Computing the Newton Polytope of Specialized Resultants

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Abstract

We consider sparse (or toric) elimination theory in order to describe, by combinatorial means, the monomials appearing in the (sparse) resultant of a given overconstrained algebraic system. A modification of reverse search allows us to enumerate all mixed cell configurations of the given Newton polytopes so as to compute the extreme monomials of the Newton polytope of the resultant. We consider specializations of the resultant to a polynomial in a constant number of variables (typically up to 3) and propose a combinatorial algorithm for computing its Newton polytope; our algorithm need only examine the silhoutte of the secondary polytope with respect to an orthogonal projection in a space of as many dimensions. We describe the Newton polygon of the implicit equation of a rational parametric curve in a self-contained manner by purely combinatorial arguments; the complexity of our method is almost linear in the cardinality of the supports of the parametric polynomials. We extend certain of these results to describing the Newton polytope of the implicit equation of a polynomial parametric surface.

Classification: Algebraic geometry, Discrete geometry.

Keywords: Sparse (toric) resultant, implicitization, triangulation, secondary polytope, Newton polytope, mixed subdivision, convex geometry.

1 Extended abstract

Consider the resultant of an overconstrained system of polynomial equations with corresponding Newton polytopes $P_0, \dots, P_n \subset \mathbb{R}^n$. The (sparse) resultant is a polynomial in all coefficients of an algebraic system of $n+1$ polynomials in n variables which are to be eliminated. We use sparse (toric) elimination theory in order to exploit the sparseness of the polynomials. In many applications, the coefficients are themselves polynomials in a few parameters, and we wish to compute the resultant as a polynomial in these

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parameters. This paper studies the problem of computing the Newton polytope of interesting specializations of the (sparse) resultant.

There exists a surjection between the mixed cell configurations of the Minkowski sum $P = P_0 + \ldots + P_n$ and the extreme monomials. By means of the Cayley trick the problem of enumerating all regular tight mixed subdivisions of P is reduced to enumerating regular triangulations. The latter are in bijection with the vertices of the secondary polytope. We briefly examine the problem of describing, by purely combinatorial means, the Newton polytope of the (sparse) resultant in all symbolic coefficients. We concentrate on mixed-cell configurations following [13], instead of considering all mixed subdivisions. Then, we examine the Newton polytope of the resultant after specializing all but a few (typically one to three) coefficients. Our motivation comes from the u-resultant, which is used to compute all real roots of a system by the primitive-element method [4], see also the Rational Univariate Representation [14].

Another motivation is to describe, without any algebraic computation, the Newton polytope of the implicit equation of a parameterized curve or surface. Implicitization is a crucial problem in geometric applications and can be reduced to linear algebra, once the implicit support is known. Our presentation is self-contained and examines triangulations of point sets in the plane, for the case of rational parametric curves. This allows us to give discrete algorithms that do not rely on any symbolic computation, with complexity linear in the cardinality of the support of the parametric polynomials. We also specify certain coefficients in the implicity equation. We extend certain of these results to describing the Newton polytope of the implicit equation of a polynomial parametric surface.

1.1 Previous work

The most closely related work is in [8, 9], where sparse elimination is applied to predict the Newton polytope of any implicit equation. That method had to compute all mixed subdivisions by enumerating all vertices on the secondary polytope. Our paper improves that approach by focusing on mixed-cell configurations and the vertices on the silhouette of the secondary polytope.

More recently, [18] offered algorithms to compute the Newton polytope of the implicit equation of any hypersurface parameterized by Laurent polynomials. Their approach is based on tropical geometry and the ensuing algorithms rely solely on combinatorial geometry and linear algebra. It covers arbitrary implicit ideals thus including the object of our study which is principal implicit ideals. On the other hand, our approach handles rational parameterizations hence supercedes Laurent parameterizations.

In [11], the extreme terms of the Sylvester resultant are described. Here, we emphasize on giving a self-contained and straightforward description of the Newton polygon of implicit curves by exploiting the theory of sparse elimination.

1.2 Main results

We describe an algorithm that enumerates only the vertices of the secondary polytope corresponding to mixed-cell configurations thus allowing us to compute efficiently the extreme monomials in all symbolic coefficients. The algorithm combines ideas from [13, 12] and uses reverse search techniques for space efficiency.

We offer an algorithm that enumerates only the vertices on the silhouette of the secondary polytope, instead of computing all of its vertices, when computing the Newton polytope of the resultant under a specialization of all but a few coefficients. This algorithm can be used to attack the problem of implicitization of polynomial parametric curves or surfaces. Second, we give a full description of a polygon containing the Newton polygon of rational parametric curves by an elementary method, and discuss its extension to parametric surfaces. This polygon is optimal if the actual coefficients are sufficiently generic. We are able to determine the coefficients of the extreme monomials in the implicit equation of a curve as a corollary of [11], where the extreme terms in the (classical) resultant of two polynomials are described.

We describe a polygon containing the Newton polygon of the implicit equation of a rational parametric curve. We illustrate our method with the following theorem and an example. The proof of the theorem along with the rest of the results can be found in subsection 5.1.

Consider the parameterization $x = P_0(t)/Q(t), y = P_1(t)/Q(t)$ which leads to polynomials $f_0 = xQ(t) - P_0(t), f_i = yQ(t) - P_1(t)$ in $\mathbb{C}[t]$. The corresponding supports are denoted by $\{a_i\}, \{b_j\} \in \mathbb{Z}$. To bound the upper hull, with respect to direction $(1,1)$, of the implicit polygon, we *select* those points in the supports of f_i which have coefficients in $\mathbb{C}[x]$ or $\mathbb{C}[y]$, respectively. Now, we give an instance of our results for bounding the powers of x, y in the implicit equation.

Theorem 1.1. *(i) The maximum power of x in the implicit equation is $b_m - b_0 = b_m$. When this is attained, the maximum power of y is*

$$(a_R^+ - a_L^+) + \mathcal{X}(b_m^+) \cdot (a_n - a_R^+),$$

where a_R^+, a_L^+ are the rightmost and leftmost selected points (not necessarily distinct) in A_0, and $\mathcal{X}(b_m^+) = 1$ if b_m is selected and $\mathcal{X}(b_m^+) = 0$ otherwise. A similar result holds for y, with the roles of x and y, A_0 and A_1 exchanged.

(ii) If the extreme values of the powers of x and y in case (i) do not coincide to b_m and a_n, respectively, then the upper right corner of the polygon

containing the Newton polygon consists of either a two-edge polygonal line connecting the points having these values as coordinates if none of the four points a_0, b_0, a_n, b_m is selected or just an edge connecting these points otherwise.

A different selection criterion shall lead to a description of the lower hull wrt $(1,1)$ of the implicit polygon.

Example 1.1. For the folium of Descartes ([8, Exam.6.2]) $x = 3t^2/(t^3+1),\ y = 3t/(t^3+1)$ with implicit equation $\phi = x^3 + y^3 - 3xy = 0$, we have $f_0 = xt^3 - 3t^2 + x,\ f_1 = yt^3 - 3t + y$ and supports $A_0 = \{0^+, 2^-, 3^+\},\ A_1 = \{0^+, 1^-, 3^+\}$. The denoted selection is the same under both selection criteria (see subsection 5.1), and satisfies the assumptions of the lemmas relevent for computing the lower hull of the polygon. The set $C = \kappa(A_0, A_1)$ has fourteen triangulations. Our method yields vertices $(3,3), (0,3), (3,0), (1,1)$. By degree bounds we end up with vertices $(0,3), (3,0), (1,1)$ which are optimal. The polygon predicted by degree bounds alone contains the additional vertex $(0,0)$ which leads to a possible implicit support with five more vertices.

We extend our results to describing the Newton polytope of the implicit equation of a polynomial parametric surface.

1.3 Paper structure

The paper is organized as follows. The next section introduces our main concepts from sparse elimination and focuses on the Newton polytope of the sparse resultant. We use the Cayley trick to reduce the computation of its vertices to computing certain triangulations in higher-dimensional space, thus introducing the secondary polytope. Section 3 focuses on the enumeration of mixed cell configurations, whereas section 4 proposes methods for enumerating the vertices on the silhouette of the secondary polytope. Section 5 fully describes the Newton polytope of the implicit equation of rational parametric curves, with a discussion of the problem for parametric surfaces. The appendix contains ancillary results.

2 Sparse Elimination

The central object of study in sparse (or toric) elimination theory is the sparse (or toric) resultant. The sparse resultant depends only on the monomials of the equations with nonzero coefficients therefore for sparse systems it has lower degree than its classical (or projective) counterpart; see [10] for more information.

