Mean Delay Analysis of a Statistical Multiplexer with Batch Arrival Processes - A Generalization to Viterbi's Formula *

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Abstract

The delay analysis of a statistical multiplexer with heterogeneous input sources is presented in this work. The traffic generated by an individual source is described in terms of a discrete-time Batch Markovian arrival process with geometrically distributed idle periods. An exact closed form expression for the mean packet delay in the system is obtained by exploiting the stochastic structure of the packet arrival processes. The derived expression is a generalization of the formula developed originally by Viterbi.

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1 Introduction

The delay analysis of a statistical multiplexer with heterogeneous input sources is considered in this paper. The performance analysis of statistical multiplexers has been the focus of numerous analytical studies, both in continuous and discrete time. One of the early studies on the delay performance of a discrete-time multiplexer with correlated input sources is presented in [1]. An individual source delivers a single packet each time the underlying two-state (ON-OFF) Markov chain visits the active (ON) state. The major contribution of this work by Viterbi is the derivation of a simple explicit formula which demonstrates the dependence of the mean delay on the correlation of the sources.

A generalization of Viterbi's model is developed in [2], where source activity periods are assumed to have a phase type distribution. The number of phases determines the cardinality of the state space of the underlying Markov chain and can be different for different sources. In this case, the packet arrival process delivers a single packet every time the underlying Markov chain visits an active state. The analysis is based on matrix analytic techniques and is a refinement of the approach developed in [1].

The expression for the mean delay obtained in [2] can be derived as a special case of the model analyzed in [3], where source activity periods are assumed to have a general distribution. The analysis is based on the theory of alternating renewal processes and applies to both continuous and discrete time input sources. A random function defined over active periods determines the source data rate.

In this paper, a generalization of the model studied originally by Viterbi and later by Neuts is presented. Individual input sources are assumed to deliver batches of packets—instead of single packets—during periods of activity. The resulting arrival process belongs to the class of discrete-time Batch Markovian arrival processes (D-BMAPs), defined formally in the next section. It has geometrically distributed idle periods and active periods distributed according to a discrete phase type distribution. Matrix analytic techniques which exploit the stochastic structure of the arrival processes are applied for the derivation of the system mean queue occupancy. The derived expression can also be obtained by applying the result derived in [3] for the discrete-time case. However, this is not trivial, due to the complexity associated with the derivation of the quantities required for the application of the general formula derived in [3] to the case of D-BMAPs considered in this paper.

The analysis of queueing systems which provide service to a large number of sources that generate correlated traffic—typically represented by complex, nonrenewal processes—can be facilitated through the approximate modeling of the aggregate traffic by accurate, analytically tractable processes. For instance, the Markov Modulated Poisson Process (MMPP) has been used to approximate the superposition of packetized voice processes in continuous time, [4]. In discrete time, the superposition of packet arrival processes is modeled by processes which are capable of delivering multiple packets per time slot, [5]. The D-BMAP has been used to model the superposition of video sources in [6] and to approximate the superposition of data, voice and video traffic in [7].

The model for the packet arrival processes and their superposition is described in Section 2. The mean delay analysis of the system, presented in Section 3, is based on the matrix analytic technique for Markov chains of the M/G/1 type and the Perron-Frobenius theory for nonnegative matrices.

2 The Queueing System

Consider a discrete-time statistical multiplexer of infinite queueing capacity which serves the traffic delivered by N independent, heterogeneous input sources (Fig. 1). The arrival processes are described in Section 2.2; they belong to the class of Discrete-time Batch Markovian arrival processes (D-BMAPs), [6]-[13], which are described next.

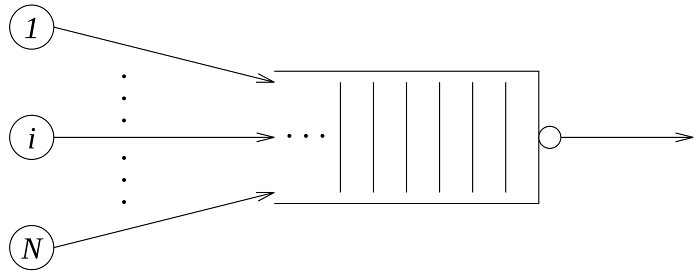

Figure 1: The queueing system.

2.1 Description of the Discrete-time Batch Markovian Arrival Process (D-BMAP)

The D-BMAP is described in terms of a modulating Markov chain and a packet arrival distribution as follows: Let $\{s_j\}_{j\geq 0}$ denote a finite Markov chain with state space $S=\{0,1,\ldots,M\}$ and transition probability matrix $\tilde{P}$. The number of packets delivered by the D-BMAP at the j-th time instant, denoted by a_j, depends on the state of $\{s_j\}_{j\geq 0}$ both at time instants j and $j+1$ and is determined by the probability distribution

$$f_{mn}(k)=\Pr\{a_j=k/s_j=m,s_{j+1}=n\},\quad m,n\in S,\quad k\geq 0,\quad \forall j\geq 0. \tag{1}$$

Equivalently, the D-BMAP may be defined as the two-dimensional Markov chain $\{L_j,s_j\}_{j\geq 1}$, where $\{L_j\}_{j\geq 1}$ is the counting process of the number of packet arrivals up to (not including) time instant j. Markov chain $\{L_j,s_j\}_{j\geq 1}$ has state space $\{0,1,2,\ldots\}\times S$ and transition probability matrix

$$\tilde{R}=\begin{bmatrix} \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_4 & \cdots \\ \tilde{0} & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \cdots \\ \tilde{0} & \tilde{0} & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \cdots \\ \tilde{0} & \tilde{0} & \tilde{0} & \tilde{A}_0 & \tilde{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{2}$$

where matrices $\tilde{A}_k$, $k\geq 0$, govern transitions accompanied by the arrival of a batch of size k. Their elements $[A_k]_{mn}$, $m,n\in S$, are given by

$$\begin{aligned} [A_k]_{mn} &= \Pr\{a_j=k,s_{j+1}=n/s_j=m\} \\ &= \Pr\{a_j=k/s_{j+1}=n,s_j=m\}\Pr\{s_{j+1}=n/s_j=m\} \\ &= f_{mn}(k)p_{mn},\quad m,n\in S,\quad k\geq 0,\quad \forall j\geq 0, \end{aligned} \tag{3}$$

