Cryptographic Boolean Functions with Maximum Algebraic Immunity

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Talk Outline

1. Introduction
   - Problem Statement
   - Definitions
   - Previous work

2. New constructions of functions with maximum AI
   - Annihilators as codewords of punctured RM codes
   - Secondary constructions
     - Application to the Carlet–Feng construction
     - Behavior w.r.t other cryptographic criteria

3. Conclusions
Stream ciphers

Simplest Case: **Binary additive stream cipher**

![Diagram of a binary additive stream cipher]

- Suitable in environments characterized by a limited computing power or memory, and the need to encrypt at high speed.
- The seed of the keystream generators constitutes the secret key.
- Security depends on
  - Pseudorandomness of the keystream $k_i$.
  - Properties of the underlying functions (mainly Boolean functions) that form the keystream generator.
Problem Statement

Cryptographic criteria

- Several criteria to assess the resistance against attacks
  - balancedness
  - algebraic degree
  - correlation immunity
  - nonlinearity
- Much research effort has been put during last decades on achieving these properties

Cryptanalytic Advances

- Many cryptographic functions failed to thwart more recent attacks
  - (fast) algebraic attacks (Courtois-Meier, 2003)
- Design of functions being tolerant against these attacks, achieving all main cryptographic criteria, is still an active research area
A **Boolean function** \( f \) on \( n \) variables is a mapping from \( \mathbb{F}_2^n \) onto \( \mathbb{F}_2 \)

- The vector \( f = (f(0, 0, \ldots, 0), f(1, 0, \ldots, 0), \ldots, f(1, 1, \ldots, 1)) \) of length \( 2^n \) is the **truth table** of \( f \)
- The **Hamming weight** of \( f \) is denoted by \( \text{wt}(f) \)
  - \( f \) is **balanced** if and only if \( \text{wt}(f) = 2^{n-1} \)
- The **support** \( \text{supp}(f) \) of \( f \) is the set \( \{ b \in \mathbb{F}_2^n : f(b) = 1 \} \)

**Example:** Truth table of balanced \( f \) with \( n = 3 \)

| \( x_1 \) | 0 1 0 1 0 1 0 1 |
| \( x_2 \) | 0 0 1 1 0 0 1 1 |
| \( x_3 \) | 0 0 0 0 1 1 1 1 |
| \( f(x_1, x_2, x_3) \) | 0 1 0 0 0 1 1 1 |
Algebraic Normal Form and degree of functions

- **Algebraic Normal Form (ANF) of** $f$:

  $$f(x) = \sum_{v \in \mathbb{F}_2^n} a_v x^v,$$
  where $x^v = \prod_{i=1}^n x_i^{v_i}$.

  - The sum is performed over $\mathbb{F}_2$ (XOR addition)
  - The **degree** $\text{deg}(f)$ of $f$ is the highest number of variables that appear in a product term in its ANF.
  - In the previous example: $f(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_1$.
    - $\text{deg}(f) = 2$
  - If $\text{deg}(f) \leq r$, then $f$ is a codeword of the $r$th order binary Reed–Muller code $\text{RM}(r, n)$
  - The **punctured Reed–Muller** code $\text{RM}^*(r, n)$ is known to be cyclic having as zeros the elements $\alpha^t$, for all nonzero $t \in \mathbb{Z}_N$ satisfying $\text{wt}(t) < n - r$. 

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Univariate representation of Boolean functions

- $\mathbb{F}_2^n$ is isomorphic to the finite field $\mathbb{F}_{2^n}$,

- $\Rightarrow$ Any function $f \in \mathcal{B}_n$ can also be represented by a univariate polynomial, mapping $\mathbb{F}_{2^n}$ onto $\mathbb{F}_2$, as follows

$$f(x) = \sum_{i=0}^{2^n-1} \beta_i x^i$$

where $\beta_0, \beta_{2^n-1} \in \mathbb{F}_2$ and $\beta_{2^i} = \beta_i^2 \in \mathbb{F}_{2^n}$ for $1 \leq i \leq 2^n - 2$

- The coefficients of the polynomial are associated with the Discrete Fourier Transform (DFT) of $f$

- The degree of $f$ can be directly deduced by the univariate representation - i.e. by the DFT of $f$