Definition 2.1. The sparse (or toric) resultant $\mathcal{R}$ of polynomials $f_i \in \mathbb{C}[x_1, \ldots, x_n],\ i = 0, \ldots, n$, is the unique (up to sign) irreducible polynomial in $\mathbb{Z}[c_{i,j}]$, which vanishes iff the f_i have a common root in the corresponding toric variety.

We recall now some crucial notions of sparse elimination theory. The *support* $A(f)$ of a polynomial f is the set of the exponent vectors of its monomials with nonzero coefficients. The *Newton polytope* $N(f)$ of f is the convex hull of its support. The *Minkowski sum* $A+B$ of convex polytopes $A, B \subset \mathbb{R}^n$ is the set $A+B = \{a+b \mid a \in A, b \in B\} \subset \mathbb{R}^n$.

Let $f_0, \ldots, f_n$, be $n+1$ Laurent polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ with symbolic coefficients $c_{i,j}$ and corresponding Newton polytopes $P_0, \ldots, P_n \subset \mathbb{R}^n$. Suppose that the Minkowski sum $P = P_0 + \ldots + P_n \subset \mathbb{R}^n$ is a n-dimensional convex polytope; otherwise, we consider an essential subset of the polytopes.

Definition 2.2. [10, 17] A *tight mixed subdivision* of P, is a collection of n-dimensional convex polytopes R, called (Minkowski) *cells*, st.: (1) They form a polyhedral complex that partitions P, and (2) Every cell R is a Minkowski sum of subsets F_i of P_i:

$$R = F_0 + \cdots + F_n, \; F_i \subset P_i, \quad dim(R) = dim(F_0) + \cdots + dim(F_n) = n.$$

A cell R is called *i-mixed*, or *v_i-mixed*, if it is the Minkowski sum of n one-dimensional segments $E_j \subset P_j$ and one vertex $v_i \in P_i$: $R = E_0 + \cdots + v_i + \cdots + E_n$.

A mixed subdivision is called *regular* if it is obtained as the projection of the lower hull of the Minkowski sum of lifted polytopes $\hat{P}_i := \{(p_i, \omega_i(p_i)) \mid p_i \in P_i\}$. If the lifting function $\omega := \{\omega_i \ldots, \omega_n\}$ is sufficiently generic, the induced mixed subdivision is tight.

Two mixed subdivisions are equivalent if they have the same mixed cells. The equivalence classes are called *mixed cell configurations* [13].

A monomial of the sparse resultant is called *extreme* if its exponent vector is a vertex of the Newton polytope $N(\mathcal{R})$ of the resultant.

Theorem 2.1. *[17] For every sufficiently generic lifting function ω, we obtain an extreme monomial of $\mathcal{R}$, of the form*

$$\pm \cdot \prod_{i=0}^{n} \prod_{R} c_{i,v_i}^{\mathrm{Vol}(R)}, \tag{1}$$

where the second product is over all i-mixed cells R of the regular tight mixed subdivision of $P = \sum_{i=0}^{n} P_i$, induced by ω, and c_{i,v_i} is the coefficient of the monomial of f_i corresponding to vertex v_i.

Corollary 2.2. *There exists a surjection between the mixed cell configurations and the extreme monomials of the sparse resultant.*

Theorem 2.1 reduces the problem of computing the extreme monomials of the sparse resultant to the problem of computing all mixed cell configurations of the Minkowski sum of the Newton polytopes. Given supports $A_0, \ldots, A_n$, the Cayley embedding κ introduces a new point set

$$C := \kappa(A_0, A_1, \ldots, A_n) = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{R}^{2n+1},$$

where e_i are an affine basis of $\mathbb{R}^n$. The dimension of the convex hull of C is $d \leq 2n$.

Theorem 2.3. [The Cayley Trick] [13, 15] *There exists a bijection between the regular tight mixed subdivisions of the Minkowski sum P and the regular triangulations of C.*

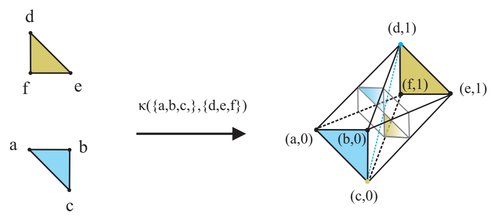


Figure 1: Application of the Cayley Trick for two triangles.

3 Enumeration of Mixed Cell Configurations

The set of regular triangulations of C is very well understood thanks to a bijection to the vertices of a polytope the so called *secondary polytope* $\Sigma(C)$ of C. In particular, for every regular triangulation of C there is a vertex in $\Sigma(C)$, and two vertices in $\Sigma(C)$ are connected by an edge if they can be obtained from each other by a local modification called *bistellar flip*.

Bistellar flips are the generalization of edge flips in the two dimensional case. They are based on certain subsets of C called circuits.

We now switch to a more general notation. Let $A \subset \mathbb{R}^d$ be a set of points and $\mathcal{T}_1$ a regular triangulation of A. A circuit $Z = \{z_1, \ldots, z_k\}$ is a minimal affinely dependent subset of A, satisfying a unique (up to a constant) affine equation $\lambda_1 z_1 + \ldots + \lambda_k z_k = 0$ where all λ_i are non zero and $\sum \lambda_i = 0$. Z can be written in the form $Z = (Z_+, Z_-)$, where $Z_+ = \{z_i \mid \lambda_i > 0\}$ and $Z_- = \{z_i \mid \lambda_i < 0\}$ This is usually called Radon's property.

A circuit Z has exactly two triangulations $\mathcal{T}_+^Z = \{Z \setminus \{z_i\} \mid z_i \in Z_+\}$ and $\mathcal{T}_-^Z = \{Z \setminus \{z_i\} \mid z_i \in Z_-\}$.

The link of a set $\sigma \subset A$ in a triangulation $\mathcal{T}$ of A is defined as

$$\text{link}_{\mathcal{T}}\sigma := \{\rho \subset A \mid \rho \cap \sigma = \emptyset, \rho \cup \sigma \in \mathcal{T}\}.$$

Now the bistellar flip on Z can be defined as follows:

Definition 3.1. Let $\mathcal{T}_1$ be a triangulation of A that contains one of the two triangulations of Z, say $\mathcal{T}_+^Z$. Suppose that all the cells $\sigma \in \mathcal{T}_+^Z$ have the same link L in $\mathcal{T}_1$. Then the circuit Z supports a flip in $\mathcal{T}_1$ which gives the triangulation $\mathcal{T}_2$:

$$\mathcal{T}_2 := \mathcal{T}_1 \setminus \{\rho \cup \sigma \mid \rho \in L, \sigma \in \mathcal{T}_+^Z\} \cup \{\rho \cup \sigma \mid \rho \in L, \sigma \in \mathcal{T}_-^Z\}.$$

If all the cells $\sigma \in \mathcal{T}_+^Z$ do not have the same link L in $\mathcal{T}_1$, then the circuit Z does not support a flip in $\mathcal{T}_1$.

The following theorem allows us to explore the set of regular triangulations of a point set using bistellar flips.

Theorem 3.1. *[10] For every set A of points affinely spanning $\mathbb{R}^d$ there is a polytope $\Sigma(A)$ in $\mathbb{R}^{|A|-d-1}$ such that its vertices correspond to the regular triangulations of A and there is an edge between two vertices if and only if the two corresponding triangulations are obtained one from the other by a bistellar flip.*

There are two standard methods to construct the secondary polytope of a point set A. The first one, due to Gelfand, Kapranov and Zelevinskii [10], gives for each triangulation $\mathcal{T}$ of A (not necessarily regular), coordinates:

$$(v_{\mathcal{T}})_i = \sum_{\sigma:\sigma\in T, i\in \text{Vert}(\sigma)} \text{Vol}(\sigma), \quad i = 1, \ldots, |A|.$$

The $|A|$-dimensional vector $v_{\mathcal{T}}$ corresponding to every triangulation $\mathcal{T}$ of A, is called the *volume vector* of $\mathcal{T}$. Then, $\Sigma(A) \subset \mathbb{R}^{|A|}$ is defined as the convex hull of all the volume vectors. Volume vectors of triangulations that are not regular fall into the interior of some face of $\Sigma(A)$. However, the secondary polytope constructed this way is not full-dimensional but resides in an $(|A| - d - 1)$ - dimensional subspace.

