Fractal Interpolation Surfaces. Theory and Applications in Image Compression.

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Abstract. In this dissertation, the problem of the construction of Fractal Interpolation Surfaces and their application in image compression is studied. We give exact conditions, so that this construction is valid and we introduce some free parameters that make our model as flexible as possible. In addition, we compute the box-counting dimension of a Fractal Interpolation Surface. Finally, we give a new image compression algorithm, based on fractal interpolation that differs from other fractal compression techniques.

1 Introduction

Fractal theory has been drawing considerable attention of researchers in various scientific areas. The application of fractals created by *iterated function systems* (IFSs) in the area of image compression is probably the most known one. Applications of fractal surfaces have been also found in computer graphics, metallurgy, geology, chemistry, medical sciences and several other areas where there is great need to construct extremely complicated objects; see for example [10], [14].

Mazel and Hayes (see [12]) use Fractal Interpolation Functions (FIFs) (introduced by Barnsley in [1]) to approximate discrete sequences of data (like one-dimensional signals). They demonstrated the effectiveness of their method by modelling seismic and electrocardiogram data. Recently, Navascues and Sebastian (in [13]) gave a generalisation of Hermite functions using FIFs.

Fractal Interpolation Surfaces (FISs) were used to approximate surfaces of rocks, metals, terrains, planets and to compress images. Self-affine FISs were first introduced in [11] in the case where the domains are triangular and the interpolation points on the boundary of the domain are coplanar. A few years later Geronimo and Hardin [8] generalised the construction of Massopust to allow for more general boundary data and domains.

Some problems in the construction by [11] remained unsolved, amongst which was the lack of free contractivity factors, which are necessary in modelling complicated surfaces. A general constructive method of generating affine FISs is presented in [17]. Xie and Sun in [15] and Xie, Sun, Ju, Feng in [16] presented a construction of a compact set that contains the interpolation points defined in

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a rectangular domain. A special construction of bivariate FIS (BFIS) is given in [9]. Dalla [7] gives some conditions in such a way that the bivariate IFS gives a FIS.

In this dissertation we introduce recurrent BFISs as a generalisation of the aforementioned methods in order to gain more flexibility in natural-shape generation or in image compression. The main advantage of this new class of FIS is that they are neither self-affine nor self-similar in contrast to all the previously mentioned constructions. Our method is used for image reconstruction and offers the advantage of a more flexible fractal modelling compared to previous fractal techniques (based on affine transformations). Finally we introduce closed fractal interpolation surfaces and vector valued fractal interpolation surfaces that offer even more free modelling parameters.

2 Main Results

2.1 Fractal Interpolation Surfaces

We consider the metric space $X = [0,1] \times [0,p] \times \mathbb{R}$. Let $\Delta = \{(x_i, y_j, z_{ij}) : i = 0,1,\ldots,N; j = 0,1,\ldots,M\}$ be an interpolating set with $(N+1)\cdot(M+1)$ *interpolation points*, such that $0 = x_0 < x_1 < \cdots < x_N = 1$ and $0 = y_0 < y_1 < \cdots < y_M = p$. Furthermore let $Q = \{(\hat{x}_k, \hat{y}_l, \hat{z}_{kl}) : k = 0,1,\ldots,K; l = 0,1,\ldots,L\}$ be a set with $(K+1)\cdot(L+1)$ points with $Q \subset \Delta$ $(Q \neq \Delta)$, such that $0 = \hat{x}_0 < \hat{x}_1 < \cdots < \hat{x}_K = 1$ and $0 = \hat{y}_0 < \hat{y}_1 < \cdots < \hat{y}_L = p$. The interpolation points divide $[0,1] \times [0,p]$ into $N \cdot M$ rectangles $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1,\ldots,N$ and $j = 1,\ldots,M$, which we call *sections*, while the points of Q divide $[0,1] \times [0,p]$ to $K \cdot L$ rectangles $J_{kl} = [\hat{x}_{k-1}, \hat{x}_k] \times [\hat{y}_{l-1}, \hat{y}_l], k = 1,\ldots,K$ and $l = 1,\ldots,L$ which we simply call *intervals*. It is evident that for every interval J_{kl} there are some sections lying inside.