where p_{mn} denotes the transition probability from state m to state n of $\{s_j\}_{j\geq 0}$. By the definition of matrices $\tilde{A}_k$, $k \geq 0$, it follows that

$$\sum_{k=0}^{\infty} \tilde{A}_k = \tilde{P}. \tag{4}$$

An alternative representation for the D-BMAP which will be used in the analysis is provided through the matrix generating function defined by

$$\tilde{A}(z) = \sum_{k=0}^{\infty} \tilde{A}_k z^k. \tag{5}$$

The elements of $\tilde{A}(z)$ are given by

$$\left[\tilde{A}(z)\right]_{mn} = \sum_{k=0}^{\infty} f_{mn}(k) z^k p_{mn} = F_{mn}(z) p_{mn}, \quad m, n \in S, \tag{6}$$

where $F_{mn}(z)$ is the probability generating function of the distribution $f_{mn}(k)$, $k \geq 0$. Clearly, $\tilde{A}(z) \mid_{z=1} = \tilde{P}$.

2.2 The arrival process

The model for the input processes considered in this work is a special case of the D-BMAP described above. It is assumed that there is a *single* state—without loss of generality, state zero—in which the arrival process may not deliver any packets. A number of packets greater than or equal to one is delivered when the modulating chain is in any state m other than zero,

$$f_{mn}(0) = 0 \qquad m \in S \setminus \{0\}, \quad n \in S. \tag{7}$$

It should be emphasized that assumption (7) is necessary for the applicability of the developed analysis, as it will become clear in the next section. In the following, it will be assumed that the arrival process delivers *no packets* in state zero. That is, for $n \in S$,

$$f_{0n}(0) = 1, \tag{8a}$$
$$f_{0n}(k) = 0, \quad k \geq 1. \tag{8b}$$

Note that, as a result of conditions (7), (8), only the first row of matrix $\tilde{A}_0$, denoted by vector $\bar{b}$, has nonzero elements. Furthermore, (8b) implies that the first row of matrices $\tilde{A}_k$, $k \geq 1$, consists of zero elements. Consequently, the first row of matrix $\tilde{A}_0$, $\bar{b}$, is equal to the first row of matrix $\tilde{A}(1) = \tilde{P}$. The presented analysis can be applied in a straightforward manner to the study of the general case where the number of packets delivered when $\{s_j\}_{j\geq 0}$ is in state zero is determined according to some general distributions $f_{0n}(k)$, $k \geq 0$, $n \in S$.

In view of the above, the state space S of $\{s_j\}_{j\geq 0}$ may be partitioned into the sets of active $S^\alpha = \{1, 2, \ldots, M\}$ and idle states $S^0 = \{0\}$. The matrix generating function $\tilde{A}(z)$ may then be written as

$$\tilde{A}(z) = \begin{bmatrix} 1 - \phi & \phi\bar{\alpha} \\ \bar{\Phi}_0(z) & \tilde{\Phi}(z) \end{bmatrix}, \tag{9}$$

where

ϕ : probability that $\{s_j\}_{j\geq 0}$ makes a transition from state 0 to some active state n, $n \in S^\alpha$, i.e., ϕ is equal to the probability that $\{s_j\}_{j\geq 0}$ enters an active period. Clearly, ϕ^{-1} is equal to the average sojourn time in state zero (average duration of the geometrically distributed idle period).

$\bar{\alpha}$: M-dimensional probability vector, that is, $\bar{\alpha}\bar{e} = 1$, where $\bar{e}$ denotes a unit column vector of proper dimensionality. The n-th element of vector $\bar{\alpha}$, $n \in S^\alpha$, is equal to the probability that $\{s_j\}_{j\geq 0}$ visits state n at the beginning of an active period.

$\tilde{\Phi}(z)$: M-dimensional matrix; matrix $\tilde{\Phi}(1) \stackrel{\triangle}{=} \tilde{\Phi}$ contains the transition probabilities associated with transitions of $\{s_j\}_{j\geq 0}$ within an active period.

$\bar{\Phi}_0(z)$: M-dimensional column vector, equal to $[F_{10}(z)p_{10}, F_{20}(z)p_{20}, \ldots, F_{M0}(z)p_{M0}]^T$. Clearly, $\bar{\Phi}_0(1) \stackrel{\triangle}{=} \bar{\Phi}_0 = (\tilde{I} - \tilde{\Phi})\bar{e}$.

It can be easily seen that the duration of an active period has a phase type distribution, [2],

$$\tau_k = \bar{\alpha}\tilde{\Phi}^{k-1}\bar{\Phi}_0, \quad k \geq 1. \tag{10}$$

It follows that the mean active period, μ, is equal to

$$\mu = \bar{\alpha}(\tilde{I} - \tilde{\Phi})^{-1}\bar{e}. \tag{11}$$

Let $\bar{\pi} = [\pi_0, \pi_1, \ldots, \pi_M]$ denote the stationary probability vector of $\{s_j\}_{j\geq 0}$. It is easily established that the stationary probability that the modulating chain is in state zero is given by

$$\pi_0 = \frac{\phi^{-1}}{\phi^{-1} + \mu} = \frac{1}{1 + \phi\mu}. \tag{12}$$

The stationary probability that the chain is in an active state, π_α, is then determined by

$$\pi_\alpha = \bar{\pi}_\alpha\bar{e} = [\pi_1, \pi_2, \ldots, \pi_M]\bar{e} = 1 - \pi_0 = \frac{\phi\mu}{1 + \phi\mu}. \tag{13}$$

Vector $\bar{\pi}_\alpha$ can be expressed in terms of π_α and the normalized vector $\bar{\pi}_{norm} = \mu^{-1}\bar{\alpha}(\tilde{I} - \tilde{\Phi})^{-1}$, $\bar{\pi}_{norm}\bar{e} = 1$,

$$\bar{\pi}_\alpha = \pi_\alpha\bar{\pi}_{norm} = \frac{\phi\mu}{1 + \phi\mu}\bar{\pi}_{norm} = \frac{\phi}{1 + \phi\mu}\bar{\alpha}(\tilde{I} - \tilde{\Phi})^{-1}. \tag{14}$$