The second method, due to Billera and Sturmfels [2], describes the secondary polytope as the Minkowski integral of the fibers of the affine projection $\pi : \Delta_A \to \text{conv}(A)$, where Δ_A is a simplex with $|A|$ vertices of dimension $|A| - 1$, and π bijects the vertices of Δ_A to A.

Figure 2: Secondary polytope of a quadrilateral.

We can enumerate all regular triangulations of the set C indroduced by the Cayley embedding κ (see section 2), by computing a spanning tree of the secondary polytope $\Sigma(C)$. For efficiency we use reverse search techniques as proposed in [12].

Following [13], we exploit the surjection from the mixed-cell configurations onto the vertices of the Newton polytope of the resultant, and enumerate only a subset of the vertices of $\Sigma(C)$.

Bistellar flips can be defined over the mixed-cell configurations. Enumeration of mixed-cell configurations is based on circuits of a regular triangulation with certain properties; these are characterized in [13]. They are called odd and even circuits, based on their cardinality as tuplets of subsets of the initial supports A_i. The simplices of these circuits are images, or subsets of images when they are not full dimensional, via κ of mixed cells of the Minkowski sum of the supports A_i. We allow bistellar flips only on these circuits.

The algorithm runs in time $O(D^2 s^2 LP(|C| - D - 1, s)|R|)$ and space $O(Ds)$, where $D = 2n + 1$, s is the number of any dimensional simplices in a triangulation of C, and $|R|$ is the number of mixed-cell configurations, see [12] for details.

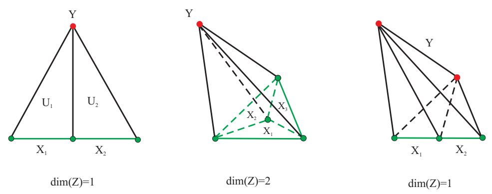


Figure 3: Odd circuits (left and right figures), and a non suitable circuit.

4 Enumerating silhouettes of $\Sigma(C)$

Applications such as the computation of the u-resultant or implicitization of polynomial parametric curves or surfaces call for the computation of the Newton polytope of the resultant after a specialization of some coefficients. This is equivalent to enumerating the vertices lying on the silhouette of the secondary polytope $\Sigma(C)$ with respect to some suitably defined projection. For example, the projection of $\Sigma(C)$ to $\mathbb{R}^2$ solves the problem of implicitization of polynomial curves, the projection to $\mathbb{R}^3$ the one of polynomial surfaces etc. Interestingly, the approache of this section and of section 5 give the same result for the case of polynomial parametric curves, althought they use different criteria, the first based on volume vectors and the latter on mixed volumes. The silhouette can be obtained naively by computing all the vertices of $\Sigma(C)$ corresponding to mixed cell configurations and then projecting them to the subspace of smaller dimension. For efficiency we want to enumerate only a subset of the previous vertices lying on a silhouette of $\Sigma(C)$ with respect to a projection to be defined by the problem.

4.1 The projection of $\Sigma(C)$ in dimension one.

Suppose that we project $\Sigma(C)$, of dimension D, to a line by deleting all coordinates except the first one $(v_{\mathcal{T}})_1$. Then the convex hull of the projection of $\Sigma(C)$ has only two vertices $v_{\mathcal{T}_{max}}, v_{\mathcal{T}_{min}}$ corresponding to the triangulations $\mathcal{T}_{max}$ and $\mathcal{T}_{min}$ of the secondary polytope which maximize and minimize coordinate $(v)_1$ respectively. Translating the problem to its algebraic counterpart, we wish to specialize all but one coefficient appearing in a single monomial. The Newton polytope of the specialized resultant is a possible degenerate segment. We utilize the algorithm of section 3 modified so as to apply the following criteria:

Lemma 4.1. [Maximization criterion] *Let $\mathcal{T}_{init}$ be a triangulation of C, $Z \subset C$ a circuit of $\mathcal{T}_{init}$ supporting a flip and $a_1 \in C$. Suppose that the induced triangulation $\mathcal{T}_{init}^Z$ of Z has simplices $\sigma_i,\ i \in I$. Then the following criterion decides if flipping on circuit Z, leads to a new triangulation $\mathcal{T}$ satisfying $(v_{\mathcal{T}})_1 > (v_{\mathcal{T}_{init}})_1$* (R)*:*

$$\forall i \in I\ [a_1 \in \mathrm{Vert}(\sigma_i)] \Longleftrightarrow \neg \mathrm{R}$$

Proof. Let T_{init}^Z, T^Z be the triangulations of Z induced by $\mathcal{T}_{init}$ and T respectively, L the common link of all simplices σ_i, and $\sigma_j',\ j \in J$ the simplices of T^Z.

Suppose that $\forall i \in I\ [a_1 \in \mathrm{Vert}(\sigma_i)]$. This implies that $\forall i \in I\ [a_1 \in \mathrm{Vert}(\sigma_i \cup \rho)],\ \rho \in L$ and thus $(v_{\mathcal{T}_{init}})_1 = \sum_{\forall i \in I, \rho \in L} \mathrm{Vol}(\mathrm{CH}(\sigma_i \cup \rho)) = \mathrm{Vol}(\mathrm{CH}(Z \cup L))$. Since there is a unique triangulation of Z such that a_1 is a vertex of all its simplices, the conclusion follows.

Now suppose that there is a simplex $\sigma_k,\ k \in I$ such that a_1 is not a vertex of it. Then a_1 is not a vertex of $\sigma_k \cup \rho,\ \rho \in L$ and $(v_{\mathcal{T}_{init}})_1 = \sum_{\forall i \in I \setminus k, \rho \in L} \mathrm{Vol}(\mathrm{CH}(\sigma_i \cup \rho)) < \mathrm{Vol}(\mathrm{CH}(Z \cup L)) = (v_{\mathcal{T}})_1$. ☐

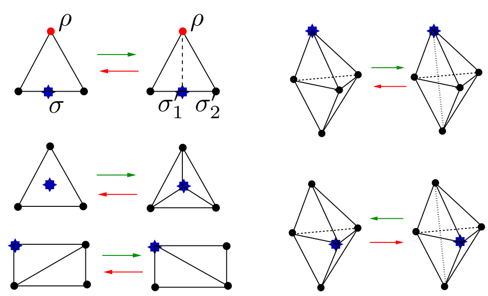


Figure 4: Application of the maximization criterion, a_1 is star shaped vertex.

The previous lemma allows us to compute a set of candidate circuits with the property that flipping on each one of them increases coordinate $(v)_1$ of the volume vector. Now we wish to choose among the candidate circuits, the one with maximum increase in $(v)_1$ coordinate.

Lemma 4.2. *Let* $Z_1, \ldots, Z_s$ *be a set of circuits satisfying* (R) *with links* $L_1, \ldots, L_s$ *respectively and* $\mathcal{T}_1, \ldots, \mathcal{T}_s$ *the corresponding triangulations obtained by perfoming a bistellar flip on them. Then the triangulation* $\bar{\mathcal{T}}$ *with* $(v_{\bar{\mathcal{T}}})_1 = \max\{(v_{\mathcal{T}_j})_1 \mid j = 1, \ldots, s\}$ *is the one obtained by perfoming a bistellar flip on circuit* $\bar{Z}$ *such that*

$$\mathrm{Vol}(\mathrm{CH}(\bar{Z} \cup \bar{L})) = \max\{\mathrm{Vol}(\mathrm{CH}(Z_j \cup L_j)) \mid j = 1, \ldots, s\},$$

where $\mathrm{Vol}(\mathrm{CH}(Z \cup L)) = \sum_{\sigma \in \mathcal{T}^Z, \rho \in L} \mathrm{Vol}(\mathrm{CH}(\sigma \cup \rho))$.

Proof. A flip on one of the candidate circuits Z_j in the triangulation $\mathcal{T}_{init}$ results in a new triangulation $\mathcal{T}_j$, in which point a_1 is a vertex of all the simplices in the induced triangulation $\mathcal{T}_j^{Z_j}$ of Z_j. This suggests that the best circuit among all candidates is the circuit with maximum associated volume. □

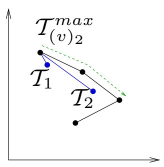
The previous results can be modified accordingly to provide the vertex $\hat{\mathcal{T}}$ with minimum $(v)_1$-coordinate among all the neighbours of vertex $\mathcal{T}_{init}$. or the vertex with minimum increase/decrease in $(v)_1$-coordinate.