Define a labelling map $\mathbb{J}\colon \{1,2,\ldots,N\} \times \{1,2,\ldots,M\} \to \{1,2,\ldots,K\} \times \{1,2,\ldots,L\}$ with $\mathbb{J}(i,j) = (k,l)$, such that $\hat{x}_k - \hat{x}_{k-1} > x_i - x_{i-1}$ and $\hat{y}_l - \hat{y}_{l-1} > y_j - y_{j-1}$ for $i = 1,2,\ldots,N$, $j = 1,2,\ldots,M$ and contractive mappings $w_{ij}\colon X \to X$ satisfying

$$\begin{aligned}
w_{ij}\begin{pmatrix}\hat{x}_{k-1}\\ \hat{y}_{l-1}\\ \hat{z}_{k-1,l-1}\end{pmatrix} = \begin{pmatrix}x_{i-1}\\ y_{j-1}\\ z_{i-1,j-1}\end{pmatrix}, w_{ij}\begin{pmatrix}\hat{x}_{k}\\ \hat{y}_{l-1}\\ \hat{z}_{k,l-1}\end{pmatrix} = \begin{pmatrix}x_{i}\\ y_{j-1}\\ z_{i,j-1}\end{pmatrix}, \\
w_{ij}\begin{pmatrix}\hat{x}_{k-1}\\ \hat{y}_{l}\\ \hat{z}_{k-1,l}\end{pmatrix} = \begin{pmatrix}x_{i-1}\\ y_{j}\\ z_{i-1,j}\end{pmatrix} \quad \text{and} \quad w_{ij}\begin{pmatrix}\hat{x}_{l}\\ \hat{y}_{l}\\ \hat{z}_{kl}\end{pmatrix} = \begin{pmatrix}x_{i}\\ y_{j}\\ z_{i,j}\end{pmatrix}
\end{aligned} \tag{1}$$

for $i = 1,\ldots,N$ and $j = 1,\ldots,M$. The w_{ij} map the vertices of the interval $J_{kl} = J_{\mathbb{J}(i,j)}$ to the vertices of the section I_{ij}. Finally, let $\Phi\colon \{1,\ldots,N\}\times\{1,\ldots,M\} \to \{1,\ldots,N\cdot M\}$ be a 1-1 function (i.e. an enumeration of the set $\{(i,j) : i = 1,\ldots,N; j = 1,\ldots,M\}$).

A *recurrent iterated function system* (RIFS) associated with the set of data Δ consists of the IFS $\{X; w_{i,j}, i = 1,2,\ldots,N; j = 1,2,\ldots,M\}$ (or, somewhat more briefly, as $\{X; w_{1-N,1-M}\}$) together with a row-stochastic matrix ($p_{nm} \in [0,1] :\ n,m \in \{\Phi(i,j), i=1,\dots,N; j=1,\dots,M\}$), such that

$$\sum_{m=1}^{N\cdot M} p_{nm} = 1, \quad n = 1,\dots,N\cdot M. \tag{2}$$

The recurrent structure is given by the *connection matrix* $C = (C_{nm})$, defined by

$$C_{nm} = \begin{cases} 1, \text{ if } p_{mn} > 0 \\ 0, \text{ if } p_{mn} = 0, \end{cases}$$

for $n, m = 1, 2, \dots, N \cdot M$. The transition probability for a certain discrete time Markov process is p_{nm}, which gives the probability of transfer into state m given that the process is in state n. Equation (2) says that whichever state the system is in (say n), a set of probabilities is available that sum to 1, and they describe the possible states to which the system transits at the next step.