2.3 The cumulative arrival process

The cumulative arrival process and its representation in terms of the parameters of the individual arrival processes is discussed in this section. It can be easily shown that the superposition of independent D-BMAPs may also be represented by a D-BMAP. Let $\{L_j^{(i)}, s_j^{(i)}\}_{j\geq 1}$, $1 \leq i \leq N$, denote the i-th packet arrival process; the cumulative packet arrival process is completely described by the Markov chain $\{L_j, s_j\}_{j\geq 1}$, where $L_j = \sum_{i=1}^{N} L_j^{(i)}$ and $s_j \equiv \{s_j^{(1)}, s_j^{(2)}, \ldots, s_j^{(N)}\}$,

$j \geq 1$. The notation introduced in the previous two subsections, will henceforth be adopted for the representation of the quantities associated with the *cumulative packet arrival process* $\{L_j, s_j\}_{j\geq 1}$. Unless otherwise noted, a *superscript* (i) will be used to indicate the respective quantity associated with the i-th arrival process, $\{L_j^{(i)}, s_j^{(i)}\}_{j\geq 1}$.

Let $\tilde{P}^{(i)}$ and $\bar{\pi}^{(i)}$ denote the transition probability matrix and the stationary probability vector, respectively, of the modulating chain $\{s_j^{(i)}\}_{j\geq 0}$ with state space $S^{(i)} = \{0, 1, \ldots, M^{(i)}\}$. Markov chain $\{s_j\}_{j\geq 0}$ of the cumulative process is defined on state space $S = S^{(1)} \times \cdots \times S^{(N)}$ with cardinality $M + 1 = \prod_{i=1}^{N}(M^{(i)} + 1)$. Under lexicographic ordering of the states, the transition probability matrix of $\{s_j\}_{j\geq 0}$ is given by the N-fold Kronecker product

$$\tilde{P} = \bigotimes_{i=1}^{N} \tilde{P}^{(i)}. \tag{15}$$

Similarly, for the stationary probability vector $\bar{\pi}$

$$\bar{\pi} = \bigotimes_{i=1}^{N} \bar{\pi}^{(i)} \tag{16}$$

and the matrix generating function $\tilde{A}(z)$

$$\tilde{A}(z) = \bigotimes_{i=1}^{N} \tilde{A}^{(i)}(z), \tag{17}$$

which provides a complete description of the cumulative arrival process.

For the special case of D-BMAPs considered in Section 2.2, assumptions (7)-(8) hold true for processes $\{L_j^{(i)}, s_j^{(i)}\}_{j\geq 1}$ as well as for process $\{L_j, s_j\}_{j\geq 1}$ and the structure of $\tilde{A}^{(i)}(z)$ is preserved in $\tilde{A}(z)$. That is, the first row of matrix $\tilde{A}_0 = \tilde{A}(1)$, is the only row of $\tilde{A}_0$ with nonzero elements and is determined by

$$\bar{b} = [1 - \phi\ ,\ \phi\bar{\alpha}] = \bigotimes_{i=1}^{N} \bar{b}^{(i)}, \tag{18}$$

where

$$1 - \phi = \prod_{i=1}^{N}(1 - \phi^{(i)}).$$

3 Queueing Analysis

The derivation of the explicit formula for the system mean queue occupancy is presented in this section.

Let q_j, $j \geq 0$, denote the queue occupancy at the end of the j-th slot. The slot is equal to the packet transmission time and will be considered as the system time constant. All events of interest—packet arrivals and departures—will be defined at the end of slots; time instant j is defined at the end of the j-th slot. Process $\{q_j\}_{j\geq 0}$ evolves according to the state equation

$$q_{j+1} = \max\{0, q_j - 1\} + a_j, \quad j \geq 0,$$

where a_j is the number of packets delivered by the cumulative arrival process at the end of the j-th slot. The system state is described by the two-dimensional Markov chain $\{q_j, s_j\}_{j\geq 0}$ with transition probability matrix of the M/G/1 type

$$\tilde{Q} = \begin{bmatrix} \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_4 & \cdots \\ \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_4 & \cdots \\ \tilde{0} & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \cdots \\ \tilde{0} & \tilde{0} & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\tilde{A}_k$, $k \geq 0$, are $M+1$-dimensional matrices, associated with the cumulative arrival process.

The stationary probability distribution of $\{q_j, s_j\}_{j\geq 0}$,

$$x_{km} = \lim_{j\to\infty} \Pr\{q_j = k, s_j = m\} \quad k \geq 0, \quad m \in S,$$

can be derived by applying the general theory of Markov chains of the M/G/1 type, [14]. $\{q_j\}_{j\geq 0}$ and $\{s_j\}_{j\geq 0}$ are identified as the level and phase process, respectively.

The matrix analytic approach involves the computation of the $(M+1)$-dimensional matrix $\tilde{G}$, the solution to the nonlinear matrix equation $\tilde{G} = \sum_{k=0}^{\infty} \tilde{A}_k \tilde{G}^k$, through an iterative convergence scheme. The numerical complexity increases with the state space of the phase process and, thus, the applicability of the method is limited to systems of relatively small dimensionality. It is possible though, under certain conditions regarding the stochastic structure of the cumulative arrival process, to derive an explicit closed form expression for the mean queue occupancy E[q] of the system as will be shown in the following.