In order to compute vertices $\mathcal{T}_{max}$ and $\mathcal{T}_{min}$, we have to apply these results to an initial triangulation $\mathcal{T}_{init}$. Depending on the setting, the output of these criteria might be the empty set. This means that relation (R) (or its analogous one for the case of minimization) does not hold for every adjacent vertex of $\mathcal{T}_{init}$. In such a case, the next vertex to be enumerated is decided by the criteria of the initial algorithm of section 3. The space and time complexities of the new algorithm are the same with those of section 3, but for settings where there exists a path from $\mathcal{T}_{init}$ to $\mathcal{T}_{max}$, $\mathcal{T}_{min}$ consisting of vertices with an absolutely monotonic sequence of $(v)_1$-coordinates, then the projection algorithm provides the shortest path.

4.2 The projection of $\Sigma(C)$ in two and three dimensions.

Suppose that we project $\Sigma(C)$ of dimension D, to the plane defined by the first two coordinates $(v)_1, (v)_2$. Initially we apply the criteria of the previous section in order to find the vertices of $\Sigma(C)$ that are extreme with respect to each of the variables we project to; thus we can compute triangulations $\mathcal{T}_{(v)_1}^{max}, \mathcal{T}_{(v)_1}^{min}, \mathcal{T}_{(v)_2}^{max}$ and $\mathcal{T}_{(v)_2}^{min}$. Now we have to compute the vertices that fill the rest of the silhouette of $\Sigma(C)$. Unfortunately every combination of our criteria is not sufficient for this, as it is illustrated by the following example:

Example 4.1. Starting with a triangulation $\mathcal{T}_{init}$ we compute the vertex $\mathcal{T}^{max}_{(v)_2}$ with maximum $(v)_2$-coordinate. From $\mathcal{T}^{max}_{(v)_2}$ we wish to flip towards vertex $\mathcal{T}^{max}_{(v)_1}$ with maximum $(v)_1$-coordinate. If the circuits supporting a flip in $\mathcal{T}^{max}_{(v)_2}$ are $Z_1, \ldots, Z_k$, then we have to choose on which circuit to flip in order to compute the next vertex of $\Sigma(C)$ that lies on the silhouette. In the setting shown in figure 5, there is no combination of our criteria applied to $(v)_1$ and $(v)_2$ that can characterize the vertex under consideration.



$$\max_{(v)_2}\{\mathcal{T} \mid (v)_1^{\mathcal{T}} > (v)_1^{\mathcal{T}_{init}}\} = \mathcal{T}_1$$
$$\min_{(v)_2}\{\mathcal{T} \mid (v)_1^{\mathcal{T}} > (v)_1^{\mathcal{T}_{init}}\} = \mathcal{T}_2$$

Figure 5: A case where every combination of combinatorial criteria fails.

We can overcome this difficulty by switching from combinatorial to geometric criteria. In particular, we utilize the well known CCW determinant [7], which decides the relative orientation of any three points in a plane. Suppose that $\mathcal{T}_1, \ldots, \mathcal{T}_r$ are the neighbours of vertex $\mathcal{T}^{max}_{(v)_2}$, with greater $(v)_1$ coordinate. A vertex $\mathcal{T}_i$ for which $\mathrm{CCW}(\mathcal{T}^{max}_{(v)_2}, \mathcal{T}_i, \mathcal{T}_j)$ holds, for some $\mathcal{T}_j \in \{\mathcal{T}_1, \ldots, \mathcal{T}_k\}$, cannot lie on the silhouette. This is essentially Jarvis' algorithm for computing the Convex Hull of points in the plane; it is an instance of the gift-wrapping paradigm [7].

In general, we follow the gift-wrapping algorithm. To fully describe it we need to compute the first edge, and to specify the procedure for discovering new edges. The former can be done by applying the above methods for projecting to a line. Specifically, by applying the combinatorial criteria from the one dimensional projection, we can obtain from the set of all adjacent vertices to $\mathcal{T}_{init}$, the ones with greater $(v)_1$-coordinate. Then we apply Jarvis' algorithm to this set of vertices to obtain the next vertex on the silhouette. For moving to neighbouring edges, we use the previous discussion of selecting candidates and selecting among them by use of CCW. This is a well-known algorithm with output sensitive complexity. It suffers from high space requirements; for this, we can use reverse search to minimize memory consumption [1, 12].

The previous discussion can be generalized for the case where we project to a subspace of dimension three.

5 The Newton polytope of the Implicit equation

Implicitization is the problem of switching from a parametric representation of a hypersurface to an algebraic one, as the zero set of a polynomial equation.

Let $h_0, \dots, h_n \in \mathbb{C}[t_1, \dots, t_r]$ be polynomials in n parameters t_i. The implicitization problem is to compute the prime ideal I of all polynomials $\phi \in \mathbb{C}[x_0, \dots, x_n]$ which satisfy $\phi(h_0, \dots, h_n) \equiv 0$ in $\mathbb{C}[t_1, \dots, t_r]$. We are interested in the case where $r = n$, and h_i are rational expressions in $\mathbb{C}(t_1, \dots, t_n)$. Then $I = \langle \phi \rangle$ is a principal ideal. In this case we have a rational parameterization of a hypersurface defined by

$$x_i = \frac{P_i(\mathrm{t})}{Q(\mathrm{t})}, \ \gcd(P_i(\mathrm{t}), Q(\mathrm{t})) = 1, \quad i = 0, \dots, n, \ \mathrm{t} = (t_1, \dots, t_n). \quad (2)$$

Notice that $\phi \in \mathbb{C}[x_0, \dots, x_n]$ is uniquely defined up to sign. The x_i are called implicit variables and the support of ϕ is the implicit support.

Let us define polynomials $f_i(t) = x_i Q(t) - P_i(t) \in (\mathbb{C}[x_i])[\mathrm{t}]$, and let c_{ij} $(0 \leq j \leq m_i)$, q_i $(0 \leq i \leq k)$ denote the coefficients of polynomials $P_i(\mathrm{t})$ and $Q(t)$ respectively. The support of polynomial f_i is of the form $A_i = \{a_{i0}, \dots, a_{im_i}\} \subset \mathbb{Z}^n$. The assumption $\gcd(P_i(\mathrm{t}), Q(\mathrm{t})) = 1$ implies that all $P_i(t)$ have a nonzero constant term or $Q(t)$ has a nonzero constant term, hence we have that $a_{i0} = 0$ for every i, hence there is always a nonzero constant term in all f_i.

There exist several algorithms for this problem, mostly based on resultants and Groebner bases; see e.g. [3, 5, 6, 8, 9, 18] and references thereof. We focus on computing the Newton polytope of the implicit equation, or implicit polytope under the assumption of generic coefficients relative to the given supports. The motivation is that, knowledge of a good superset of the implicit support reduces the computation of the implicit equation to a a problem in linear algebra; see, e.g. [5, 9, 18].

The work in [18] is based on geometric characterizations of the tropical variety of the prime ideal I. The authors consider Laurent polynomial parameterizations; this is a special case of rational parameterizations, where the denominator $Q(\mathrm{t})$ is a single monomial t^M, M being the largest negative exponent of Laurent monomials. In [8, 9] tools from toric elimination theory lead to an algorithm for obtaining a superset of the implicit support by computing the Newton polytope of the toric resultant by means of theorem 2.1.

The methods of section 4 offer an algorithm to compute the Newton polytope of polynomial parameterizations. For instance, in the case of parametric curves we project upon a_{00}, a_{10}. In this section though, we offer direct methods, in conjuction with the application of degree bounds (cf Prop. 5.1) in order to specify a polytope guaranteed to contain the implicit polytope. Proper containment occurs when the actual coefficients are not sufficiently generic so terms are cancelled; otherwise, the polytope we obtain is optimal. It is typically smaller than the one predicted solely by degree bounds.

Proposition 5.1. *Let $S \subset \mathbb{Z}^n$ be the union of the supports of polynomials f_i. Then, the total degree of the implicit equation ϕ is bounded by the volume* *of the convex hull $CH(S)$ multiplied by $n!$. The degree of ϕ in x_j is bounded by the mixed volume of the f_i, $i \neq j$.*

The polygon constructed below should be intersected with that predicted by the degree bounds of Proposition 5.1.

We now focus on polynomial and rational parametric curves and explicitly describe the implicit polygon. Our method has complexity linear in the cardinality of supports. Then, we extend our approach to polynomial parametric surfaces and offer results leading to an algorithm for the Newton polytope. The polytope we obtain is the same as in [8, 9] but we improve upon the complexity of the latter in all cases.