We study the special case where w_{ij} are transformations of the form

$$w_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{ij}x + b_{ij} \\ c_{ij}y + d_{ij} \\ e_{ij}x + f_{ij}y + g_{ij}xy + s_{ij}z + k_{ij} \end{pmatrix} = \begin{pmatrix} \phi_{ij}(x) \\ \psi_{ij}(y) \\ F_{ij}(x,y,z) \end{pmatrix}. \tag{3}$$

From Eq. (3) eight linear equations arise which can always be solved for a_{ij}, b_{ij}, c_{ij}, d_{ij}, g_{ij}, e_{ij}, f_{ij}, k_{ij} in terms of the coordinates of the interpolation points and the *vertical scaling* (or *contractivity*) factor s_{ij} (see [7], [15]).

If the vertical scaling factors obey $|s_{ij}| < 1$, then there is a metric d on X equivalent to the Euclidean metric, such that w_{ij} is a contraction with respect to d (i.e. $\exists \hat{s}_{ij} : 0 \le \hat{s}_{ij} < 1 : d(w_{ij}(\bar{x}), w_{ij}(\bar{y})) \le \hat{s}_{ij} d(\bar{x}, \bar{y})$, $\forall \bar{x}, \bar{y} \in X$). The corresponding RIFS is called *recurrent bivariate IFS* (RBIFS).

It has been proved in [2] that there is a fixed point (attractor) A of this RBIFS. We also assume that pN, $pK \in \mathbb{N}$, the sections (defined by the interpolation points) are squares of side $\delta = 1/N$, while the intervals are squares of side $\psi = 1/K$ (thus $M = pN$, $L = pK$) and the number

$$a = \frac{\psi}{\delta} = \frac{N}{K}$$

is an integer greater than one. The number a^2 expresses how many sections lie inside any interval.

If we define the enumeration $\Phi(i,j) = (i-1)M + j$, $i = 1,\dots,N$ and $j = 1,\dots,M$, then $\Phi^{-1}(n) = ((n-1) \operatorname{div} M + 1,\ (n-1) \bmod M + 1)$, $n = 1,\dots,N\cdot M$ and the $NM \times NM$ stochastic matrix (p_{nm}) is defined by

$$p_{nm} = \begin{cases} \frac{1}{q_n}, \text{ if } I_{\Phi^{-1}(n)} \subseteq J_{\mathbb{J}(\Phi^{-1}(m))} \\ 0, \quad \text{otherwise}, \end{cases}$$

where q_n is the number of non zero elements of the n-th row of the stochastic matrix (p_{nm}). This means that p_{nm} is positive iff there is a transformation, which maps an interval containing the nth section (i.e. $I_{\Phi^{-1}(n)} = I_{(n-1)\,\mathrm{div}\,M+1,(n-1)\bmod M+1}$) to the mth section (i.e. $I_{(m-1)\,\mathrm{div}\,M+1,(m-1)\bmod M+1}$). Let us take a point in $I_{ij} \times \mathbb{R}$, $i = 1, \dots N$, $j = 1, \dots, M$. We say that we are in state n, if $n = \Phi(i, j)$. The matrix (p_{nm}) shows the probability of applying the map $w_{\Phi^{-1}(m)}$ to that point, so that the system transits to state m. Finally, we define the enumeration $\hat{\Phi}(k, l) = (k-1)K + l$, $k = 1, \dots, K$, $l = 1, \dots, L$ and the *connection vector* $C^v = \{c_1^v, c_2^v, \dots, c_{NM}^v\}$ as follows:

$$c_n^v = \hat{\Phi}(\mathbb{J}(\Phi^{-1}(n))), \quad n = 1, 2, \dots, NM.$$

If the attractor A of the above RBIFS is the graph of a continuous function, then it is called a *recurrent bivariate fractal interpolation surface* (RBFIS). Some sufficient assumptions will be given so that the above can be fulfilled.

