# Cryptographic Properties of Boolean Functions: Recent Developments and Open Problems

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# Talk Outline

#### Introduction - Definitions

- Stream ciphers
- Block ciphers

#### Properties of cryptographic functions

- Correlation immunity
- Nonlinearity
  - Higher-order nonlinearity
- Algebraic immunity
  - Fast algebraic immunity

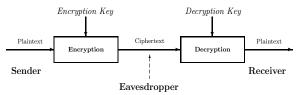
#### Occursions - Open problems

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Symmetric ciphers Stream ciphers Block ciphers

# Symmetric ciphers

#### A typical cryptosystem



#### Symmetric cryptography

- Encryption Key = Decryption Key
- The key is only shared between the two parties
  - The security rests with the secrecy of the key (Kerchoffs principle)

#### Two types of symmetric ciphers

- Stream ciphers
- Block ciphers

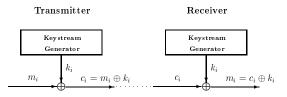
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## Stream ciphers

#### Simplest Case: Binary additive stream cipher



- Suitable in environments characterized by a limited computing power or memory, and the need to encrypt at high speed
- The seed of the keystream generators constitutes the secret key
- Security depends on
  - Pseudorandomness of the keystram  $k_i$
  - Properties of the underlying functions that form the keystream generator

### The optimal cipher: one-time pad

#### Description

- If  $M = m_1 m_2 \dots m_n$ , then  $k = k_1 k_2 \dots k_n$  satisfying
  - $\bullet \ k$  is trully random
  - $\bullet$  k is aperiodic
  - For each different message, we use different key
- Encryption:  $c_i = m_i \oplus k_i$ ,  $i = 1, 2, \ldots, n$
- Decryption:  $m_i = c_i \oplus k_i$ ,  $i = 1, 2, \dots, n$
- Such cipher is perfectly secure (Claude Shannon 1949)

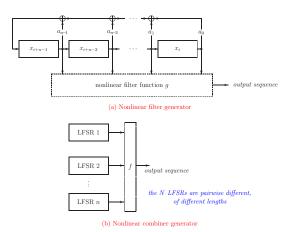
• 
$$p(M|C) = p(M)$$
 for any pair  $M, C$ 

- However both randomness as well as aperiodicity can not be ensured in a realistic model
- Designing of stream ciphers strives to resemble the one-time pad

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## Classical Keystream Generators



- Unpredictability of keystreams is ensured by appropriately choosing the underlying Boolean functions
- If these functions though do not satisfy certain properties, the system may be vulnerable to attacks
- More recently, nonlinear FSRs are preferred (although their mathematics are not well-known)

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# Known stream ciphers

#### • RC4

- $\bullet\,$  Used in WEP, WPA, SSL/TLS
- A5/1
  - Used in mobile telephony (GSM)
- E0
  - Used in Bluetooth protocol

eStream project (2004–2008): Effort to promote the design of efficient and compact stream ciphers suitable for widespread adoption.

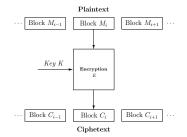
- Finalists:
  - Software implementation: HC-128, Rabbit, Salsa20/12, SOSEMANUK
  - Hardware implementation: Grain v1, MICKEY 2.0, Trivium

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# Block ciphers

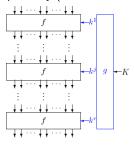
Simplest Case: Electronic Codebook Mode (ECB)



- Encryption on a per-block basis (typical block size: 128 bits)
- The encryption function E performs key-dependent substitutions and permutations (Shannon's principles)
- Security depends on
  - Generation of the sub-keys used in E
  - Properties of the underlying functions of  ${\boldsymbol E}$

## The encryption function E in a block cipher

- Iterative structure
  - Several rounds occur
- A sub-key is being used in each round
- The round function *f* performs substitution and permutations, via multi-output Boolean functions (S-boxes, P-boxes)
  - S-boxes and P-boxes provide the cryptographic properties of diffusion and confusion respectively (Claude Shannon 1949)



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## Known block ciphers

#### • Advanced Encryption Standard (AES)

- NIST's standard since 2001 (initial submission: Rijndael cipher)
- Supported key lengths: 128, 192, 256 bits
- Widespread adoption (SSL/TLS, IPSec, commercial products,...)
- Data Encryption Standard (DES)
  - The predecessor of AES (1976-1996)
  - Official withdrawing: 2004 (although it is still being met today)
  - Key size: 56 bits (actually, the only flaw of the algorithm)
- 3DES
  - Modification of DES, to use key of 112 or 168 bit length
  - Still in use today although not very efficient
- Other block ciphers: IDEA, MARS, RC6, Serpent, Twofish