5.1 Polynomial parametric curves

We consider polynomial parameterizations of curves. In this case $f_0 = x - P_0(t)$, $f_1 = y - P_1(t) \in (\mathbb{C}[x,y])[t]$, and the supports of f_0, f_1 are of the form $A_0 = \{a_0, a_1, \ldots, a_m\}$ and $A_1 = \{b_0, b_1, \ldots, b_n\}$ where points a_i and b_j are sorted in ascending order. Points a_0, b_0 are always equal to zero. The new point set

$$C = \kappa(A_0, A_1) = \{(a_0, 0), \ldots, (a_n, 0), (b_0, 1), \ldots, (b_m, 1)\},$$

introduced by the Cayley embedding κ is a subset of $\mathbb{Z}^2$. For convenience, we shall abuse notation omitting the extra coordinate. The convex hull of the set C is a quadrangle and $\Sigma(C)$ is a polytope in $\mathbb{R}^{m+n}$ of dimension $m+n-3$, usually called associahedron [10, 16]. Every circuit of $\Sigma(C)$ is either even or odd due to the structure of C, and every triangulation of this set is regular, and corresponds to a mixed cell configuration of $A_0 + A_1$.

The resultant $\mathcal{R}(f_0, f_1, t)$ is a polynomial in $(\mathbb{C}[x,y])[c_{ij}]$. We consider the specialization of coefficients c_{ij} in the resultant in order to study the implicit equation of the curve; generically, this specialization yields the implicit equation. The Newton polytope of the implicit equation is a subset of $\mathbb{Z}^2$. Its vertices, as indicated by theorem 2.1, are obtained from those extreme monomials of $\mathcal{R}(f_0, f_1, t)$ which are associated with points a_0 and b_0. Since every triangle of a triangulation $\mathcal{T}$ of C corresponds to a mixed cell of a mixed subdivision of $A_0 + A_1$, we can rewrite relation (1) as:

$$\pm \prod_{i=0}^{1} \prod_{R} c_{i,p}^{\mathrm{Vol}(R)}, \tag{3}$$

where R is an i-mixed cell with vertex $p \in A_i$ and $c_{i,p}$ is the coefficient of the monomial with exponent p.

After the specialization of the coefficients of f_0, f_1, the terms of (3) associated with mixed cells having a vertex p other than a_0, b_0 contribute only a coefficient to the corresponding term of the implicit equation. This implies that the only mixed cells that we need to consider are the ones with vertex

a_0 or b_0 (or both). For any triangulation $\mathcal{T}$, these mixed cells correspond to triangles with vertices a_0, b_l, b_r where $l, r \in \{0, \ldots, m\}$, or b_0, a_l, a_r, where $l, r \in \{0, \ldots, n\}$.

The following lemmas determine the polytope containing the Newton polytope of the implicit equation.

Lemma 5.2. *If P_0 or P_1 (or both) contain a constant term, then the Newton polygon of the implicit equation is the triangle with vertices $(0,0), (b_m, 0), (0, a_n)$.*

Proof. To compute vertices $(b_m, 0)$ and $(0,0)$ consider the triangulation $\mathcal{T}$ of C obtained by drawing edge (a_0, b_m). The only 0-mixed cell with vertex a_0 corresponding to $\mathcal{T}$ is $R = a_0 + (b_0, b_m)$ with volume equal to b_m; there are no 1-mixed cells with vertex b_0. The etxreme monomial associated with such a triangulation is of the form $(x - c_{00})^{b_m} c_{1m}^{a_m}$, which after specializing c_{00}, c_{1m} and expanding gives monomials in x with exponents $b_m, b_{m-1}, \ldots, 0$.

For vertex $(0, a_n)$ consider the triangulation $\mathcal{T}'$ obtained by drawing edge (b_0, a_n). The only 1-mixed cell with vertex b_0 corresponding to $\mathcal{T}'$ is $R = b_0 + (a_0, a_n)$ with volume equal to a_n; there are no 0-mixed cells with vertex a_0. The etxreme monomial associated with this triangulation is of the form $(y - c_{10})^{a_n} c_{0n}^{b_m}$, which after specializing c_{10}, c_{0n} and expanding gives monomials in y with exponents $a_n, a_{n-1}, \ldots, 0$.

To complete the proof it suffices to observe that every triangulation of C having edges of the form (a_0, b_j), $0 < j < m$ and (b_j, a_i), $i > 0$ leads to an extreme monomial which specializes to a polynomial in x of the implicit equation of degree b_j. Therefore we obtain monomials of the implicit equation with exponents $(b_j, 0), \ldots, (0,0)$ which all lie in the interior of the triangle.

Similarly, every triangulation of C having edges of the form (b_0, a_i), $0 < i < n$ and (a_i, b_j), $j > 0$ leads to an extreme monomial which specializes to a polynomial in y of the implicit equation of degree a_i. Therefore we obtain monomials of the implicit equation with exponents $(0, a_i), \ldots, (0,0)$ which all lie in the interior of the triangle (see Figure (6). $\square$

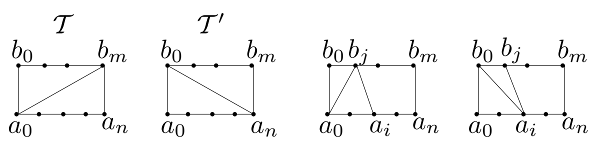


Figure 6: The triangulations of C that give the vertices of the Newton polytope of the implicit equation.

Lemma 5.3. *If P_0, P_1 contain no constant terms, then the Newton polygon of the implicit equation is the quadrilateral with vertices $(b_1, 0), (b_m, 0)$, $(0, a_n), (0, a_1)$.*

Proof. If c_{00}, c_{10} are both equal to zero, then the extreme monomials associated with points a_0 and b_0 are specialized to monomials of the implicit equation in x or y respectively thus not producing any constant terms which would imply that point $(0,0)$ is a vertex of the Newton polytope of the implicit equation. The proof of the previous lemma implies that, when $y = 0$, the smallest exponent of x is b_1, which is obtained by a triangulation containing edges (a_0, b_1) and (b_1, a_i), $i > 0$. Similarly, the smallest exponent of y is a_1. □

Now we use [11, Prop.15] to arrive at the following; recall that the implicit equation is defined up to a sign.

Corollary 5.4. *The coefficient of x^{b_m} is $c(-1)^{(1+a_n)b_m} c_{1m}^{a_n}$ and that of y^{a_n} is $c(-1)^{a_n(1+b_m)} c_{0n}^{b_m}$, where $c \in \{-1, 1\}$.*

Example 5.1. Parameterization $x = 2t^3 - t + 1, y = t^4 - 2t^2 + 3$ yields implicit equation $\phi = 608 - 136x + 569y + 168y^2 - 72x^2 - 32xy - 4x^3 - 16x^2y - x^4 + 16y^3$ Our method yields the vertices $(0,0), (4,0), (0,3)$ which are optimal. The degree bounds describe a larger quadrilateral with vertices $(0,0), (4,0), (1,3), (0,3)$. Corollary 5.4 predicts, for x^4, coefficient $(-1)^{16} = 1$, and for y^3, coefficient $(-1)^{15}2^4 = -16$, up to a fixed sign which equals -1 here.

Example 5.2. Parameterization $x = t + t^2, y = 2t - t^2$ yields implicit equation $\phi = 6x - 3y + x^2 + 2xy + y^2$. The previous lemma yields vertices $(1,0), (2,0), (0,2), (0,1)$, which defines the actual implicit polygon. Here the degree bounds imply a larger triangle, with vertices $(0,0), (2,0), (0,2)$. Corollary 5.4 predicts, for x^2 and y^2, coefficients $(-1)^6(-1)^2 = 1$ and $(-1)^6(1)^2 = 1$ respectively.

Example 5.3. For the Fröberg-Dickenstein example [9, Exam.3.3], $x = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, y = t^{32}$, our method yields vertices $(32,0), (0,48), (0,63)$, which define the actual implicit polygon. Here the degree bounds describe the larger quadrilateral with vertices $(0,0), (32,0), (32,31), (0,63)$.

5.2 Rational parametric curves

Now we turn to the case of rational parameteric curves. In this case $f_0(t) = xQ(t) - P_0(t)$, $f_1(t) = yQ(t) - P_1(t) \in (\mathbb{C}[x,y])[t]$, and the supports of f_0, f_1 are of the form $A_0 = \{a_0, a_1, \ldots, a_n\}$ and $A_1 = \{b_0, b_1, \ldots, b_m\}$ where points a_i and b_j are sorted in ascending order; $a_0 = b_0 = 0$. Points in A_0, A_1 are embedded by κ in $\mathbb{R}^2$. The embedded points are denoted by $(a_i, 0), (b_i, 1)$; by abusing notation, we will ommit the extra coordinate.