Proposition 1. *Assume that for every interval $J_{kl}, k = 1, 2, \dots, K,\ l = 1, 2, \dots, L$, the points of each of the sets*

$$\begin{array}{l}
\{(x_{(k-1)a} = \hat{x}_{k-1},\ y_{(l-1)a+\nu},\ z_{(k-1)a,(l-1)a+\nu}); \nu = 0, 1, 2, \dots, a\}, \\
\{(x_{ka} = \hat{x}_k,\ y_{(l-1)a+\nu},\ z_{ka,(l-1)a+\nu}); \nu = 0, 1, 2, \dots, a\}, \\
\{(x_{(k-1)a+\nu},\ y_{(l-1)a} = y_{l-1},\ z_{(k-1)a+\nu,(l-1)a}); \nu = 0, 1, 2, \dots, a\}, \\
\{(x_{(k-1)a+\nu},\ y_{la} = y_l,\ z_{(k-1)a+\nu,la}); \nu = 0, 1, 2, \dots, a\}
\end{array}$$

are collinear. Then there exists a continuous function $f\colon [0,1] \times [0,p] \to \mathbb{R}$ that interpolates the given data $P = \{(x_i, y_j, z_{ij}) : i = 1, 2, \dots, N, j = 1, 2, \dots, M\}$ and its graph $\{(x, y, f(x, y)) : (x, y) \in [0,1] \times [0,p]\}$ is the attractor A of the RBIFS.

More generally we have the following.

Proposition 2. *Let the RBIFS be as defined above. Consider the interval J_{kl}, $k = 1, \dots, K, l = 1, \dots, L$, and let, for $\kappa \in \mathbb{N}$,*

$\mathcal{L}_{kl}^{\kappa}[\nu]$, $\nu = 1, \dots, a^{\kappa+1} - 1$,
denote the vertical distance of each one of the points computed in the step κ of the construction, with $x = \hat{x}_{k-1}$ and $y \in [\hat{y}_{l-1}, \hat{y}_l]$, from the line defined by the points $(x_{(k-1)a}, y_{(l-1)a}, z_{(k-1)a,\,(l-1)a})$ and $(x_{(k-1)a}, y_{la}, z_{(k-1)a,\,la})$,

$\mathcal{R}_{kl}^{\kappa}[\nu]$, $\nu = 1, \dots, a^{\kappa+1} - 1$,
denote the vertical distance of each one of the points computed in the step κ of the construction, with $x = \hat{x}_k$ and $y \in [\hat{y}_{l-1}, \hat{y}_l]$, from the line defined by the points $(x_{ka}, y_{(l-1)a}, z_{ka,\,(l-1)a})$ and $(x_{ka}, y_{la}, z_{ka,\,la})$,

$\mathcal{D}_{kl}^{\kappa}[\nu]$, $\nu = 1, \dots, a^{\kappa+1} - 1$,
denote the vertical distance of each one of the points computed in the step κ of the construction with $y = \hat{y}_{l-1}$ and $x \in [\hat{x}_{k-1}, \hat{x}_k]$, from the line defined by the points $(x_{(k-1)a}, y_{(l-1)a}, z_{(k-1)a,\,(l-1)a})$ and $(x_{ka}, y_{(l-1)a}, z_{ka,\,(l-1)a})$,

$\mathcal{U}_{kl}^{\kappa}[\nu]$, $\quad \nu = 1, \ldots, a^{\kappa+1} - 1$,
denote the vertical distance of each one of the points computed in the step κ of the construction with $y = \hat{y}_l$ and $x \in [\hat{x}_{k-1}, \hat{x}_k]$, from the line defined by the points $(x_{(k-1)a}, y_{la}, z_{(k-1)a,\, la})$ and $(x_{ka}, y_{la}, z_{ka,\, la})$.