- The univariate representation is more convenient in several cases
Milestones

- Algebraic attacks (Courtois-Meier, 2003)
- Fast algebraic attacks (Courtois, 2003)
- The basic idea is to reduce the degree of the mathematical equations employing the secret key
- Known cryptographic Boolean functions failed to thwart these attacks
- The notion of **algebraic immunity** has been introduced (Meier-Pasalic-Carlet, 2004), to assess the strength of a function against such attacks
**Annihilators and algebraic immunity**

**Definition**

Given $f \in \mathbb{B}_n$, we say that $g \in \mathbb{B}_n$ is an annihilator of $f$ if and only if $g$ lies in the set

$$\mathcal{AN}(f) = \{ g \in \mathbb{B}_n : f \ast g = 0 \}$$

**Definition**

The algebraic immunity $\text{Al}(f)$ of $f \in \mathbb{B}_n$ is defined by

$$\text{Al}(f) = \min_{g \neq 0} \{ \text{deg}(g) : g \in \mathcal{AN}(f) \cup \mathcal{AN}(f + 1) \}$$

- A high algebraic immunity is prerequisite for preventing algebraic attacks (**Meier-Pasalic-Carlet, 2004**)
- Well-known upper bound: $\text{Al}(f) \leq \lceil \frac{n}{2} \rceil$
Extensions of the conventional algebraic attacks

Aiming at identifying \( g, h \in \mathbb{B}_n \), for a given function \( f \in \mathbb{B}_n \), such that \( fg = h \) with \( \deg(g) = e < \text{AI}(f) \), \( \deg(h) = d \) and \( e + d < n \)

- A pair \((e, d)\) with \( e + d \geq n \) always exists

We say that \( f \) admits a \((e, d)\) pair if there exist functions \( g, h \) with the aforementioned properties.

Functions that have no \((e, d)\) pair such that \( e + d < n \) are called perfect algebraic immune

Maximum AI does not imply resistance to fast algebraic attacks

- A perfect algebraic immune function though has always maximum AI (Pasalic, 2008)
Constructions of functions with maximum AI

- **Dalai-Maitra-Sarkar, 2006**: Majority function
- **Carlet-Dalai-Gupta-Maitra-Sarkar, 2006**: Iterative construction
- **Li-Qi, 2006, Su-Tang-Zeng, 2014**: Modification of the majority function
- **Sarkar-Maitra, 2007**: Rotation Symmetric Boolean functions (RSBF) of odd $n$
  - **Su-Tang, 2014**: RSBF for arbitrary $n$
- **Carlet, 2008**: Based on properties of affine subspaces
  - Further investigation in **Carlet-Zeng-Li-Hu, 2009**
  - Generalization (for odd $n$) in **Limniotis-Kolokotronis-Kalouptsidis, 2011**
- Balanceness and/or high nonlinearity are not always attainable, whereas they do not behave well w.r.t. fast algebraic attacks
The Carlet-Feng (CF) construction

- **Carlet-Feng, 2008**: \( \text{supp}(f) = \{1, \alpha, \alpha^2, \ldots, \alpha^{2^n-1}-1\} \), where \( \alpha \) a primitive element of the finite field \( \mathbb{F}_{2^n} \).
  - Degree \( n - 1 \) (i.e. the maximum possible)
  - High nonlinearity is ensured
    - Best currently known lower bound (Tang et. al., 2013)
    \[
    \text{nl}(f) \geq 2^n - 1 - \left( \frac{n \ln(2)}{\pi} + 0.74 \right)\left(2^{n/2} - 1\right)
    \]
  - Experiments show that the actual values of nonlinearities are much higher
  - Optimal against fast algebraic attacks, as subsequently shown (Liu-Zhang-Lin, 2012)
- Other important constructions have been also recently proved (e.g. Tang-Carlet-Tang, 2013, Li-Carlet-Zeng-Li-Hu-Shan, 2014)
Generalizations of Carlet-Feng construction

- **Rizomiliotis, 2010**: A new construction based on the univariate representation
  - Associate the AI with the rank of a well-determined matrix
  - For $n$ odd, equivalent to the CF construction

- **Zeng-Carlet-Shan-Hu, 2011**: Modifications of the Rizomiliotis construction

- **Further generalizations in Limniotis-Kolokotronis-Kalouptsidis, 2013**:  
  - Finding swaps between $\text{supp}(f)$ and $\text{supp}(f + 1)$ that preserve maximum AI
  - $\Rightarrow$ Algorithm singleswap($f$ **for** $n$ **odd**)
  - Why restricted to odd $n$?
    - If $n$ is odd, then $f \in \mathbb{B}_n$ has maximum algebraic immunity $\frac{n+1}{2}$ if and only if $f$ is balanced and has no nonzero annihilators of degree at most $\frac{n-1}{2}$. 