# A common approach for block and stream ciphers

- Despite their differences, a common study is needed for their building blocks (multi-output and single-output Boolean functions respectively)
- The attacks in block ciphers are, in general, different from the attacks in stream ciphers and vice versa. However:
  - For both cases, almost the same cryptographic criteria of functions should be in place
- Challenges:
  - There are tradeoffs between several cryptographic criteria
  - The relationships between several criteria are still unknown
  - Constructing functions satisfying all the main criteria is still an open problem

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# **Boolean Functions**

A Boolean function f on n variables is a mapping from  $\mathbb{F}_2^n$  onto  $\mathbb{F}_2$ 

- The vector  $f = (f(0,0,\ldots,0), f(1,0,\ldots,0),\ldots,f(1,1,\ldots,1))$  of length  $2^n$  is the truth table of f
- The Hamming weight of f is denoted by  $\operatorname{wt}(f)$ 
  - f is balanced if and only if  $wt(f) = 2^{n-1}$

• The support  $\mathrm{supp}(f)$  of f is the set  $\{\pmb{b}\in\mathbb{F}_2^n:f(\pmb{b})=1\}$  Example: Truth table of balanced f with n=3

$x_1$	0	1	0	1	0	1	0	1
$x_2$	0	0	1	1	0	0	1	1
$x_3$	0	0	0	0	1	1	0 1 1	1
$f(x_1, x_2, x_3)$	0	1	0	0	0	1	1	1

A vectorial Boolean function f on n variables is a mapping from  $\mathbb{F}_2^n$  onto  $\mathbb{F}_2^m, \ m>1$ 

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#### Algebraic Normal Form and degree of functions

• Algebraic Normal Form (ANF) of f:

$$f(x) = \sum_{\boldsymbol{v} \in \mathbb{F}_2^n} a_{\boldsymbol{v}} x^{\boldsymbol{v}}, \quad \text{where } x^{\boldsymbol{v}} = \prod_{i=1}^n x_i^{v_i}$$

• The sum is performed over  $\mathbb{F}_2$  (XOR addition)

- The degree deg(f) of f is the highest number of variables that appear in a product term in its ANF.
- If  $\deg(f) = 1$ , then f is called affine function
  - If, in addition, the constant term is zero, then the function is called linear
- In the previous example:  $f(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1$ .

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$$\deg(f) = 2$$

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## Univariate representation of Boolean functions

- $\mathbb{F}_2^n$  is isomorphic to the finite field  $\mathbb{F}_{2^n}$  ,
- $\Rightarrow$  Any function  $f \in \mathbb{B}_n$  can also be represented by a univariate polynomial, mapping  $\mathbb{F}_{2^n}$  onto  $\mathbb{F}_2$ , as follows

$$f(x) = \sum_{i=0}^{2^n - 1} \beta_i x^i$$

where  $\beta_0, \beta_{2^n-1} \in \mathbb{F}_2$  and  $\beta_{2i} = \beta_i^2 \in \mathbb{F}_{2^n}$  for  $1 \le i \le 2^n - 2$ 

- The coefficients of the polynomial determine the Discrete Fourier Transform of f
- The degree of f can be directly deduced by the univariate representation
- The univariate representation is more convenient in several cases

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#### Walsh transform

#### Definition

The Walsh transform  $\widehat{\chi}_f(a)$  at  $a\in\mathbb{F}_2^n$  of  $f:\mathbb{F}_2^n\to\mathbb{F}_2$  is

$$\widehat{\chi}_f(a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + ax^T} = 2^n - 2 \operatorname{wt}(f + \phi_a)$$

where  $\phi_a(x) = ax^{\tau} = a_1x_1 + \dots + a_nx_n$ 

- Computational complexity:  $\mathcal{O}(n2^n)$  (via fast Walsh transform)
- Parseval's theorem:  $\sum_{a \in \mathbb{F}_2^n} \widehat{\chi}_f(a)^2 = 2^{2n}$