Each one of these vertical distances is taken positive if the corresponding interpolation point is above the corresponding line; otherwise it is taken negative. If $\kappa = 0$, then $\mathcal{L}_{kl}^{0}[\nu]$ denotes the vertical distance of the interpolation points from the straight line defined above etc. If we can select the vertical scaling factors so that

$$\begin{aligned} s_{i,j} \cdot \mathcal{R}_{\mathbb{J}(i,j)}^{\kappa}[\nu] &= s_{i+1,j} \cdot \mathcal{L}_{\mathbb{J}(i+1,j)}^{\kappa}[\nu] \\ s_{i,j} \cdot \mathcal{U}_{\mathbb{J}(i,j)}^{\kappa}[\nu] &= s_{i,j+1} \cdot \mathcal{D}_{\mathbb{J}(i,j+1)}^{\kappa}[\nu] \end{aligned}$$

for $i,\ j,\ \nu \in \mathbb{N}:\ i = 1, \ldots, N-1,\ \ j = 1, \ldots, M-1,\ \ \nu = 1, \ldots, a^{\kappa+1} - 1, \kappa \in \mathbb{N}$, then there exists a continuous function $f\colon [0,1] \times [0,p] \to \mathbb{R}$ that interpolates the given data $\Delta = \{(x_i, y_j, z_{ij}) : i = 1, 2, \ldots, N; j = 1, 2, \ldots, M\}$ and its graph $\{(x, y, f(x, y)) : (x, y) \in [0,1] \times [0,p]\} = A$.

Theorem 1. *Let the above RBIFS be defined by an* irreducible *connection matrix C. Let S be the $N \cdot M \times N \cdot M$ diagonal matrix*

$$S = \operatorname{diag}(|s_{\Phi^{-1}(1)}|, |s_{\Phi^{-1}(2)}|, \ldots, |s_{\Phi^{-1}(NM)}|)$$

with $0 < |s_{ij}| < 1,\ i = 1, \ldots, N$ and $j = 1, \ldots, M$. Suppose that the attractor A of the RBIFS is the graph of a continuous function f that interpolates Δ and that the interpolation points of every interval are not x-collinear or not y-collinear. Then, the box-counting dimension of A is given by

$$D(A) = \begin{cases} 1 + \log_a \lambda, & \text{if } \lambda > a \\ 2, & \text{if } \lambda \leq a \end{cases}$$

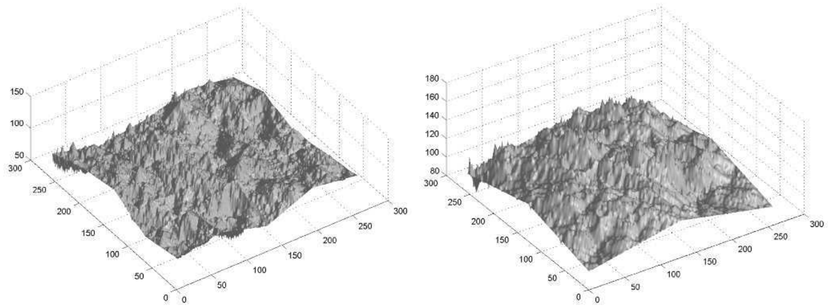
where $\lambda = \rho(SC) > 0$, the spectral radius of the irreducible matrix $S \cdot C$.

More details (and also the proofs of the theorems) may be found in [4].

2.2 Closed Fractal Interpolation Surfaces

A well known (and in many areas useful) system of coordinates are the spherical coordinates. This system is ideal for describing positions on a sphere or spheroid. We let

$$0 \leq \theta < 2\pi, \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}, \quad r > 0$$



(a) (b)

Fig. 1. Two fractal interpolation surfaces constructed according to conditions described in proposition 1. The first (a) (where $N = M = 8$, $K = L = 4$) has box counting dimension 2.2769 and the second (b) (where $N = M = 4$, $K = L = 2$) 2.3325.

and define $\boldsymbol{g} = (g_1, g_2, g_3)$ to be the transformation from spherical coordinates to cartesian coordinates, where

$$x = g_1(\theta, \phi, r) = r \cos\phi \cos\theta$$
$$y = g_2(\theta, \phi, r) = r \cos\phi \sin\theta$$
$$z = g_3(\theta, \phi, r) = r \sin\phi.$$

We can construct a closed fractal surface using the next theorem.