Let $\bar{x}_k$ denote the stationary probability vector associated with level k,

$$\bar{x}_k = [x_{k0}, x_{k1}, \ldots, x_{kM}].$$

By definition, vectors $\bar{x}_k$, $k \geq 0$, satisfy the equation

$$[\bar{x}_0 \; \bar{x}_1 \; \bar{x}_2 \; \cdots] = [\bar{x}_0 \; \bar{x}_1 \; \bar{x}_2 \; \cdots]\tilde{Q}\,.$$

Consequently,

$$\bar{x}_k = \bar{x}_0 \tilde{A}_k + \sum_{\ell=1}^{k+1} \bar{x}_\ell \tilde{A}_{k+1-\ell}, \quad k \geq 0. \tag{19}$$

It follows easily from (19) that the vector generating function $\bar{X}(z) = \sum_{k=0}^{\infty} \bar{x}_k z^k$ satisfies the equation

$$\bar{X}(z)[z\tilde{I} - \tilde{A}(z)] = (z-1)\bar{x}_0 \tilde{A}(z). \tag{20}$$

The mean queue occupancy can be derived from (20) and explicitly evaluated provided that the boundary probability vector $\bar{x}_0$ be known. In general, the computation of $\bar{x}_0$ requires that matrix $\tilde{G}$ be evaluated. However, due to the stochastic structure of the M/G/1 system under study, vector $\bar{x}_0$ can be obtained explicitly without pursuing the computationally intense standard methodology. Specifically, as was first noted by Viterbi, [15], equation

$$\bar{x}_0 = \bar{x}_0 \tilde{A}_0 + \bar{x}_1 \tilde{A}_0$$

yields that vector $\bar{x}_0$ is proportional to the nonzero row of matrix $\tilde{A}_0$. Thus, $\bar{x}_0 = c\bar{b}$, where c is a constant of proportionality and $\bar{b}$ is the probability vector given by (18), $\bar{b}\bar{e} = 1$.

By setting $z = 1$ in (20) it can be easily obtained that $\bar{X}(1) = \bar{\pi}$. The same equation, upon differentiation, yields the probability that the queue is empty, $\bar{x}_0\bar{e}$, $\bar{x}_0\bar{e} = 1 - \bar{\pi}\tilde{A}'(1)\bar{e} = 1 - \rho = c$, where ρ is the system utilization (cumulative packet arrival rate), [14].

An expression for the mean queue occupancy E[q] can be derived based on the general methodology developed in [14]

$$\mathrm{E}[\mathrm{q}] = \frac{1}{2(1-\rho)} \left\{ 2\bar{x}_0 \tilde{A}'(1)\bar{e} + 2\left[\bar{x}_0 \tilde{A}(1) - \bar{\pi} + \bar{\pi}\tilde{A}'(1)\right] \left(\tilde{I} - \tilde{A}(1) + \bar{e}\bar{\pi}\right)^{-1} \tilde{A}'(1)\bar{e} + \bar{\pi}\tilde{A}''(1)\bar{e} \right\}. \tag{21}$$

This approach requires the inversion of the $(M+1)$-dimensional matrix $\tilde{I} - \tilde{A}(1) + \bar{e}\bar{\pi}$ and does not illustrate the relationship between E[q] and the characteristics of the individual packet arrival processes.

An alternative formula for E[q] may be obtained by exploiting the Kronecker product structure of $\tilde{A}(z)$, [14], and by applying the Perron-Frobenius theory for nonnegative matrices. Let $\delta(z)$ denote the Perron-Frobenius eigenvalue of the nonnegative matrix $\tilde{A}(z)$, $0 < z \leq 1$,

$$\det\left[\delta(z)\tilde{I} - \tilde{A}(z)\right] = 0.$$

The left and right eigenvectors of $\tilde{A}(z)$, denoted by vector $\bar{u}(z)$ and column vector $\bar{v}(z)$, respectively,

$$\bar{u}(z)\tilde{A}(z) = \delta(z)\bar{u}(z), \tag{22a}$$
$$\tilde{A}(z)\bar{v}(z) = \delta(z)\bar{v}(z), \tag{22b}$$

may be chosen to satisfy the following normalizing equations:

$$\bar{u}(z)\bar{v}(z) = 1,$$
$$\bar{u}(z)\bar{e} = 1,$$
$$\bar{u}(1) = \bar{\pi},$$
$$\bar{v}(1) = \bar{e}.$$

Vectors $\bar{u}(z)$ and $\bar{v}(z)$ are uniquely determined by the above equations, [14].

Postmultiplication of Eq. (20) by $\bar{v}(z)$ and the definition of the right eigenvector of $\tilde{A}(z)$ (Eq. (22b)) yield

$$(z - \delta(z))\bar{X}(z)\bar{v}(z) = (1-\rho)(z-1)\delta(z)\bar{b}\bar{v}(z). \tag{23}$$

The following relationship is obtained from (23) by differentiating twice and setting $z = 1$

$$2(1-\rho)(\mathrm{E}[\mathrm{q}]-\rho) = \delta''(1) + 2(1-\rho)\left[\bar{b}\bar{v}'(1) - \bar{\pi}\bar{v}'(1)\right]. \tag{24}$$

Due to the structure of $\tilde{A}(z)$, $\delta(z)$ and $\bar{v}(z)$ can be expressed in terms of $\delta^{(i)}(z)$ and $\bar{v}^{(i)}(z)$, $1 \le i \le N$, the respective quantities associated with the matrix generating functions of the individual arrival processes $\tilde{A}^{(i)}(z)$:

$$\begin{aligned}\delta(z) &= \prod_{i=1}^{N} \delta^{(i)}(z), &(25)\\ \bar{v}(z) &= \bigotimes_{i=1}^{N} \bar{v}^{(i)}(z). &(26)\end{aligned}$$

An expression for $\delta''(1)$ is obtained from (25) by differentiation and by applying the Perron eigenvalue property, $\delta^{(i)}(1) = 1$, $1 \le i \le N$,

$$\delta''(1) = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}[\delta^{(i)}(1)]'[\delta^{(j)}(1)]' + \sum_{i=1}^{N}[\delta^{(i)}(1)]''. \tag{27}$$

Application of the Kronecker product properties yields

$$\bar{b}\bar{v}'(1) = \sum_{i=1}^{N}\bar{b}^{(i)}[\bar{v}^{(i)}(1)]'. \tag{28}$$