Recall that each $p \in A_0 \cup A_1$ corresponds to a monomial of f_0, f_1. The corresponding coefficient, if we consider the first polynomial, lies either in $\mathbb{C}$ or is a linear polynomial in $\mathbb{C}[x]$; the latter is a monomial $q_i x$ or a binomial

$q_i x + c_{0i}$, where $q_i, c_{0i} \in \mathbb{C}$. An analogous description holds for the second polynomial.

Definition 5.1. Let V, W be non-empty subsets of $\mathbb{Z}$. A *selection* is a pair of sets S, T such that $S \subseteq V$ and $T \subseteq W$.

With respect to the sets A_0 and A_1, we now define two selection criteria:

- The first selects the exponents in A_0 or A_1 corresponding to coefficients which are non-constant polynomials in $\mathbb{C}[x]$ or $\mathbb{C}[y]$, respectively; hence, they are either linear monomials or linear binomials. The selected points are those in the support of the denominator $Q(t)$; if $Q(t), P_0(t), P_1(t)$ have the same support, then all points are selected.
- The second selection picks the exponents in A_0 or A_1 corresponding to coefficients which are monomials in $\mathbb{C}[x]$ or $\mathbb{C}[y]$ respectively. In this case there is at least one non-selected point coming from the numerator $P_i(t)$.

In order to denote that a point $a_i \in A_0$ or $b_i \in A_1$ is selected (non-selected), we write a_i^+ or b_i^+ (resp. a_i^- or b_i^-).

The use of the first criterion on the sets A_0 and A_1 produces a selection S_0, S_1 that has the following important properties: (i) $|S_0| \geq 1$ and $|S_1| \geq 1$, i.e., at least one exponent from both A_0 and A_1 is selected since $Q(t) \neq 0$; (ii) because we have assumed that the denominators of the rational parameterization are the same polynomial, $S_0 = S_1$, i.e., a_i is selected iff there exists a selected point with equal coordinate $b_j = a_i$, for indices $i, j > 0$ not necessarily equal (in particular, the same argument is true for a_0, b_0 because, if a_0 is selected, then $q_0 \neq 0$ which implies that b_0 is also selected). Unfortunately, the second criterion does not guarantee any of these two properties. In general, using either criterion, there may exist several selected points, and a_0, b_0 need not be selected. For example, the case of polynomial parameterizations yields $A_0 = \{a_0^+, a_1^-, \ldots, a_n^-\}$, $A_1 = \{b_0^+, b_1^-, \ldots, b_m^-\}$.

We shall consider only i-mixed cells associated with a selected vertex in A_i. For any triangulation $\mathcal{T}$, these mixed cells correspond either to triangles with vertices $\{a_i^+, b_l, b_r\}$, where $l, r \in \{0, \ldots, m\}$, or to $\{a_l, a_r, b_j^+\}$, where $l, r \in \{0, \ldots, n\}$. Given a selection and a triangulation, we set

$$x = \sum_{i,l,r} \mathrm{Vol}(a_i^+, b_l, b_r), \ y = \sum_{l,r,j} \mathrm{Vol}(a_l, a_r, b_j^+), \tag{4}$$

where i, j range over all selected points in A_0 and A_1 respectively, and we sum up the normalized volumes of mixed triangles.

In the following result, we use the concept of upper (resp. lower) hull of a convex polytope in $\mathbb{R}^d$ wrt some direction $v \in \mathbb{R}^d$: it is the subset of facets (i.e. faces of maximum-dimension) whose inner normal vector has a non-positive (resp. non-negative) inner product with v. Notice that we can define a new convex polytope by "gluing" together the upper hull of some arbitrary polytope in $\mathbb{R}^d$ and the lower hull of another polytope in $\mathbb{R}^d$, as long as they are defined wrt the same direction $v \in \mathbb{R}^d$.

The resultant $\mathcal{R}(f_0, f_1, t)$ lies in $(\mathbb{C}[x,y])[c_{ij}, q_k]$. We consider the specialization of coefficients c_{ij}, q_k in order to study ϕ; generically, this specialization yields the implicit equation. The vertices of $N(\mathcal{R})$ are given by theorem 2.1 and expression (3). The vertices of the implicit polygon are exponents of extreme monomials of $\mathcal{R}(f_0, f_1, t)$ which have been specialized.

Theorem 5.5. *Consider points (x, y) defined by expressions (4). The polygon defined by the upper hull of points (x, y) under the first selection and the lower hull of points (x, y) under the second selection contains the implicit polygon $N(\phi)$.*

Proof. Consider the extreme terms of the resultant, given by theorem 2.1 and expression (3). After the specialization of the coefficients, those associated with i-mixed cells having a non-selected vertex $p \in A_i$ contribute only a coefficient in $\mathbb{C}$ to the corresponding term of ϕ. This is why they are not taken into account in (4).

Now consider the first selection. By maximizing x or y, as defined in (4), it is clear that we shall obtain the maximum possible powers in the terms which are polynomials in x and y respectively, hence the largest degrees in x, y in ϕ. Under certain genericity assumptions, we shall obtain all vertices in the implicit polygon, which appear in its upper hull with respect to vector $(1, 1)$. If genericity fails, the implicit polygon will contain vertices with smaller coordinates.

The second selection minimizes the powers of coefficients corresponding to monomials in the implicit variables. All other coefficients are in $\mathbb{C}$ or are binomials in x (or y), so they contain a constant term, hence their product will contain a constant, assuming generic coefficients in the parametric equations. Therefore these are vertices on the lower hull with respect to $(1, 1)$. If genericity fails, then fewer terms appear in ϕ and the implicit polygon is interior to the lower hull computed. $\square$

For any $p \in P$, we define functions $\mathcal{X}(p^+)$ and $\mathcal{X}(p^-)$ where $\mathcal{X}(p^+) = 1$ if p is selected and $\mathcal{X}(p^+) = 0$ otherwise, and $\mathcal{X}(p^-) = 1$ if there exists some non-selected point $p^- \in P$ and $\mathcal{X}(p^-) = 0$ otherwise.

The following results determine the polygon that contains $N(\phi)$.

Lemma 5.6. *The maximum power of x in the implicit equation is generically*

$$b_m - b_0 = b_m.$$

When this is attained, the maximum power of y is generically

$$(a_R^+ - a_L^+) + \mathcal{X}(b_m^+) \cdot (a_n - a_R^+),$$

where a_R^+, a_L^+ are the rightmost and leftmost selected points (not necessarily distinct) in A_0. A similar result holds for y with the roles of x and y, and A_0 and A_1 exchanged.

Proof. There is always at least one selected point in each of A_0 and A_1. This implies that the maximum power of x is equal to $b_m - b_0 = b_m$ and is attained by any triangulation with at least two edges $(a_i^+, b_0), (a_j^+, b_m)$, where $i \leq j$.

In order to obtain the maximum power of y possible when the maximum power of x is attained, we must choose from the previous set of triangulations, one where a maximum part of segment (a_0, a_n) is visible by some selected points $b_k^+ \in A_1$. Such a triangulation must contain edges (a_L^+, b_0) and (a_R^+, b_m) in order to maximize the power of x and y simultaneously.

Assume that point b_0 is selected. Then point a_0 must also be selected and $a_L^+ = a_0$. If all other selected points in A_1 (if any) lie inside (b_0, b_m), then $\mathcal{X}(b_m^+) = 0$; the maximum value of y is $a_R^+ - a_L^+ = a_R^+$ which reduces to zero if $a_R^+ = a_0$ (point a_0 is the only one selected in A_0; polynomial case). It is obtained by drawing edge (b_0, a_R^+). If b_m is selected then $\mathcal{X}(b_m^+) = 1$ and segment (a_R^+, a_n) is also visible from selected points in A_1 (namely b_m) hence the maximum power of y is $a_R^+ + (a_n - a_R^+) = a_n$.