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# Correlation immunity

- If the output of a Boolean function *f* is correlated to at least one of its inputs, then it is vulnerable to correlation attacks (Siegenthaler, 1984).
- The  $f \in \mathbb{B}_n$  is *t*-th correlation immune if it is not correlated with any *t*-subset of  $\{x_1, \ldots, x_n\}$ ; namely if

$$Pr(f(\boldsymbol{x}) = 0 | x_{i_1} = b_{i_1}, \dots, x_{i_t} = b_{it}) = Pr(f(\boldsymbol{x}) = 0)$$

for any t positions  $x_{i_1}, \ldots, x_{i_t}$  and any  $b_{i_1}, \ldots, b_{i_t} \in \mathbb{F}_2$ 

• If a *t*-th order correlation immune function is also balanced, then it is called *t*-th order resilient.

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## Properties of correlation immunity

- Siegenthaler, 1984: A known trade-off: If f is k-th order resilient for  $1 \le k \le n-2$ , then  $\deg(f) \le n-k-1$ .
- Xiao-Massey, 1988: A function  $f \in \mathbb{B}_n$  is t-th order correlation immune iff its Walsh transform satisfies

$$\widehat{\chi}_f(a) = 0, \forall \ 1 \le \operatorname{wt}(a) \le t$$

- Note that f is balanced iff  $\widehat{\chi}_f(\mathbf{0}) = 0$ .
- $\Rightarrow$  A function  $f \in \mathbb{B}_n$  is t-th order resilient iff its Walsh transform satisfies  $\widehat{\chi}_f(a) = 0, \forall \ 0 \le \operatorname{wt}(a) \le t$
- Siegenthaler also proposed a recursive procedure to construct *m*-th order resilient Boolean functions, for any desired *m*, with the maximum possible degree
- Several other constructions are currently known

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## Linear approximation attacks

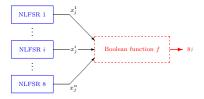
- Cryptographic functions need to be balanced, as well as of high degree
  - $\bullet\,$  The maximum possible degree of a balanced Boolean function with n variables is n-1
- High degree though is not adequate to prevent linear cryptanalysis (in block ciphers - Matsui, 1992) or best affine approximation attacks (in stream ciphers - Ding et. al., 1991)
  - A function should not be well approximated by a linear/affine function
  - Any function of degree 1 that best approximates f is a best affine/linear approximation of f

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#### Example of approximation attacks

The Achterbahn cipher [Gammel-Göttfert-Kniffler,2005] (candidate in eSTREAM project)



- Lengths of nonlinear FSRs: 22-31
- $f(x_1, \ldots, x_8) = \sum_{i=1}^4 x_i + x_5 x_7 + x_6 x_7 + x_6 x_8 + x_5 x_6 x_7 + x_6 x_7 x_8$
- Johansson-Meier-Muller, 2006: cryptanalysis via the linear approximation  $g(x_1, \ldots, x_8) = x_1 + x_2 + x_3 + x_4 + x_6$ , satisfying wt(f + g) = 64 (p(f = g) = 3/4)

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### The notion of nonlinearity

• The minimum distance between *f* and all affine functions is the nonlinearity of *f*:

$$\mathsf{nl}(f) = \min_{l \in \mathbb{B}_n: \deg(l) = 1} \operatorname{wt}(f + l)$$

• Relathionship with Walsh transform

$$\mathsf{nl}(f) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} |\widehat{\chi}_f(a)|$$

 $\bullet\,\Rightarrow$  Nonlinearity is computed via the Fast Walsh Transform

• High nonlinearity is prerequisite for thwarting attacks based on affine (linear) approximations

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## Known results on nonlinearity of Boolean functions

- For even n, the maximum possible nonlinearity is  $2^{n-1} 2^{n/2-1}$ , achieved by the so-called bent functions
  - Many constructions are known (not fully classified yet)
  - But bent functions are never balanced!
- $\bullet\,$  For odd n, the maximum possible nonlinearity is still unknown
  - By concatenating bent functions, we can get nonlinearity  $2^{n-1} 2^{\frac{n-1}{2}}$ . Can we impove this?
    - For  $n \leq 7$ , the answer is no
    - For n ≥ 15, the answer is yes (Patterson-Wiedemann, 1983 -Dobbertin, 1995 - Maitra-Sarkar, 2002)
    - For n = 9, 11, 13, such functions have been found more recently (Kavut, 2006)
- Several constructions of balanced functions with high nonlinearity exist (e.g. Dobbertin, 1995). However:
  - Finding the highest possible nonlinearity of balanced Boolean functions is still an open problem