By differentiating the normalizing conditions $\bar{u}(z)\bar{v}(z) = \bar{u}(z)\bar{e} = 1$, it can be shown that

$$\bar{\pi}\bar{v}'(1) = 0. \tag{29}$$

Based on Eq. (27)-(29), Eq. (24) becomes

$$2(1-\rho)(\mathrm{E}[\mathrm{q}]-\rho) = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}[\delta^{(i)}(1)]'[\delta^{(j)}(1)]' + \sum_{i=1}^{N}[\delta^{(i)}(1)]'' + 2(1-\rho)\sum_{i=1}^{N}\bar{b}^{(i)}[\bar{v}^{(i)}(1)]'. \tag{30}$$

The closed form expression for E[q] is obtained following the derivations of $[\delta^{(i)}(1)]'$ (A.10), $[\delta^{(i)}(1)]''$ (A.11) and $\bar{b}^{(i)}[\bar{v}^{(i)}(1)]'$ (B.14) which are presented in Appendices A and B:

$$\begin{aligned}[\delta^{(i)}(1)]' &= \bar{\pi}_{\alpha}^{(i)}[\bar{Y}^{(i)}(1)]' = \rho^{(i)}, &(31)\\ [\delta^{(i)}(1)]'' &= \bar{\pi}_{\alpha}^{(i)}\Big\{2[\rho^{(i)}]^2(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}\bar{e} - 2\rho^{(i)}[\tilde{\Phi}^{(i)}(1)]'(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}\bar{e} + [\bar{Y}^{(i)}(1)]'' \\ &\quad -2\rho^{(i)}(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}[\bar{Y}^{(i)}(1)]' + 2[\tilde{\Phi}^{(i)}(1)]'(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}[\bar{Y}^{(i)}(1)]'\Big\}, &(32)\\ \bar{b}^{(i)}[\bar{v}^{(i)}(1)]' &= \rho^{(i)} + \bar{\pi}_{\alpha}^{(i)}(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}\left\{\rho^{(i)}\bar{e} - [\bar{Y}^{(i)}(1)]'\right\}, &(33)\end{aligned}$$

where $\bar{Y}^{(i)}(z) \triangleq \tilde{\Phi}^{(i)}(z)\bar{e} + \bar{\Phi}_0^{(i)}(z)$. Note that the n-th element of vector $\bar{Y}^{(i)}(z)$, $1 \le n \le M^{(i)}$, is equal to the generating function of the number of packets delivered in a slot by the i-th arrival process given that the modulating chain $\{s_j^{(i)}\}_{j\ge 0}$ is in state n.

The mean delay of a packet in the system can be readily obtained by application of Little's law

$$\mathrm{E[D]} = \frac{\mathrm{E[q]}}{\rho}.$$

After several algebraic manipulations, the following formula is derived

$$\begin{aligned}
2(1-\rho)(\mathrm{E[q]}-\rho) &= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \rho^{(i)}\rho^{(j)}\bar{\pi}_{\alpha}^{(i)}\bar{w}^{(i)} \\
&+ 2\sum_{i=1}^{N}\bar{\pi}_{\alpha}^{(i)}\left\{[\tilde{\Phi}^{(i)}(1)]' - \tilde{I}\right\}(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}\left\{[\bar{Y}^{(i)}(1)]' - \rho^{(i)}\bar{e}\right\} \\
&+ \sum_{i=1}^{N}\bar{\pi}_{\alpha}^{(i)}\left\{[\bar{Y}^{(i)}(1)]'' + 2[\bar{Y}^{(i)}(1)]' - 2\rho^{(i)}[\bar{Y}^{(i)}(1)]'\right\},
\end{aligned} \tag{34}$$

where

$$\bar{w}^{(i)} = (\tilde{I}-\tilde{\Phi}^{(i)})^{-1}\left\{[\rho^{(i)}]^{-1}(\tilde{I}+\tilde{\Phi}^{(i)})[\bar{Y}^{(i)}(1)]' - 2\bar{e}\right\}. \tag{35}$$

Finally, Eq. (34) may be rewritten in the form

$$\begin{aligned}
\mathrm{E[q]} &= \rho + \frac{1}{1-\rho}\sum_{i=1}^{N}\sum_{j>i}^{N}\rho^{(i)}\rho^{(j)}\left\{1+\frac{\gamma^{(i)}}{1-\gamma^{(i)}}+\frac{\gamma^{(j)}}{1-\gamma^{(j)}}\right\} \\
&+ \frac{1}{(1-\rho)}\sum_{i=1}^{N}\bar{\pi}_{\alpha}^{(i)}\left\{[\tilde{\Phi}^{(i)}(1)]' - \tilde{I}\right\}(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}\left\{[\bar{Y}^{(i)}(1)]' - \rho^{(i)}\bar{e}\right\} \\
&+ \frac{1}{2(1-\rho)}\sum_{i=1}^{N}\bar{\pi}_{\alpha}^{(i)}\left\{[\bar{Y}^{(i)}(1)]'' + 2[\bar{Y}^{(i)}(1)]' - 2\rho^{(i)}[\bar{Y}^{(i)}(1)]'\right\},
\end{aligned} \tag{36}$$

where

$$\gamma^{(i)} = 1 - \frac{1}{\bar{\pi}_{\alpha}^{(i)}(\tilde{I}-\tilde{\Phi}^{(i)})^{-1}\left([\rho^{(i)}]^{-1}[\bar{Y}^{(i)}(1)]' - \bar{e}\right)}. \tag{37}$$

Eq. (36) yields the mean queue occupancy, E[q], in terms of the parameters of the individual arrival processes, unlike Eq. (21) which involves the parameters of the cumulative arrival process. The dimensionality of the vectors and matrices in (36) is at most as large as the greatest value of $M^{(i)}$, $1 \leq i \leq N$. On the other hand, the matrices and vectors involved in (21) are $(M+1)$-dimensional, where $M+1=\prod_{i=1}^{N}(M^{(i)}+1)$. Thus, the derived expression not only illustrates the effect of the statistical characteristics of the distinct processes on E[q], but it is also computationally less complex.