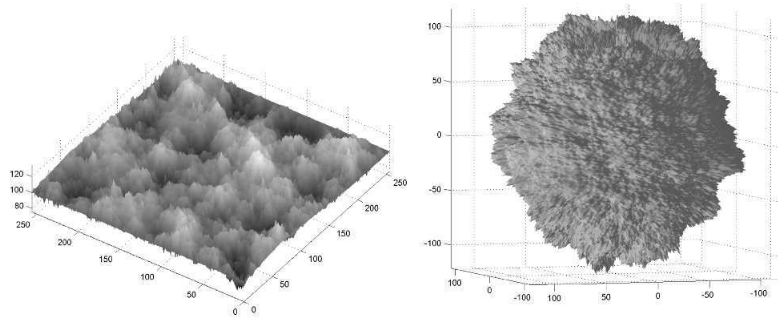
Theorem 2. *Consider a set of equidistant interpolation points*

$$\Delta_S = \{(\theta_i, \phi_j, r_{ij}) : \ i = 0, 1, \ldots, N; \ j = 0, 1, \ldots, M\}$$

given in spherical coordinates, such that $\theta_0 = 0$, $\theta_N = 2\pi$, $\theta_i - \theta_{i-1} = \delta$, $\phi_0 = -\frac{\pi}{2}$, $\phi_M = \frac{\pi}{2}$, $\phi_j - \phi_{j-1} = \delta$. *Consider, also,* $Q_S \subset \Delta_S = \{(\hat{\theta}_k, \hat{\phi}_l, \hat{r}_{kl}) : \ k = 0, 1, \ldots, K; \ l = 0, 1, \ldots, L\}$ *such that* $\hat{\theta}_0 = 0$, $\hat{\theta}_N = 2\pi$, $\hat{\theta}_i - \hat{\theta}_{i-1} = \psi$, $\hat{\phi}_0 = -\frac{\pi}{2}$, $\hat{\phi}_M = \frac{\pi}{2}$, $\hat{\phi}_j - \hat{\phi}_{j-1} = \psi$ *and a labelling map* $\mathbb{J}$ *as defined above. the points are taken such that they obey the conditions of proposition 1. Let* G *be the graph of the function* $r(\theta, \phi)$ *which arises as the attractor of this RIFS. Then* $\boldsymbol{g}(G)$ *is a continuous closed fractal interpolation surface if and only if the following conditions apply.*

1. $r_{i,0} = r_{i,M} = R, \ i = 0, 1, \ldots, N.$
2. $r_{0,j} = r_{N,j}, \ j = 0, 1, \ldots, M.$
3. *The contraction factors are chosen such that* $G \subset [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [\epsilon, +\infty)$, *for given* $\epsilon > 0$.

The details of this construction along with some results on the Hausdorff dimension of the closed surfaces are given in [3].



(a) (b)

Fig. 2. The fractal surface (a) (spherical coordinates) interpolates 9×9 points and it is transformed through $\boldsymbol{g}$ into a closed fractal interpolation surface (b) (cartesian coordinates).

2.3 Vector Valued Fractal Interpolation Functions

Consider the set $\{(x_n, y_m) = \boldsymbol{x}_{n,m} : n = 0, \ldots, N, m = 0, \ldots, M\} \subseteq [a,b] \times [c,d] = B$, where $a = x_0 < x_1 < \cdots < x_N = b$, $c = y_0 < y_1 < \cdots < y_M = d$ and the data $\Delta = \{(x_n, y_m, z_{n,m}, t_{n,m}) = (\boldsymbol{x}_{n,m}, \boldsymbol{z}_{n,m})\} \subseteq B \times \mathbb{R}^2 \subseteq \mathbb{R}^4$. The aim is to construct a continuous vector valued function $\boldsymbol{f} = (f_1, f_2) : B \to \mathbb{R}^2$ such that $\boldsymbol{f}(\boldsymbol{x}_{n,m}) = \boldsymbol{z}_{n,m}$, $n = 0, \ldots, N$, $m = 0, \ldots, M$ (i.e. $\boldsymbol{f}$ be an interpolation function for the data Δ) and such that its graph be the attractor of an interpolation function system. We define $\boldsymbol{w}_{n,m} : [a,b] \times [c,d] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$:

$$\begin{aligned}
\boldsymbol{w}_{n,m}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} &= \begin{pmatrix} a_n x + b_n \\ c_m y + d_m \\ e_{n,m}x + f_{n,m}y + g_{n,m}xy + s_{n,m}z + s'_{n,m}t + k_{n.m} \\ \tilde{e}_{n,m}x + \tilde{f}_{n,m}y + \tilde{g}_{n,m}xy + \tilde{s}_{n,m}z + \tilde{s}'_{n,m}t + \tilde{k}_{n,m} \end{pmatrix} \\
&= \begin{pmatrix} \phi_n(x) \\ \psi_m(y) \\ \Phi_{n,m}(x,y) + s_{n,m}z + s'_{n,m}t \\ \tilde{\Phi}_{n,m}(x,y) + \tilde{s}_{n,m}z + \tilde{s}'_{n,m}t \end{pmatrix} + \boldsymbol{c}_{n,m}.
\end{aligned}$$

The constants of $s_{n,m}, s'_{n,m}, \tilde{s}_{n,m}, \tilde{s}'_{n,m}$ are fixed. The remaining constants are defined by the equations

$$\begin{aligned}
\boldsymbol{w}_{n,m}\begin{pmatrix} \boldsymbol{x}_{0,0} \\ \boldsymbol{z}_{0,0} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{x}_{n-1,m-1} \\ \boldsymbol{z}_{n-1,m-1} \end{pmatrix} \quad \boldsymbol{w}_{n,m}\begin{pmatrix} \boldsymbol{x}_{N,0} \\ \boldsymbol{z}_{N,0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{n,m-1} \\ \boldsymbol{z}_{n,m-1} \end{pmatrix} \\
\boldsymbol{w}_{n,m}\begin{pmatrix} \boldsymbol{x}_{0,M} \\ \boldsymbol{z}_{0,M} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{x}_{n-1,m} \\ \boldsymbol{z}_{n-1,m} \end{pmatrix} \quad \boldsymbol{w}_{n,m}\begin{pmatrix} \boldsymbol{x}_{N,M} \\ \boldsymbol{z}_{N,M} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{n,m} \\ \boldsymbol{z}_{n,m} \end{pmatrix}
\end{aligned} \tag{4}$$

for $n = 1, \ldots, N$, $m = 1, \ldots, M$ and they depend on the data Δ and the contraction factors $s_{n,m}, s'_{n,m}, \tilde{s}_{n,m}, \tilde{s}'_{n,m}$ (where $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\boldsymbol{z} = \begin{pmatrix} z \\ t \end{pmatrix}$). The conditions for this construction to be valid are given in [6].

2.4 Image Compression

One may use RBIFSs to approximate any discrete natural surface, as complicated as may be, using the methodology described below. This methodology is based on ideas similar to the ones presented by Mazel and Hayes in [12], where affine FIFs where used to model single-valued discrete sequences.

Proposition 2 gives the general conditions that the interpolation points and the contraction factors must satisfy so that the attractor of the corresponding RIFS be a continuous function. If the interpolation points are collinear on the boundary of each interval J_{kl}, then any selection of the contractivity factors s_{ij} will be sufficient (Proposition 1). On the other hand, if the interpolation points are not collinear on the boundary of each interval J_{kl}, then it is almost impossible to find contractivity factors that satisfy the conditions of Proposition 2 for an arbitrary selection of interpolation points. However, for $\kappa = 0$ it is relatively easy to find contractivity factors such that