Alg. singleswap

- Basic tool: The \((2^{n-1}) \times (2^n - 1)\) binary matrix \(R_{(n+1)/2,n-1}\) (Rizomiliotis, 2010)

\[
R_{(n+1)/2,n-1} = \begin{pmatrix}
e_0 & e_1 & \ldots & e_E & 0 & \ldots & 0 \\
0 & e_0 & \ldots & e_{E-1} & e_E & \ldots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & \ldots & \vdots & \vdots & \ldots & 0 \\
0 & 0 & \ldots & \vdots & \vdots & \ldots & e_E
\end{pmatrix}
\]

- \(E = 2^{n-1} - 1\)
- \(e_0 + e_1 x + \ldots + e_E x^E\): the generator polynomial of \(RM^*(\frac{n-1}{2}, n)\)
- For any \(0 \leq r < 2^n - 1\) each column vector \(v^r\) of \(R_{(n+1)/2,n-1}\) is

\[
v^r = \begin{cases} 
(e_r \cdots e_1 e_0 \mathbf{0}_{E-r})^T, & \text{if } r \leq E \\
(\mathbf{0}_{r-E} e_E \cdots e_{r-E})^T, & \text{otherwise}
\end{cases}
\]
Alg. singleswap (Cont.)

- Goal: For $\alpha^m$, $m > 2^{n-1} - 1$, find $\alpha^j$, $j \leq 2^{n-1} - 1$, such that replacing (swapping) $\alpha^j$ with $\alpha^m$ in the support of the CF function retains the maximum AI.

- Limniotis-Kolokotronis-Kalouptsidis, 2013: Consider the left-hand square upper-diagonal sub-matrix $R'$

$$
\begin{pmatrix}
e_0 & e_1 & \ldots & e_E & | & 0 & \ldots & 0 \\
0 & e_0 & \ldots & e_{E-1} & | & e_E & \ldots & 0 \\
\vdots & \vdots & \ldots & \vdots & | & \vdots & \ldots & \vdots \\
0 & 0 & \ldots & e_1 & | & \vdots & \ldots & 0 \\
0 & 0 & \ldots & e_0 & | & \vdots & \ldots & e_E
\end{pmatrix}
$$

- Solve the system $R'z = v^m$
  - Via backward substitution
  - Each $0 \leq j \leq 2^{n-1} - 1$ such that $z_j = 1$ is an answer.
We may simply find $k$ entries of $z$, for any $k \ll 2^{n-1}$

- The algorithm computes the last $k$ entries $z_E, \ldots, z_{E-k+1}$ in decreasing order

- The overall computational complexity is described by $\mathcal{O}(k^2)$
New approach

Limniotis-Kolokotronis, 2015

- Generalization of the above, so as to find arbitrary number of swaps retaining the maximum AI (for odd $n$)
- Properties of punctured Reed–Muller codes $\text{RM}^*(\frac{n-1}{2}, n)$ are employed
  - Due to Alg. singleswap, efficient application to the CF function

Useful terminology

- For two codewords (polynomials) of a binary code
  
  $$h(x) = \sum_{i=0}^{N-1} h_i x^i \quad \text{and} \quad c(x) = \sum_{i=0}^{N-1} c_i x^i$$

  we have $h \preceq c \iff h_i \leq c_i$ for all $i$.