Special Cases

a. If the number of packets delivered per slot depends on the state of the modulating chain in the current slot only, then $\tilde{\Phi}^{(i)}(z)$ and $\bar{\Phi}_0^{(i)}(z)$ can be expressed by the matrix products

$$\begin{aligned}
\tilde{\Phi}^{(i)}(z) &= \tilde{F}^{(i)}(z)\tilde{\Phi}^{(i)}(1) = \tilde{F}^{(i)}(z)\tilde{\Phi}^{(i)}, \quad 1 \leq i \leq N, \\
\bar{\Phi}_0^{(i)}(z) &= \tilde{F}^{(i)}(z)\bar{\Phi}_0^{(i)}(1) = \tilde{F}^{(i)}(z)\bar{\Phi}_0^{(i)}, \quad 1 \leq i \leq N,
\end{aligned}$$

where $\tilde{F}^{(i)}(z)$ is the diagonal matrix

$$\tilde{F}^{(i)}(z) = \text{diag}\,[F_1^{(i)}(z), F_2^{(i)}(z), \ldots, F_{M^{(i)}}^{(i)}(z)] \tag{38}$$

and $F_n^{(i)}(z)$ is the generating function of the number of packet arrivals given that $\{s_j^{(i)}\}_{j\geq 0}$ is in state n.

In this case Eq. (36) becomes

$$\begin{aligned} \text{E[q]} &= \rho + \frac{1}{1-\rho}\sum_{i=1}^{N}\sum_{j>i}^{N}\rho^{(i)}\rho^{(j)}\left\{1 + \frac{\gamma^{(i)}}{1-\gamma^{(i)}} + \frac{\gamma^{(j)}}{1-\gamma^{(j)}}\right\} \\ &+ \frac{1}{(1-\rho)}\sum_{i=1}^{N}\bar{\pi}_\alpha^{(i)}\left\{[\tilde{F}^{(i)}(1)]' - \tilde{I}\right\}(\tilde{I} - \tilde{\Phi}^{(i)})^{-1}\left\{[\tilde{F}^{(i)}(1)]' - \rho^{(i)}\tilde{I}\right\}\bar{e} \\ &+ \frac{1}{2(1-\rho)}\sum_{i=1}^{N}\bar{\pi}_\alpha^{(i)}\left\{[\tilde{F}^{(i)}(1)]'' + 2[\tilde{F}^{(i)}(1)]' - 2[\tilde{F}^{(i)}(1)]'[\tilde{F}^{(i)}(1)]'\right\}\bar{e} \end{aligned} \tag{39}$$

with

$$\gamma^{(i)} = 1 - \frac{1}{\bar{\pi}_\alpha^{(i)}(\tilde{I} - \tilde{\Phi}^{(i)})^{-1}\left([\rho^{(i)}]^{-1}[\tilde{F}^{(i)}(1)]' - \tilde{I}\right)\bar{e}}. \tag{40}$$

b. If the distribution of the number of packet arrivals is independent of the active state n, $n \in S^\alpha$, that is, if

$$F_1^{(i)}(z) = \cdots = F_n^{(i)}(z) = \cdots = F_{M^{(i)}}^{(i)}(z) = F^{(i)}(z), \quad 1 \leq i \leq N, \tag{41}$$

Eq. (39) yields

$$\begin{aligned} \text{E[q]} &= \rho + \frac{1}{1-\rho}\sum_{i=1}^{N}\sum_{j>i}^{N}\rho^{(i)}\rho^{(j)}\left\{1 + \frac{\gamma^{(i)}}{1-\gamma^{(i)}} + \frac{\gamma^{(j)}}{1-\gamma^{(j)}}\right\} \\ &+ \frac{1}{(1-\rho)}\sum_{i=1}^{N}\left\{[F^{(i)}(1)]' - 1\right\}\left\{[F^{(i)}(1)]' - \rho^{(i)}\right\}\bar{\pi}_\alpha^{(i)}(\tilde{I} - \tilde{\Phi}^{(i)})^{-1}\bar{e} \\ &+ \frac{1}{2(1-\rho)}\sum_{i=1}^{N}\pi_\alpha^{(i)}\left\{[F^{(i)}(1)]'' + 2[F^{(i)}(1)]' - 2[F^{(i)}(1)]'[F^{(i)}(1)]'\right\}, \end{aligned} \tag{42}$$

where

$$\gamma^{(i)} = 1 - \frac{2}{(1-\pi_\alpha^{(i)})\left(\frac{\mu_2^{(i)}}{\mu^{(i)}} + 1\right)} \tag{43}$$

and $\mu_2^{(i)}$ is the second moment of the active period duration

$$\mu_2^{(i)} = \mu^{(i)}\left\{2\bar{\pi}_{norm}^{(i)}(\tilde{I} - \tilde{\Phi}^{(i)})^{-1}\bar{e} - 1\right\}. \tag{44}$$

c. If every active state delivers a single packet, that is, if

$$F^{(i)}(z) = z, \quad 1 \leq i \leq N,$$

Eq. (42) reduces to

$$\mathrm{E[q]} \;=\; \rho + \frac{1}{1-\rho}\sum_{i=1}^{N}\sum_{j>i}^{N}\rho^{(i)}\rho^{(j)}\left\{1+\frac{\gamma^{(i)}}{1-\gamma^{(i)}}+\frac{\gamma^{(j)}}{1-\gamma^{(j)}}\right\}, \tag{45}$$

where $\gamma^{(i)}$ is given by the expression in (43).

This case has been studied by Neuts, [2], in an elegant generalization of the formula derived by Viterbi [1] for 2-state Markov chains ($M^{(1)} = M^{(2)} = \cdots = M^{(i)} = 2$). In the model studied by Viterbi matrix $\tilde{\Phi}^{(i)}$ reduces to a scalar $\Phi^{(i)}$ equal to the transition probability $p_{11}^{(i)}$. The mean queue occupancy is given by (45) where $\gamma^{(i)} = \Phi^{(i)} - \phi^{(i)} = p_{11}^{(i)} - p_{01}^{(i)}$.

Conclusions

An explicit formula for the mean packet delay at a discrete-time statistical multiplexer, which generalizes previous work by Viterbi and Neuts, has been derived. The packet arrival processes belong to the class of discrete-time Batch Markovian arrival processes. Unlike the arrival process models adopted in [1, 2], multiple packets may be delivered in each active state of the modulating Markov chain. Consequently, the developed analysis allows for the study of statistical multiplexing under the more general class of discrete-time Batch Markovian arrival processes which can be used to model the superposition of packet arrival streams.