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### Nonlinearity and correlation immunity

- If  $f \in \mathbb{B}_n$  is *m*-th order resilient, then  $nl(f) \le 2^{n-1} 2^{m+1}$ (Sarkar-Maitra, 2000)
- $\bullet\,$  If the upper bound is achieved, the Walsh transform takes 3 values
- For n even, the best possible nonlinearity is  $2^{n-1} 2^{n/2-1}$  (bent functions)
  - If  $f \in \mathbb{B}_n$  is *m*-th order resilient, *n* even, with  $m \le n/2 2$ , then  $nl(f) \le 2^{n-1} 2^{n/2-1} 2^{m+1}$
- For n odd, the maximum possible nonlinearity  $nI_{max}$  is unknown
  - The above bound holds if  $2^{m+1} \leq 2^{n-1} \mathsf{nl}_{max}$

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### The Maiorana-McFarland class of functions

- A widely known class of functions with nice cryptographic properties
- $f \in \mathbb{B}_{k+s}$  satisfying the following:

$$f(y,x) = F(y)x + h(y), \ x \in \mathbb{F}_2^k, \ y \in \mathbb{F}_2^s$$

- F is any mapping from  $\mathbb{F}_2^k$  to  $\mathbb{F}_2^s$
- $h \in \mathbb{B}_s$
- If k = s and F is a permutation over  $\mathbb{F}_2^k \Rightarrow f$  is bent (e.g. Dillon, 1974)
- For injective F, if  $wt(F(\tau)) \ge t + 1$  for all  $\tau \in \mathbb{F}_2^s$ , then f is t-resilient (Camion et. al., 1992).

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# Higher-order nonlinearity

- Approximating a function by a low-order function (not necessarily linear) may also lead to cryptanalysis (Non–linear cryptanalysis -Knudsen-1996, low-order approximation attacks - Kurosawa et. al. -2002)
- The rth order nonlinearity of a Boolean function  $f \in \mathbb{B}_n$  is given by

$$\mathsf{nl}_r(f) = \min_{g \in \mathbb{B}_n : \deg(g) \le r} \operatorname{wt}(f+g)$$

- The rth order nonlinearity remains unknown for r > 1
  - Recursive lower bounds on  $nl_r(f)$  (Carlet, 2008)
  - Specific lower and upper bounds for nl<sub>2</sub>(*f*) (Cohen, 1992 Carlet, 2007)
  - More recent lower bounds for 2-nd order nonlinearity: Gangopadhyay et. al. 2010, Garg et. al. 2011, Singh 2011, Singh et. al. 2013

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## Computing best low order approximations

- Computing even the best 2-nd order approximations is a difficult task
  - Efficient solution for specific class of 3-rd degree functions (Kolokotronis-Limniotis-Kalouptsidis, 2009)
    - The problem is appropriately reduced in computing best affine approximation attacks of the underlying 2-nd degree sub-functions
  - For the Achterbahn's combiner function:

 $q(x) = x_5x_7 + x_6x_8 + x_1 + x_2 + x_3 + x_4$  is a best 2-nd approximation

- wt(f+q) = 32 (p(f=q) = 7/8 > 3/4)
- No much is known regarding constructions of functions with high r-th nonlinearity, for  $r\geq 2$ 
  - Even if a high lower bound on the nonlinearity is proved, best *r*-th order approximations can not be computed

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A case of highly nonlinear function f with  $nl_2(f) = nl(f)$ 

Cubic functions in the general Maiorana-McFarland class

 $f(x,y) = F(x)y^{\mathsf{T}}, \qquad (x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^m.$ 

- $F: \mathbb{F}_2^n \to \mathbb{F}_2^m$ : quadratic vectorial Perfect Nonlinear (PN) function
- $\bullet\,$  All linear combinations of the m underlying Boolean functions are bent
- (Kolokotronis-Limniotis, 2012): Let f ∈ B<sub>n+m</sub> be cubic function of the above form, where each linear combination of the Boolean functions of F is bent of minimal weight, and m ≤ L<sup>n</sup>/<sub>4</sub>. Then,

$$\mathsf{nl}_2(f) = 2^{n+m-1} - 2^{n/2-1}(2^{n/2} + 2^m - 1) = \mathsf{nl}(f)$$

• Best 2-nd order approximations are also efficiently computed

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# More recent attacks: Algebraic attacks