- A minimal codeword is any codeword $v(x)$ such that there is no nonzero codeword $v'(x)$ of the code with $v' \prec v$. 
Annihilators as codewords

In the sequel: \( n \) odd, \( \alpha \) a primitive element of \( \mathbb{F}_{2^n} \)

**Theorem**

Let \( f \in \mathbb{B}_n \) be balanced with \( \text{supp}(f) = \{ \alpha^{r_0}, \alpha^{r_1}, \ldots, \alpha^{r_E} \} \) and \( r_0 = 0 \). Then, \( \text{AI}(f) = \frac{n+1}{2} \) if and only if there is no nonzero even weight codeword \( v(x) \) of the code \( \text{RM}^*(\frac{n-1}{2}, n) \) such that \( v(x) \preceq c(x) = 1 + x^{r_1} + \cdots + x^{r_E} \).

**Proof (Sketch)**

- We consider the DFT representation of any annihilator \( g \) of \( f + 1 \)
- If \( \deg(g) \leq \frac{n-1}{2} \), then specific DFT coefficients should be zero
  - Such a requirement leads to the proof of the claim
Annihilators as minimal codewords

Proposition

Let \( f \in \mathbb{B}_n \) have \( \text{Al}(f) = \frac{n+1}{2} \), where \( \text{supp}(f) = \{\alpha^{r_0}, \ldots, \alpha^{r_E}\} \) and \( r_0 = 0 \). For all \( \alpha^j \notin \text{supp}(f) \), there exists a unique nonzero even weight minimal codeword \( v(x) \) of \( \text{RM}^*(\frac{n-1}{2}, n) \) such that \( x^j \prec v(x) \) and
\[
v(x) \preceq c(x) = 1 + x^{r_1} + \cdots + x^{r_E} + x^j.
\]

If \( f \) is the CF function:

- For any \( j > E \), there exists a unique nonzero even-weight minimal codeword \( u_j(x) \) of \( \text{RM}^*(\frac{n-1}{2}, n) \), with \( x^j \prec u_j(x) \) and
\[
u_j(x) \preceq c(j)(x) = \sum_{i=0}^{E} x^i + x^j.
\]

- Direct corollary from the previous Proposition

- The codewords \( u_j \) have a main role in developing new construction of functions with maximum AI, as shown next
Proposition

Let  \( c(x) = c_1(x) + c_2(x) \), where  \( c_1(x) \leq \sum_{i=0}^{E} x^i \),  \( c_2(x) \leq \sum_{i=E+1}^{N-1} x^i \).

If \( \exists \) nonzero even weight codeword \( v(x) \) of \( \text{RM}^*(\frac{n-1}{2}, n) \) with 
\( v(x) \preceq c(x) \), then \( v(x) \) necessarily has the form

\[
v(x) = \sum_{j \in J} \delta_j u_j(x), \quad \delta_j \in \mathbb{F}_2, \quad J \subseteq \{E < i < N : x^i \leq c_2(x)\}
\]

Proof (Sketch)

- Suppose there exists minimal codeword \( v'(x) \preceq c(x) \) not having the above form.
- It holds \( v'(x) = v'_1(x) + v'_2(x) \), where \( v'_1(x) \preceq c_1(x) \), \( v'_2(x) \preceq c_2(x) \).
- Let \( J' = \{E < i < N : x^i \leq v'_2(x)\} \). Then \( u' + v' \) is also an even weight codeword of \( \text{RM}^*(\frac{n-1}{2}, n) \), where \( u' = \sum_{j \in J'} u_j(x) \).
- But \( u' + v' \preceq \sum_{i=0}^{E} x^i \Rightarrow \deg(u' + v') \leq E - a \) contradiction.
A property that ensures maximum AI

Theorem

Let \( g \in \mathbb{B}_n \), where

- \( \text{supp}(g) = \{\alpha^0, \alpha^1, \ldots, \alpha^E\} \cup A \setminus B \),
- \( A = \{\alpha^{j_1}, \ldots, \alpha^{j_r}\} \subset \text{supp}(f + 1) \) and \( B = \{\alpha^{i_1}, \ldots, \alpha^{i_r}\} \subset \text{supp}(f) \), where
  - a. \( i_s \neq 0 \), for all \( 1 \leq s \leq r \),
  - b. \( x^{i_s} \prec u_{j_s}(x) \) for all \( 1 \leq s \leq r \),
  - c. \( x^{i_s} \nprec u_{j_t}(x) \) for all \( 1 \leq t \leq r \) with \( t \neq s \).