APPENDICES

A Derivation of the Perron-Frobenius eigenvalue $\delta^{(i)}(z)$ and the derivatives $[\delta^{(i)}(1)]'$, $[\delta^{(i)}(1)]''$, $1 \le i \le N$.

The Perron-Frobenius eigenvalue of $\tilde{A}^{(i)}(z)$, $1 \le i \le N$, and its two first derivatives are derived in this appendix. In the following, superscript (i) will be dropped for notational convenience; all the quantities involved in the calculations are associated with an individual packet arrival process.

Recall that by definition of the left eigenvector $\bar{u}(z)$ of $\tilde{A}(z)$,

$$\bar{u}(z)\tilde{A}(z) = \delta(z)\bar{u}(z). \tag{A.1}$$

In view of the partitioning of state space S, $\bar{u}(z)$ may be written as

$$\bar{u}(z) = [u_0(z)\,,\; \bar{u}_\alpha(z)], \tag{A.2}$$

where $\bar{u}_\alpha(z)$ is the M-dimensional vector $\bar{u}_\alpha(z) = [u_1(z), u_2(z), \cdots, u_M(z)]$. Similarly for the right eigenvector $\bar{v}(z)$:

$$\bar{v}(z) = [v_0(z)\,,\; \bar{v}_\alpha^T(z)]^T, \qquad \bar{v}_\alpha(z) = [v_1(z), v_2(z), \ldots, v_M(z)]^T. \tag{A.3}$$

Eq. (A.1), by (A.2), results in

$$u_0(z)(1-\phi) + \bar{u}_\alpha(z)\bar{\Phi}_0(z) \;=\; \delta(z)u_0(z), \tag{A.4}$$

$$u_0(z)\phi\bar{\alpha} + \bar{u}_\alpha(z)\tilde{\Phi}(z) \;=\; \delta(z)\bar{u}_\alpha(z). \tag{A.5}$$

From (A.5), $\bar{u}_\alpha(z)$ can be expressed in terms of $u_0(z)$

$$\begin{aligned}\bar{u}_\alpha(z) &= u_0(z)\phi\bar{\alpha}\left(\delta(z)\tilde{I}-\tilde{\Phi}(z)\right)^{-1} \\ &= u_0(z)\phi\bar{\alpha}\tilde{W}^{-1}(z), \end{aligned} \tag{A.6}$$

where $\tilde{W}(z) \triangleq \delta(z)\tilde{I}-\tilde{\Phi}(z)$. By substituting $\bar{u}_\alpha(z)$ in (A.4) by the above expression, an implicit relationship is derived for $\delta(z)$

$$\delta(z) = 1-\phi+\phi\bar{\alpha}\tilde{W}^{-1}(z)\bar{\Phi}_0(z) \tag{A.7}$$

which upon differentiation yields

$$\delta'(z) = \phi\bar{\alpha}\left\{-\tilde{W}^{-1}(z)\tilde{W}'(z)\tilde{W}^{-1}(z)\bar{\Phi}_0(z)+\tilde{W}^{-1}(z)[\bar{\Phi}_0(z)]'\right\}. \tag{A.8}$$

Note that $\tilde{W}(1) = \tilde{I}-\tilde{\Phi}$ since by properties of the Perron eigenvalue, $\delta(1)=1$. Thus,

$$\begin{aligned}\delta'(1) &= \phi\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}\left\{-\tilde{W}'(1)(\tilde{I}-\tilde{\Phi})^{-1}\bar{\Phi}_0+\bar{\Phi}_0'(1)\right\} \\ &= \phi\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}\left\{-\tilde{W}'(1)\bar{e}+\bar{\Phi}_0'(1)\right\} \\ &= \phi\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}\left\{-\delta'(1)\bar{e}+\tilde{\Phi}'(1)\bar{e}+\bar{\Phi}_0'(1)\right\} \\ &= -\phi\mu\delta'(1)+\phi\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}\bar{Y}'(1), \end{aligned} \tag{A.9}$$

where $\bar{Y}'(1) \triangleq \tilde{\Phi}'(1)\bar{e}+\bar{\Phi}_0'(1)$. Note the n-th element of vector $\bar{Y}'(1)$, $1 \le n \le M$, is equal to the conditional average number of packets generated in a slot given that the modulating chain is in state n.

Eq. (A.9) yields the following explicit expression for $\delta'(1)$

$$\delta'(1) = \frac{\phi}{1+\phi\mu}\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}\bar{Y}'(1) = \bar{\pi}_\alpha\bar{Y}'(1). \tag{A.10}$$

Clearly, $\delta'(1)$ is equal to the packet arrival rate, $\rho = \delta'(1)$.

The second derivative of the Perron eigenvalue at $z=1$, $\delta''(1)$, is obtained from (A.8) by differentiation

$$\begin{aligned}\delta''(1) = \bar{\pi}_\alpha\Big\{&2[\delta'(1)]^2(\tilde{I}-\tilde{\Phi})^{-1}\bar{e}-2\delta'(1)\tilde{\Phi}'(1)(\tilde{I}-\tilde{\Phi})^{-1}\bar{e} \\ &-2\delta'(1)(\tilde{I}-\tilde{\Phi})^{-1}\bar{Y}'(1)+2\tilde{\Phi}'(1)(\tilde{I}-\tilde{\Phi})^{-1}\bar{Y}'(1)+\bar{Y}''(1)\Big\}, \end{aligned} \tag{A.11}$$

where $\bar{Y}''(1) \triangleq \tilde{\Phi}''(1)\bar{e}+[\bar{\Phi}_0(1)]''$.

B Derivation of the left and right eigenvectors of $\tilde{A}^{(i)}(z)$ and their derivatives $[\bar{u}^{(i)}(1)]'$, $[\bar{v}^{(i)}(1)]'$, $1 \le i \le N$.