#### Milestones

- Algebraic attacks (Courtois-Meier, 2003)
- Fast algebraic attacks (Courtois, 2003)
- The basic idea is to reduce the degree of the mathematical equations employing the secret key
- Known cryptographic Boolean functions failed to thwart these attacks
- Some applications of algebraic attacks
  - Six rounds of DES, with only one known plaintext (Courtois-Bard, 2006)
  - Keeloq block cipher (Courtois-Bard-Wagner, 2008)
  - Hitag2 stream cipher (Courtois et. al., 2009)

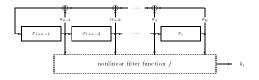
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# Algebraic attacks

#### Example

• Stream cipher based on a nonlinear filter generator



- $k_i = f(L^i(x_0, x_1, \dots, x_{N-1}))$  the filter function f has high degree
- Assume that there exists  $g \in \mathbb{B}_n$  of low degree such that f \* g = h, where h is also of low degree. Then,

$$k_i g(L^i(x_0, x_1, \dots, x_{N-1})) = h(L^i(x_0, x_1, \dots, x_{N-1}))$$

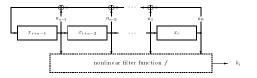
• Several other proper choices of g, h may also reduce the degree of the system

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# An example: The Toyocrypt cipher

• A submission to a Japanese government call



• The nonlinear filter function is

$$f(x_1, \dots, x_{128}) = q(x_1, \dots, x_{128}) + x_{10}x_{23}x_{32}x_{42} + \prod_{i=1}^{62} x_i +$$

 $+ x_1 x_2 x_9 x_{12} x_{18} x_{20} x_{23} x_{25} x_{26} x_{28} x_{33} x_{38} x_{41} x_{42} x_{51} x_{53} x_{59}$ 

where  $\deg(q) = 2$ .

• By multiplying f with the affine functions  $1 + x_{23}$  or  $1 + x_{42}$ , we get two functions of degree only 3

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#### How to proceed with algebraic attacks

- Once the degree of the equations have been reduced, several algebraic techniques have been proposed for solving the (still nonlinear) system:
  - Linearization of the system
  - Use of Gröbner bases
  - More specific techniques: XL, XSL
- Hence, the core of the algebraic attacks is the transformation of the initial system to a new one having low degree

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# Annihilators and algebraic immunity

#### Definition

Given  $f \in \mathbb{B}_n$ , we say that  $g \in \mathbb{B}_n$  is an annihilator of f if and only if g lies in the set

$$\mathcal{AN}(f) = \{g \in \mathbb{B}_n : f * g = 0\}$$

#### Definition

The algebraic immunity  $AI_n(f)$  of  $f \in \mathbb{B}_n$  is defined by

$$\mathsf{Al}_n(f) = \min_{g \neq 0} \{ \deg(g) : g \in \mathcal{AN}(f) \cup \mathcal{AN}(f+1) \}$$

- A high algebraic immunity is prerequisite for preventing algebraic attacks (Meier-Pasalic-Carlet, 2004)
- Well-known upper bound:  $AI_n(f) \leq \lceil \frac{n}{2} \rceil$

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### Algebraic immunity and nonlinearity

• Carlet et. al., 2006:  $nl_r(f) \ge \sum_{i=0}^{Al_n(f)-r-1} \binom{n}{i}$ 

Low nonlinearity implies low algebraic immunity

- Especially for r = 1 (Lobanov, 2005):  $nl(f) \ge 2 \sum_{i=0}^{Al_n(f)-2} \binom{n-1}{i}$
- Mesnager, 2008: Improvements one the above bounds:

$$\mathsf{nl}_r(f) \geq \sum_{i=0}^{\mathsf{Al}_n(f)-r-1} \binom{n}{i} + \sum_{i=\mathsf{Al}_n(f)-2r}^{\mathsf{Al}_n(f)-r-1} \binom{n-r}{i} \,,$$

• Rizomiliotis, 2010: Further improvements

• The notion of complementary algebraic immunity is defined

$$\overline{\mathsf{AI}_n(f)} \triangleq \max\{\min_{\deg(g)}\{f \ast g = 0\}, \min_{\deg(g)}\{(f+1) \ast g = 0\}\}$$