$$\begin{aligned} s_{i,j} \cdot \mathcal{R}^0_{\mathbb{J}(i,j)}[\nu] &= s_{i+1,j} \cdot \mathcal{L}^0_{\mathbb{J}(i+1,j)}[\nu] \\ s_{i,j} \cdot \mathcal{U}^0_{\mathbb{J}(i,j)}[\nu] &= s_{i,j+1} \cdot \mathcal{D}^0_{\mathbb{J}(i,j+1)}[\nu] \end{aligned}$$

for $i = 1, \ldots, N-1$, $j = 1, \ldots, M-1$, $\nu = 1, \ldots, a-1$, or

$$|s_{i,j} \cdot \mathcal{R}^0_{\mathbb{J}(i,j)}[\nu] - s_{i+1,j} \cdot \mathcal{L}^0_{\mathbb{J}(i+1,j)}[\nu]| < \varepsilon \tag{5}$$

$$|s_{i,j} \cdot \mathcal{U}^0_{\mathbb{J}(i,j)}[\nu] - s_{i,j+1} \cdot \mathcal{D}^0_{\mathbb{J}(i,j+1)}[\nu]| < \varepsilon \tag{6}$$

for $\varepsilon > 0$ (relatively small). In this case the attractor of the corresponding RIFS is not graph of a continuous surface. Instead, the attractor is a compact subset of $[0,1] \times [0,p] \times \mathbb{R}$ that approximates a continuous surface.

Now, consider the data set $D = \{(n, m, f(n,m)) : n = 0, 1, \ldots, N^*; m = 0, 1, \ldots, M^*\}$ representing points of an arbitrary surface. Our goal is to choose interpolation points and contractivity factors such that the attractor of the corresponding RIFS approximates the surface. We choose $\delta, \psi \in \mathbb{N}$ a priori and form the sections I_{ij} and the intervals J_{kl}, so that each section contains $(\delta+1) \times (\delta+1)$ points of D and each interval $(\psi+1) \times (\psi+1)$ points of D. For each section I_{ij} we seek the best-mapped interval J_{kl} with respect to a metric h, using bivariate mappings. We compute the contractivity factor of that mapping with a methodology that takes into account that bivariate mappings have the property to map vertical lines (parallel to zz') to vertical lines scaled by the vertical scaling factor s. Since the contractivity factor has been chosen, the remaining parameters of the bivariate map w are computed easily and the set

$w(\{(x, y, f(x, y)) : (x, y) \in J_{kl}\}$ is formed. Then we compute the distance between the sets $\{(x, y, f(x, y)) : (x, y) \in I_{i,j}\}$ and $w(\{(x, y, f(x, y)) : (x, y) \in J_{kl}\}$ and repeat the procedure for every interval. If the contractivity factor that has been computed does not satisfy conditions (5-6) or is greater than 1, we remove the corresponding interval from the search pool. Finally, we choose the interval that minimizes the previously mentioned distance. If this distance, however, is greater than an error tolerance value (chosen a priori) we split the section to four subsections (adding new interpolation points) and repeat the procedure for each new subsection.

The collage theorem for RIFSs (see [2]) ensures that the attractor of the emerging RIFS will approximate the original surface f. The methodology is described in detail in [5], where it is successfully used to model and compress grey-scale images (see Figure 3). The compression is achieved by storing only the map parameters of the RIFS (interpolation points, contractivity factors and connection vector) instead of all the pixel values (i.e. the set D) of the image. The examination of the conditions (5-6) for each selected interval J_{kl} and corresponding contractivity factor significantly improves the speed of the procedure, as a lot of the intervals are removed from the search pool. In addition, the use of bivariate mappings instead of affine ones improves the quality of the reconstructed surface. In [5] the proposed fractal interpolation approach is compared to the previously presented fractal methods and found to give more satisfactory results.

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Fig. 3. Any grey-scale image can be viewed as a (discrete) surface. In (a) the reference image "Lena" is shown. The attractors of RIFSs that approximate the "Lena" surface (for different values of δ, ψ and error tolerance) are shown in (b),(c),(d).

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