Then \( \text{AI}(g) = \frac{n+1}{2} \).

Proof (Sketch)

- Let \( \text{supp}(g) = \{\alpha^0, \alpha^{r_1}, \ldots, \alpha^{r_E}\} \)
- The choice of sets \( A, B \) ensures that there is no \( A' \subseteq \{j_1, \ldots, j_r\} \) such that \( \sum_{j \in A'} u_j(x) < 1 + x^{r_1} + \ldots + x^{r_E} \).
Towards developing a new construction

- Having knowledge of $u_j$, we may proceed by a new construction due to the previous Theorem.

- Basic idea: Start from the CF function $f$ and swap elements between $\text{supp}(f)$ and $\text{supp}(f+1)$ such as:
  - If $A \subset \text{supp}(f+1)$ that is "swapped" to $\text{supp}(f)$, then for any $j$ such that $\alpha^j \in A$, there exists a position at the codeword polynomial $u_j(x)$ where the corresponding coefficient is nonzero, whereas the corresponding coefficients of all other $u_{j'}(x)$, $j' \in A$, are zero.

- Crucial point: Efficient identification of $u_j(x)$ for all desired $j$ is needed.

- The answer: Alg. singleswap!
  - It is easily proved that Alg. singleswap returns exactly the coefficients of $u_j(x)$. 
The new algorithm

- Putting all together...

**Algorithm 2 modifyCF\( (n, f, M, k) \)**

<table>
<thead>
<tr>
<th>Input:</th>
<th>odd integer ( n ), function ( f \in \mathbb{B}_n ) with ( \text{supp}(f) = {\alpha^0, \ldots, \alpha^E} ) set ( M = {\alpha^{m_1}, \ldots, \alpha^{m_r}} \subset \text{supp}(f + 1) ), and integer ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:</td>
<td>for ( i = 1, \ldots, r ) do</td>
</tr>
<tr>
<td>2:</td>
<td>( S^{(i)} \leftarrow \text{ss} (n, f, \alpha^{m_i}, k) )</td>
</tr>
<tr>
<td>3:</td>
<td>end</td>
</tr>
<tr>
<td>4:</td>
<td>( S = \emptyset )</td>
</tr>
<tr>
<td>5:</td>
<td>for ( i = 1, \ldots, r ) do</td>
</tr>
<tr>
<td>6:</td>
<td>Choose ( j_i \in S^{(i)} \setminus \bigcup_{p \neq i} S^{(p)} ) so that ( \forall p \neq i, \exists j'_i \in S^{(p)} ) with ( j'_i &lt; j_i )</td>
</tr>
<tr>
<td>7:</td>
<td>( S \leftarrow S \cup {j_i} )</td>
</tr>
<tr>
<td>8:</td>
<td>end</td>
</tr>
<tr>
<td>Output:</td>
<td>( S = {j_1, \ldots, j_r} \subset {0, 1, \ldots, E} : ) the function ( g \in \mathbb{B}_n ) with ( \text{supp}(g) = \text{supp}(f) \cup M \setminus {\alpha^{j_1}, \ldots, \alpha^{j_r}} ) has maximum AI</td>
</tr>
</tbody>
</table>

- In general, many choices for selecting \( j_i \) from \( S^{(i)} \)
- Its worst–case computational complexity is \( \mathcal{O}(rkL) \), for \( L = \max\{k, r \log_2 k\} \).
  - Line 2: \( \mathcal{O}(k^2) \)
  - Line 6: For each candidate element of \( S^{(i)} \), we apply binary search on at most \( r - 1 \) ordered arrays with length at most \( k \)
Other cryptographic criteria

**Proposition**
There always exists a Boolean function $g$ constructed via Alg. modifyCF such that $\deg(g) = n - 1$.

**Proposition**
It holds $nl(g) > 2^{n-1} - \left(\frac{\ln 2}{\pi} n + 0.74\right)2^{n/2} - 2r - 1$, where $r$ is the number of swapped pairs.