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## Fast algebraic attacks

- Consider again the filter generator:  $k_i = f(L^i(x_0, x_1, \dots, x_{N-1}))$
- Assume that there exists a low degree  $g \in \mathbb{B}_n$  such that h = f \* g is of reasonable degree. Then again,

$$k_i g(L^i(x_0, x_1, \dots, x_{N-1})) = h(L^i(x_0, x_1, \dots, x_{N-1}))$$

• There exists a linear combination of the first  $\sum_{i=0}^{\deg(h)} {N \choose i}$  equations that sum the right-hand part to zero  $\Rightarrow$  We get one equation of degree at most  $\deg(g)$ 

#### Comparison with conventional algebraic attacks

- $g + h \in \mathcal{AN}(f) \Rightarrow$  the degree of g + h may be greater than  $\mathsf{AI}_n(f)$ ,
  - Maximum AI does not imply resistance to fast algebraic attacks
- But: Knowledge of consecutive keystream bits is required

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# Fast Algebraic Immunity

Known result: For any pair of integers (e, d) such that  $e + d \ge n$ , there exists a nonzero function g of degree at most e such that f \* g has degree at most d.

#### Definition

The fast algebraic immunity  $FAI_n(f)$  of  $f \in \mathbb{B}_n$  is defined by

$$\mathsf{FAI}_n(f) = \min_{1 \le \deg(g) \le \mathsf{AI}_n(f)} \{ 2 \, \mathsf{AI}_n(f), \deg(g) + \deg(f * g) \}$$

• Upper bound:  $\operatorname{FAI}_n(f) \leq n$ 

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# Constructions of functions with maximum AI

- Dalai-Maitra-Sarkar, 2006: Majority function
  - For even n, a slight modification of the majority function also preserves maximum AI
- Carlet-Dalai-Gupta-Maitra-Sarkar, 2006: Iterative construction
- Li-Qi, 2006: Modification of the majority function
- Sarkar-Maitra, 2007: Rotation Symmetric functions of odd n
- Carlet, 2008: Based on properties of affine subspaces
  - Further investigation in Carlet-Zeng-Li-Hu, 2009
  - Generalization (for odd *n*) in Limniotis-Kolokotronis-Kalouptsidis, 2011
- Balanceness and/or high nonlinearity are not always attainable, whereas their fast algebraic immunity remains unknown

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#### The Carlet-Feng construction

- Carlet-Feng, 2008:  $\operatorname{supp}(f) = \{0, 1, \alpha, \alpha^2, \dots, \alpha^{2^{n-1}-2}\}$ , where  $\alpha$  a primitive element of the finite field  $\mathbb{F}_{2^n}$ .
  - Degree n-1 (i.e. the maximum possible)
  - High (first-order) nonlinearity is ensured
    - Best currently known lower bound (Tang et. al., 2013:)

$$\mathsf{nl}(f) \ge 2^{n-1} - (\frac{n\ln(2)}{\pi} + 0.74)2^{n/2} - 1$$

- Experiments show that the actual values of nonlinearities may be higher enough
- Optimal against fast algebraic attacks, as subsequently shown (Liu-Zhang-Lin, 2012)

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Preliminaries Correlation immunity Nonlinearity Algebraic immunity

## Generalizations of Carlet-Feng construction

- Rizomiliotis, 2010: A new construction based on the univariate representation
  - $\bullet\,$  For odd n, it is affine equivalent to the Carlet-Feng construction
- Zeng-Carlet-Shan-Hu, 2011: Modifications of the Rizomiliotis construction
- Further generalizations in Limniotis-Kolokotronis-Kalouptsidis, 2013:
  - $\bullet$  Finding swaps between  $\mathrm{supp}(f)$  and  $\mathrm{supp}(f+1)$  that preserve maximum AI
    - Via solving a system of linear equations, with upper-triangular coefficient matrix
    - Each nonzero entry in the solution vector indicates a swapping

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# Conclusions - Open problems

- Evaluation of known families of cryptographic functions in terms of resistance against (fast) algebraic attacks
- Construction of functions with maximum (fast) algebraic immunity
  - Much progress on constructing functions with maximum AI, but the case of maximum FAI is much more difficult
  - High r-th order nonlinearity, for r > 1, has not been studied at all
- Relationships between (fast) algebraic immunity and correlation immunity
  - A preliminary result is only known (Limniotis, 2013)
- Improve our knowledge regarding relationships between AI and nl
- Nonlinear FSRs (or other nonlinear structures) have not been studied to the same extent w.r.t. algebraic attacks

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#### Questions & Answers

## Thank you for your attention!

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