Now assume that point b_0 is not selected, hence point a_0 is not selected. If all selected points in A_1 lie inside (b_0, b_m), then $\mathcal{X}(b_m^+) = 0$ and the maximum value of y is $a_R^+ - a_L^+$. It is obtained by drawing edges $(b_i^+, a_L^+), (b_i^+, a_R^+)$, for some selected point b_i^+. If b_m is selected then $\mathcal{X}(b_m^+) = 1$ and segment (a_R^+, a_n) is also visible from selected points in A_1 (namely b_m) hence the maximum power of y is $(a_R^+ - a_L^+) + (a_n - a_R^+) = a_n - a_L^+$. $\square$

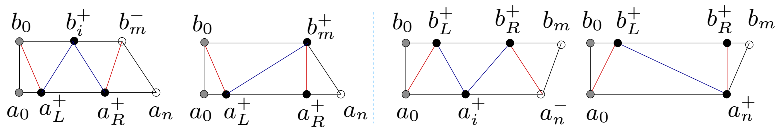


Figure 7: The triangulations of C that give the points $y_{max}|_{x=b_m}$ (left subfigure) and $x_{max}|_{y=a_n}$ (right subfigure).

In a similar fashion, we can also show the following results:

Lemma 5.7. *Suppose that the maximum power of x equal to b_m is attained; then the minimum power of y is, generically,*

$$\mathcal{X}(b_m^+) \cdot (a_n - a_R^+) + (1 - \mathcal{X}(b_i^-)) \cdot (a_R^+ - a_L^+)$$

where a_R^+, a_L^+ are the rightmost and leftmost selected points in A_0. A similar result holds for y with the roles of x and y, and A_0 and A_1 exchanged.

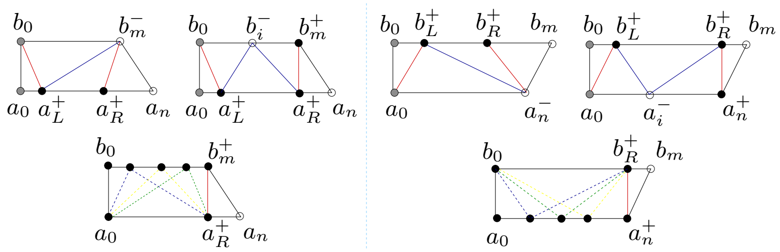


Figure 8: The triangulations of C that give the points $y_{min}|_{x=b_m}$ (left subfigure) and $x_{min}|_{y=a_n}$ (right subfigure).

Lemma 5.8. *Provided that each of the sets A_0 and A_1 contains at least one non-selected point, the minimum power of x in the implicit equation is* 0. *When this is attained, the maximum power of y is*

$$a_n - \big(1 - \mathcal{X}(b_m^+)\big) \cdot (a_n - a_R^-),$$

where a_R^- is the rightmost non-selected point in A_0.

As mentioned in Theorem 5.5, the lower part of the polygon containing the implicit polygon $N(\phi)$ is formed by the convex hull of the points defined by expressions (4) under the second selection criterion. For simplicity of the analysis that follows, we first consider some special cases which give degenerate results, as described in the following lemma.

Lemma 5.9. *Suppose that the set A_0 contains no selected point. Then, under the second selection criterion,*

(i) if no point in the set A_1 is selected, the convex hull of the points defined by expressions (4) degenerates to the point $(0, 0)$*;*

(ii) if all the points in the set A_1 are selected, the convex hull of the points defined by expressions (4) degenerates to the point $(0, a_n)$*;*

(iii) if the set A_1 contains at least one selected and at least one non-selected point, the convex hull of the points defined by expressions (4) degenerates to the line segment with endpoints $(0, 0)$ *and* $(0, a_n)$.

A similar result holds if the set A_1 contains no selected point.

In the following, we asume that each of the sets A_0 and A_1 contains at least one selected point. Then, we have:

Lemma 5.10. *Provided that each of the sets A_0 and A_1 contains at least one non-selected point, the minimum power of y in the implicit equation is* 0. *When this is attained, the maximum power of x is*

$$b_m - \mathcal{X}(a_n^-) \cdot (b_m - b_R^-) - \big(1 - \mathcal{X}(a_0^+) \cdot \mathcal{X}(b_m^+)\big) \cdot b_L^-,$$

where b_L^-, b_R^- are the leftmost and rightmost non-selected points in A_1.

Lemma 5.11. *Provided that each of the sets A_0 and A_1 contains at least one non-selected point, the minimum power of x equal to 0 is attained; then, the minimum power of y is*

$$\mathcal{X}(b_0^+) \cdot a_L^- + \mathcal{X}(b_m^+) \cdot (a_n - a_R^-),$$

where a_L^-, a_R^- are the leftmost and rightmost non-selected points in A_0. A similar result holds for y with the roles of x and y, and A_0 and A_1 exchanged.

Lemma 5.12. *Let $y_{max}|_{x=0} = a_n - \big(1 - \mathcal{X}(b_m^+)\big) \cdot (a_n - a_R^-)$ be the maximum value of the power of y when the power of x attains its minimum value 0, and $x_{min}|_{y=a_n} = \mathcal{X}(a_n^+) \cdot (b_m - b_R^+) + (1 - \mathcal{X}(a_i^-)) \cdot (b_R^+ - b_L^+)$ be the minimum value of the power of x when the power of y attains its maximum value a_n. If $y_{max}|_{x=0} \neq a_n$, which also implies that $x_{min}|_{y=a_n} \neq 0$, then the upper left corner of the polygon containing $N(\phi)$ consists of the edge connecting the points $(0, y_{max}|_{x=0})$ and $(x_{min}|_{y=a_n}, a_n)$.*

Corollary 5.13. *Let $x_{max}|_{y=0} = b_m - \mathcal{X}(a_n^-) \cdot (b_m - b_R^-) - \big(1 - \mathcal{X}(a_0^+) \cdot \mathcal{X}(b_m^+)\big) \cdot b_L^-$ be the maximum value of the power of x when the power of y attains its minimum value 0, and $y_{min}|_{x=b_m} = \mathcal{X}(b_m^+) \cdot (a_n - a_R^+) + (1 - \mathcal{X}(b_i^-)) \cdot (a_R^+ - a_L^+)$ be the minimum value of the power of y when the power of x attains its maximum value b_m. If $x_{max}|_{y=0} \neq b_m$, which also implies that $y_{min}|_{x=b_m} \neq 0$, then the lower right corner of the polygon containing $N(\phi)$ consists of the edge connecting the points $(x_{max}|_{y=0}, 0)$ and $(b_m, y_{min}|_{x=b_m})$.*

Lemma 5.14. *Let $y_{min}|_{x=0} = \mathcal{X}(b_0^+) \cdot a_L^- + \mathcal{X}(b_m^+) \cdot (a_n - a_R^-)$ be the minimum value of the power of y when the power of x attains its minimum value 0, and $x_{min}|_{y=0} = \mathcal{X}(a_0^+) \cdot b_L^- + \mathcal{X}(a_n^+) \cdot (b_m - b_R^-)$ be the minimum value of the power of x when the power of y attains its minimum value 0. If $y_{min}|_{x=0} \neq 0$, which also implies that $x_{min}|_{y=0} \neq 0$, then the lower left corner of the polygon containing $N(\phi)$ consists of the edge connecting the points $(0, y_{min}|_{x=0})$ and $(x_{min}|_{y=0}, 0)$ unless all four points a_0, b_0, a_n, b_m are selected in which case the corner consists of the edges connecting $(0, y_{min}|_{x=0})$ to point p to $(x_{min}|_{y=0}, 0)$ where $p = (b_L^-, a_n - a_R^-)$ if $\frac{b_L^-}{a_L^-} < \frac{b_m - b_R^-}{a_n - a_R^-}$ and $p = (b_m - b_R^-, a_L^-)$ if $\frac{b_L^-}{a_L^-} > \frac{b_m - b_R^-}{a_n - a_R^-}$.*

Proof. We start by noting that the values $y_{min}|_{x=0}$ and $x_{min}|_{y=0}$ are equal to 0 if at least one of the points a_0, b_0 is not selected and at least one of a_n, b_m is not selected.

Let us consider the remaining cases. First, suppose that both a_0, b_0 are selected but not both a_n, b_m are selected. In this case, $y_{min}|_{x=0} = a_L^-$ and $x_{min}|_{y=0} = b_L^-$; we will show that in this case, the interior of the axis-parallel rectangle with vertices the two points $(0, a_L^-)$ and $(b_L^-, 0)$ is empty. Suppose for contradiction that there exists a triangulation T corresponding to a point in the interior of this rectangle. Let us start at the triangle of T which touches the edge (a_0, b_0) and move from triangle to triangle by crossing edges (a_i, b_j) of T until we reach such an edge with at least one of its endpoints being a non-selected point. Suppose that this edge is $(a_{i'}, b_{j'})$ where $a_{i'}$ is non-selected; then $b_{j'}$ is selected, otherwise we would have stopped earlier since we have seen edges incident on $b_{j'}$ before the edge $(a_{i'}, b_{j'})$. Then, clearly, all the points in $\{b_0, \ldots, b_{i'}\}$ are selected points in A_1 and additionally, $a_{i'}$ either coincides with or is to the right of a_L^-. In either case, the value of the power of y is no less than a_L^-, a contradiction. The case is symmetric if we stop at an edge $(a_{i'}, b_{j'})$ where $b_{j'}$ is non-selected.

