*.
- (*s, s, s*) triplet: a triplet of consecutive slots that are in states *s, s, s*.
- Internal (*x, y, z*) triplet: An (*x, y, z*) triplet whose three slots belong to the same session.
- l_k : length of a session of multiplicity *k* (in slots).
- L_k : expected value of l_k .
- L : expected value of L_k with respect to *k*.
- $\tau_k^{u,s}$: number of internal (*u, s*) pairs in a session of multiplicity *k*.
- $T_k^{u,s}$: expected value of $\tau_k^{u,s}$.
- $T^{u,s}$: expected value of $T_k^{u,s}$ with respect to *k*.
- l_k^u : a random variable associated with the last slot of a session of multiplicity *k*; $l_k^u = 1$ if that slot is idle; $l_k^u = 0$ if that slot is involved in a successful transmission.
- L_k^u : expected value of l_k^u .
- L^u : expected value of L_k^u with respect to *k*.

- $\tau_k^{s,u,s}$ ($\tau_k^{s,s,s}$): number of internal (*s, u, s*) triplets (internal (*s, s, s*) triplets) in a session of multiplicity *k*.
- $T_k^{s,u,s}$ ($T_k^{s,s,s}$): expected value of $\tau_k^{s,u,s}$ ($\tau_k^{s,s,s}$).
- $T^{s,u,s}$ ($T^{s,s,s}$): expected value of $T_k^{s,u,s}$ ($T_k^{s,s,s}$).
- $l_k^{s,u}$ ($l_k^{s,s}$): a random variable associated with the last two slots of a session of multiplicity *k*; $l_k^{s,u} = 1$ if the last pair of slots of that session is (*s, u*); $l_k^{s,u} = 0$ otherwise ($l_k^{s,s} = 1$ if the last pair of slots of that session is (*s, s*); $l_k^{s,s} = 0$ otherwise).
- l_k^s : a random variable associated with the last slot of a session; $l_k^s = 1$ if the last slot of that session is *s*; $l_k^s = 0$ otherwise.
- $i^{u,s}$: a random variable associated with the first two slots of a session of multiplicity *k*; $i^{u,s} = 1$ if the first pair of slots of that session is (*u, s*); $i^{u,s} = 0$ otherwise.
- $L_k^{s,u}$ ($L_k^s, I_k^{u,s}, L_k^{s,s}$): expected value of $l_k^{s,u}$ ($l_k^s, i_k^{u,s}, l_k^{s,s}$).
- $L^{s,u}$ ($L^s, I^{u,s}, L^{s,s}$): expected value of $L_k^{s,u}$ ($L_k^s, I_k^{u,s}, L_k^{s,s}$).

APPENDIX B

In this appendix, we present the recursive equations and the resulting infinite-dimensional linear systems of equations with respect to the quantities of interest, defined over a session of given multiplicity. By solving a truncated version of these infinite-dimensional systems, we calculate the desired quantities. Then the corresponding mean value over any session is computed in a similar way to *L*.

A.1) Calculation of L_k^s

$$l_0^s = 0, \quad l_1^s = 1$$

$$l_k^s = l_{k-\phi_1+\phi_2}^s, \quad k \geq 2.$$

The resulting system of linear equations is the following:

$$L_0^s = 0, \quad L_1^s = 1$$

$$L_k^s = \sum_{f_2=0}^{\infty} \sum_{\phi_1=0}^{\infty} P_f(F_2 = f_2) b_k(\Phi_1 = \phi_1) L_{k-\phi_1+f_2}^s, \quad k \geq 2.$$

A.2) Calculation of L_k^u

L_k^u is the expected number of last slots of a session of multiplicity *k* which are in state *u*. L_k^u can also be described as the probability that the last slot of a session is in state *u*. As a result, we have the following equation for L_k^u .

$$L_k^u = 1 - L_k^s \quad k \geq 0.$$

A.3) Calculation of $L_k^{s,u}$

$$l_0^{s,u} = 0, \quad l_1^{s,u} = 0$$

$$l_k^{s,u} = l_{k-\phi_1+f_2}^{s,u} 1_{\{k-\phi_1+f_2 \geq 2\}} + 1_{\{\phi_1+f_1=1, k-\phi_1+f_2=0\}}, \quad k \geq 2.$$

The resulting system of linear equations is the following:

$$L_0^{s,u} = 0, \quad L_1^{s,u} = 0$$

$$L_k^{s,u} = \sum_{f_2=0}^{\infty} \sum_{\phi_1=0}^k P_f(F_2 = f_2) b_k(\Phi_1 = \phi_1) L_{k-\phi_1+f_2}^{s,u} \\ + \sum_{f_1=0}^{\infty} b_k(\Phi_1 = k) P_f(F_2 = 0) P_f(F_1 = f_1) L_{k+f_1}^{s,u}, \quad k \geq 2.$$

A.4) Calculation of $L_k^{s,s}$

$$I_0^{s,s} = 0, \quad I_1^{s,s} = 0 \\ I_k^{s,s} = I_{k-\phi_1+f_2}^{s,s} 1_{\{k-\phi_1+f_2 \geq 2\}} + 1_{\{\phi_1+f_1-1, k-\phi_1+f_2=1\}}, \quad k \geq 2.$$