**Discussion**
- Maximum possible algebraic degree is attainable
- High nonlinearity can be achieved
  - Due to the fact that the CF function has high nonlinearity
An example

- \( n = 7, \, f \in \mathbb{B}_7 \) a CF function,
  \[ M = \{ \alpha^{80}, \alpha^{81}, \alpha^{90}, \alpha^{91} \} \subset \text{supp}(f + 1) \] (random choice)

Application of Alg. singleswap to \( f \), for each element of \( M \)

<table>
<thead>
<tr>
<th>( m_i )</th>
<th>Set ( S^{(1)} ) of all possible ( j_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0 3 6–9 11–15 17 18 21–24 28 29 33 36 38–41 43 45–47 53 54 56 58 61 63</td>
</tr>
<tr>
<td>81</td>
<td>0–2 4–7 11 13 14 18 19 21 22 25 26 29 31–33 38–45 49 51 53–55 57 58–61 63</td>
</tr>
<tr>
<td>90</td>
<td>0 2 3 7 10 15–17 19 22 24 27 29 32 33 38–40 45 46 48 50 51 53–56 58 60 61 63</td>
</tr>
<tr>
<td>91</td>
<td>0–6 9 10 12 15 17 18 20 21 24–26 28 31 32 37–41 43 45 48 52–60 63</td>
</tr>
</tbody>
</table>

- All possible single swaps have been computed (Alg. singleswap has been executed for \( k = 2^{n-1} = 64 \))
  - For each \( m_i \in \{ 80, 81, 90, 91 \} \), all possible \( j_i \) such that \( g \in \mathbb{B}_7 \) with \( \text{supp}(g) = \text{supp}(f) \setminus \{ \alpha^{j_i} \} \cup \{ \alpha^{m_i} \} \) has maximum AI, are given
  - Proceed with the next step of Alg. modifyCF
Find entries that appear in exactly one row

<table>
<thead>
<tr>
<th>$m_i$</th>
<th>Set $S^{(i)}$ of all possible $j_i$</th>
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<td>80</td>
<td>0 3 6–9 11–15 17 18 21–24 28 29 33 36 38–41 43 45–47 53 54 56 58 61 63</td>
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<td>0–6 9 10 12 15 17 18 20 21 24–26 28 31 32 37–41 43 45 48 52–60 63</td>
</tr>
</tbody>
</table>

New function $g \in \mathcal{B}_7$ with maximum AI

- $\text{supp}(g) = \text{supp}(f) \setminus \{\alpha^{47}, \alpha^{49}, \alpha^{50}, \alpha^{52}\} \cup M$
  - Even if we had executed singleswap for $k = 17$ (instead of 64), we would get the same result

- For the specific example, 108 different functions can be generated
  - Possible choices:
    - $\{47, 36, 23, 8\}$ (from $S^{(1)}$),
    - $\{49, 44, 42\}$ (from $S^{(2)}$),
    - $\{50, 27, 16\}$ (from $S^{(3)}$),
    - $\{52, 37, 20\}$ (from $S^{(4)}$).
Behavior w.r.t. other cryptographic criteria

- $\deg(g) = 6$ - i.e. the maximum possible
- $nl(g) = 52$
  - Slightly lower than $nl(f) = 54$, ($f$ is the CF function)
  - Most of all possible 108 functions have also nonlinearity 52
    - Nonlinearity equal to 54 is attainable (although higher values were not observed, for the specific example)
- The same behavior w.r.t. fast algebraic attacks, as the CF function
  - $g$ does not admit any pair $(e, d)$ with $e = 1$ and $e + d \leq n - 1$, whilst for $e > 1$ there is no any pair $(e, d)$ satisfying $e + d < n - 1$. 
Conclusions - Future research

Summary

- New construction of functions with maximum AI \((n \text{ odd})\)
  - Having the CF function \(f\) as a starting point, it seems that other cryptographic criteria are also satisfied
  - Arbitrary number of swaps between \(\text{supp}(f)\) and \(\text{supp}(f + 1)\) that preserve maximum AI

Open problems

- Identify other possible swaps that satisfy the desired property
- Nonlinearity and fast algebraic attacks should be further elaborated
- Possible extension to the even case
  - Main difference: Adding an element of the \(\text{supp}(f + 1)\) into \(\text{supp}(f)\) does not necessarily reduce AI
  - However, research in progress shows that such elements can be identified for the CF function
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Questions & Answers

Thank you for your attention!