Recall that $\bar{u}_\alpha(z) = u_0(z)\phi\bar{\alpha}\tilde{W}^{-1}(z)$ (A.6). Substitution for $\bar{u}_\alpha(z)$ in the normalizing equation $\bar{u}(z)\bar{e} = u_0(z)+\bar{u}_\alpha(z)\bar{e} = 1$ yields

$$u_0(z) = \frac{1}{1+\phi\bar{\alpha}\tilde{W}^{-1}(z)\bar{e}}. \tag{B.1}$$

Then,

$$\bar{u}_\alpha(z) = \frac{\phi\bar{\alpha}\tilde{W}^{-1}(z)}{1+\phi\bar{\alpha}\tilde{W}^{-1}(z)\bar{e}}. \tag{B.2}$$

As expected,

$$u_0(1) = \frac{1}{1+\phi\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}\bar{e}} = \frac{1}{1+\phi\mu} = \pi_0 \tag{B.3}$$

and

$$\bar{u}_\alpha(1) = \frac{\phi\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}}{1+\phi\bar{\alpha}(\tilde{I}-\tilde{\Phi})^{-1}\bar{e}} = \bar{\pi}_\alpha. \tag{B.4}$$

The first derivative of $u_0(z)$ evaluated at $z = 1$ is obtained from (B.1) by differentiating and setting $z = 1$:

$$u_0'(1) = \frac{1}{1+\phi\mu}\bar{\pi}_\alpha\tilde{W}'(1)(\tilde{I}-\tilde{\Phi})^{-1}\bar{e}. \tag{B.5}$$

In the same way, $\bar{u}_\alpha'(1)$ is derived from (A.6)

$$\begin{aligned}
\bar{u}_\alpha'(1) &= (1+\phi\mu)u_0'(1)\bar{\pi}_\alpha - \bar{\pi}_\alpha\tilde{W}'(1)(\tilde{I}-\tilde{\Phi})^{-1}, \\
\bar{u}_\alpha'(1)\bar{e} &= \phi\mu u_0'(1) - \bar{\pi}_\alpha\tilde{W}'(1)(\tilde{I}-\tilde{\Phi})^{-1}\bar{e} \\
&= \phi\mu u_0'(1) - (1+\phi\mu)u_0'(1) \\
&= -u_0'(1).
\end{aligned} \tag{B.6}$$

From the defining equation of the right eigenvector (Eq. (22b)) it can be easily shown that

$$v_0(z)\bar{\Phi}_0(z) + \tilde{\Phi}(z)\bar{v}_\alpha(z) = \delta(z)\bar{v}_\alpha(z), \tag{B.7}$$

and thus,

$$\bar{v}_\alpha(z) = v_0(z)\tilde{W}^{-1}(z)\bar{\Phi}_0(z) \tag{B.8}$$

which upon differentiation and by applying the relation $v_0(1) = 1$ yields

$$\begin{aligned}
\bar{v}_\alpha'(1) &= v_0'(1)\bar{e} - (\tilde{I}-\tilde{\Phi})^{-1}\left[\tilde{W}'(1)\bar{e} - \bar{\Phi}_0'(1)\right] \\
&= v_0'(1)\bar{e} - (\tilde{I}-\tilde{\Phi})^{-1}\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right].
\end{aligned} \tag{B.9}$$

By differentiating the normalizing condition $\bar{u}(z)\bar{v}(z) = u_0(z)v_0(z) + \bar{u}_\alpha(z)\bar{v}_\alpha(z) = 1$ and noting that $\bar{v}(1) = \bar{e}$, $v_0'(1)$ is expressed in terms of $u_0'(1)$, $\bar{u}_\alpha'(1)$ and $\bar{v}_\alpha'(1)$:

$$\begin{aligned}
u_0'(1)v_0(1) + u_0(1)v_0'(1) + \bar{u}_\alpha'(1)\bar{v}_\alpha(1) + \bar{u}_\alpha(1)\bar{v}_\alpha'(1) &= 0 \\
u_0'(1) + \pi_0 v_0'(1) + \bar{u}_\alpha'(1)\bar{e} + \bar{\pi}_\alpha\bar{v}_\alpha'(1) &= 0.
\end{aligned} \tag{B.10}$$

By substituting for $\bar{v}'_\alpha(1)$ from (B.9) in Eq. (B.10),

$$(\pi_0 + \bar{\pi}_\alpha \bar{e}) v'_0(1) = -u'_0(1) - \bar{u}'_\alpha(1)\bar{e} + \bar{\pi}_\alpha(\tilde{I} - \tilde{\Phi})^{-1}\left[\tilde{W}'(1)\bar{e} - \bar{\Phi}'_0(1)\right]. \tag{B.11}$$

By using (B.6), Eq. (B.11) yields

$$v'_0(1) = \bar{\pi}_\alpha(\tilde{I} - \tilde{\Phi})^{-1}\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right]. \tag{B.12}$$

Therefore, it follows from (B.9) that

$$\bar{v}'_\alpha(1) = \bar{\pi}_\alpha(\tilde{I} - \tilde{\Phi})^{-1}\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right]\bar{e} - (\tilde{I} - \tilde{\Phi})^{-1}\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right]. \tag{B.13}$$

The inner product $\bar{b}\bar{v}'(1)$ will be used for the derivation of the mean queue occupancy E[q],

$$\begin{aligned}
\bar{b}\bar{v}'(1) &= (1-\phi)v'_0(1) + \phi\bar{\alpha}\bar{v}'_\alpha(1) \\
&= (1-\phi)v'_0(1) + \phi\bar{\alpha}v'_0(1)\bar{e} - \phi\bar{\alpha}(\tilde{I} - \tilde{\Phi})^{-1}\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right] \\
&= v'_0(1) - (1+\phi\mu)\bar{\pi}_\alpha\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right] \\
&= \bar{\pi}_\alpha(\tilde{I} - \tilde{\Phi})^{-1}\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right] - \phi\mu\delta'(1) + (1+\phi\mu)\delta'(1) \\
&= \delta'(1) + \bar{\pi}_\alpha(\tilde{I} - \tilde{\Phi})^{-1}\left[\delta'(1)\bar{e} - \bar{Y}'(1)\right].
\end{aligned} \tag{B.14}$$

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