The resulting system of linear equations is the following:

$$L_0^{s,s} = 0, \quad L_1^{s,s} = 0 \\ L_k^{s,s} = \sum_{\phi_1=0}^k \sum_{f_2=0}^{\infty} P_f(F_2 = f_2) b_k(\Phi_1 = \phi_1) L_{k-\phi_1+f_2}^{s,s} \\ + b_k(\Phi_1 = k) P_f(F_2 = 1) \sum_{f_1=0}^{\infty} P_f(F_1 = f_1) \\ + b_k(\Phi_1 = k-1) P_f(F_2 = 0) \sum_{f_1=0}^{\infty} P_f(F_1 = f_1) L_{k-1+f_1}^{s,s}.$$

A.5) Calculation of $I_k^{u,s}$

$$i_0^{u,s} = 0, \quad i_1^{u,s} = 0 \\ i_k^{u,s} = 1_{\{f_1+\phi_1=0\}}.$$

The resulting system of linear equations is the following:

$$I_0^{u,s} = 0, \quad I_1^{u,s} = 0 \\ I_k^{u,s} = P_f(F_1 = 0) b_k(\Phi_1 = 1) + P_f(F_1 = 1) b_k(\Phi_1 = 0), \quad k \geq 2.$$

A.6) Calculation of $T^{u,s}$

$$\tau_0^{u,s} = 0, \quad \tau_1^{u,s} = 0 \\ \tau_k^{u,s} = \tau_{\phi_1+f_1}^{u,s} + \tau_{k-\phi_1+f_2}^{u,s} + 1_{\{\phi_1+f_1=1, k-\phi_1+f_2=1\}} \\ + 1_{\{\phi_1+f_1=1\}}, \quad k \geq 2.$$

The truncated version of the resulting system of linear equations is of the form of (8) with the coefficients a_{kj} as in (9) and constants h_k given by

$$h_0^{u,s} = 0, \quad h_1^{u,s} = 0 \\ h_k^{u,s} = P_1(F_2 = 1) b_k(\Phi_1 = k) \sum_{f_1=0}^{J-k} P_f(F_1 = f_1) L_{k+f_1}^{u,s} \\ + P_f(F_2 = 0) b_k(\Phi_1 = k-1) \\ \cdot \sum_{f_1=0}^{J+1-k} P_f(F_1 = f_1) L_{k-1+f_1}^{u,s} \\ + P_f(F_1 = 0) b_k(\Phi_1 = 1) + P_f(f_1 = 1) b_k(\Phi_1 = 0), \\ 2 \leq k \leq J.$$

A.7) Calculation of $T^{s,u,s}$

$$\tau_0^{s,u,s} = 0, \quad \tau_1^{s,u,s} = 0 \\ \tau_k^{s,u,s} = \tau_{f_1+\phi_1}^{s,u,s} + \tau_{k-\phi_1+f_2}^{s,u,s} + 1_{\{f_1+\phi_1=1, k-\phi_1+f_2=1\}} \\ + 1_{\{f_1+\phi_1=1, i_{k-\phi_1+f_2}^{s,u,s}=1\}}.$$

The resulting system of linear equations is of the form of (7) with

coefficients a_{kj} and constants h_k given by

$$h_0^{s,u,s} = 0, \quad h_1^{s,u,s} = 0 \\ h_k^{s,u,s} = b_k(\Phi_1 = k) P_f(F_2 = 1) \sum_{f_1=0}^{\infty} P_f(F_1 = f_1) L_{f_1+k}^{s,u,s} \\ + b_k(\Phi_1 = k-1) P_f(F_2 = 0) \\ \cdot \sum_{f_1=0}^{\infty} P_f(F_1 = f_1) L_{f_1+k-1}^{s,u,s} \\ + \sum_{f_2=0}^{\infty} \sum_{f_1=0}^{\infty} \sum_{\phi_1=0}^k P_f(F_1 = f_1) \\ \cdot b_k(\Phi_1 = \phi_1) P_f(F_2 = f_2) \\ \cdot L_{f_1+\phi_1}^{s,u,s} L_{k-\phi_1+f_2}^{s,u,s}, \quad 2 \leq k \leq J.$$

A.8) Calculation of $T^{s,s,s}$

$$\tau_0^{s,s,s} = 0, \quad \tau_1^{s,s,s} = 0 \\ \tau_k^{s,s,s} = \tau_{f_1+\phi_1}^{s,s,s} + \tau_{k-\phi_1+f_2}^{s,s,s} + 1_{\{f_1+\phi_1=1, k-\phi_1+f_2=1\}}.$$

The resulting system of linear equations is of the form of (7) with coefficients a_{kj} and constants h_k given by

$$h_0^{s,s,s} = 0, \quad h_1^{s,s,s} = 0 \\ h_k^{s,s,s} = b_k(\Phi_1 = k) P_f(F_2 = 1) \sum_{f_1=0}^{\infty} P_f(F_1 = f_1) L_{f_1+k}^{s,s,s} \\ + b_k(\Phi_1 = k-1) P_f(F_2 = 0) \\ \cdot \sum_{f_1=0}^{\infty} P_f(F_1 = f_1) L_{f_1+k-1}^{s,s,s}, \quad 2 \leq k \leq J.$$

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- Demetrios Kazakos** (S'69-M'73-SM'87) for a photograph and biography, see the February 1990 issue of this TRANSACTIONS, p. 178.