Next, suppose that not both a_0, b_0 are selected but both a_n, b_m are selected. In this case, $y_{min}|_{x=0} = a_n - a_R^-$ and $x_{min}|_{y=0} = b_m - b_R^-$; This case is left-to-right symmetric to the previous one; therefore, in this case as well, the rectangle with vertices the points $(0, a_n - a_R^-)$ and $(b_m - b_R^-, 0)$ has empty interior.

Finally, suppose that all four points a_0, b_0, a_n, b_m are selected. In this case, $y_{min}|_{x=0} = a_L^- + a_n - a_R^-$ and $x_{min}|_{y=0} = b_L^- + b_m - b_R^-$; we will show that in this case, the lower hull of the polygon containing $n(\phi)$ contains a two-edge polygonal line connecting the points $(0, a_L^- + a_n - a_R^-)$ and $(b_L^- + b_m - b_R^-, 0)$. First, note that there exist triangulations corresponding to the points $(b_L^-, a_n - a_R^-)$ and $(b_m - b_R^-, a_L^-)$; the former involves the edges (a_0, b_L^-), (a_R^-, b_L^-), and (a_R^-, b_m), and the latter the (symmetric) edges (b_0, a_L^-), (b_R^-, a_L^-), and (b_R^-, a_n). The four points $(0, a_L^- + a_n - a_R^-)$, $(b_L^- + b_m - b_R^-, 0)$, $(b_L^-, a_n - a_R^-)$, and $(b_m - b_R^-, a_L^-)$ form a parallelogram which degenerates into a line segment if $\frac{b_L^-}{a_L^-} = \frac{b_m - b_R^-}{a_n - a_R^-}$; it is not difficult to see that if $\frac{b_L^-}{a_L^-} < \frac{b_m - b_R^-}{a_n - a_R^-}$, it is point $(b_L^-, a_n - a_R^-)$ that lies below the line through $(0, a_L^- + a_n - a_R^-)$ and $(b_L^- + b_m - b_R^-, 0)$, whereas if $\frac{b_L^-}{a_L^-} > \frac{b_m - b_R^-}{a_n - a_R^-}$, it is point $(b_m - b_R^-, a_L^-)$ that lies below that line. Due to the symmetries in the setting, it suffices to consider that $\frac{b_L^-}{a_L^-} < \frac{b_m - b_R^-}{a_n - a_R^-}$ and to show that there are no triangulations that correspond to points that have x-coordinate no more than b_L^- and lie below the line through the points $(0, a_L^- + a_n - a_R^-)$ and $(b_L^-, a_n - a_R^-)$. Suppose for contradiction that such a triangulation, say, T, existed. Then, by applying on T the triangle-to-triangle walking argument presented earlier in this proof, we will move from the triangle of T touching the edge (a_0, b_0) to other triangles until an edge $(a_{i'}, b_{j'})$ is reached exactly one of whose endpoints is a non-selected vertex. Let us apply the same argument starting from the triangle of T that touches the edge (a_n, b_m); in this case, we stop at an edge $(a_{i''}, b_{j''})$. Since T corresponds to a point with x-coordinate no more than b_L^- and since all four points a_0, b_0, a_n, b_m are selected, it cannot be the case that both $b_{j'}$ and $b_{j''}$ are non-selected;

moreover, since the point is below the line through $(0, a_L^- + a_n - a_R^-)$ and $(b_L^-, a_n - a_R^-)$, its y-coordinate is less than $a_L^- + a_n - a_R^-$, and hence nor both $a_{i'}$ and $a_{i''}$ can be non-selected either. Due to symmetry, suppose that $a_{i'}$ and $b_{j''}$ are non-selected. Then, the point corresponding to T has x-coordinate at least equal to $b_m - b_R^-$ and y-coordinate at least equal to a_L^-. But because $\frac{b_L^-}{a_L^-} < \frac{b_m - b_R^-}{a_n - a_R^-}$, any such point lies above the line through $(0, a_L^- + a_n - a_R^-)$ and $(b_L^-, a_n - a_R^-)$, a contradiction. $\square$

Lemma 5.15. *Let $y_{max}|_{x=b_m} = (a_R^+ - a_L^+) + \mathcal{X}(b_m^+) \cdot (a_n - a_R^+)$ be the maximum value of the power of y when the power of x attains its maximum value b_m, and $x_{max}|_{y=a_n} = (b_R^+ - b_L^+) + \mathcal{X}(a_n^+) \cdot (b_n - b_R^+)$ be the maximum value of the power of x when the power of y attains its maximum value a_n. If $y_{max}|_{x=b_m} \neq a_n$, which also implies that $x_{max}|_{y=a_n} \neq b_m$, then the upper right corner of the polygon containing $N(\phi)$ consists of the edge connecting the points $(0, y_{max}|_{x=b_m})$ and $(x_{max}|_{y=a_n}, 0)$ unless none of the four points a_0, b_0, a_n, b_m is selected in which case the corner consists of the edges connecting $(x_{max}|_{y=a_n}, a_n)$ to point p to $(b_m, y_{max}|_{x=b_m})$ where $p = (b_R^+, a_n - a_L^+)$ if $\frac{a_n - a_R^+}{b_m - b_R^+} > \frac{a_L^+}{b_L^+}$ and $p = (b_m - b_L^+, a_R^+)$ if $\frac{a_n - a_R^+}{b_m - b_R^+} < \frac{a_L^+}{b_L^+}$.*

Example 5.4. For the unit circle ([8, Exam.6.1]) $x = 2t/(t^2+1)$, $y = (1-t^2)/(t^2+1)$ we have $f_0 = xt^2 - 2t + x$, $f_1 = (y+1)t^2 + (y-1)$ and supports $A_0 = \{0^+, 1^-, 2^+\}$, $A_1 = \{0^+, 2^+\}$. The set $C = \kappa(A_0, A_1)$ has five triangulations shown in figure 9 which, after applying Theorem 2.1, give the terms $y^2 - 1$, $x^2y^2 - 2x^2y + x^2$ and $x^2y^2 + 2x^2y + x^2$. Our method yields vertices $(2,2), (2,0), (0,2), (0,0)$. By degree bounds we end up with vertices $(2,0), (0,2), (0,0)$. Interestingly, to see the cancelation of term x^2y^2 it does not suffice to consider only terms coming from extremal monomials in the resultant.

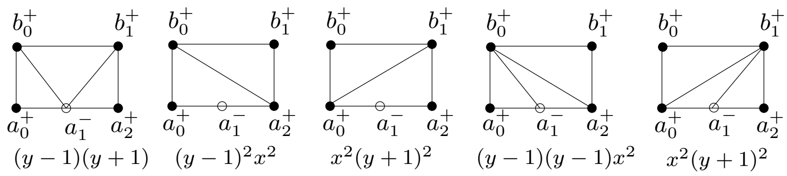


Figure 9: The triangulations of the set C of Example 5.4 and the corresponding terms.

Example 5.5. For the folium of Descartes ([8, Exam.6.2]) $x = 3t^2/(t^3+1)$, $y = 3t/(t^3+1)$ with implicit equation $\phi = x^3 + y^3 - 3xy = 0$, we have $f_0 = xt^3 - 3t^2 + x$, $f_1 = yt^3 - 3t + y$ and supports $A_0 = \{0^+, 2^-, 3^+\}$, $A_1 = \{0^+, 1^-, 3^+\}$. The denoted selection is the same under both selection criteria, and satisfies the assumptions of the lemmas relevent for computing the lower hull of the polygon. The set $C = \kappa(A_0, A_1)$ has fourteen triangulations. Our method yields vertices $(3,3), (0,3), (3,0), (1,1)$. By degree bounds we end up with vertices $(0,3), (3,0), (1,1)$ which are optimal. The polygon predicted by degree bounds alone contains the additional vertex $(0,0)$ which leads to a possible implicit support with five more vertices.

Example 5.6. Parametrization $x = (2t^3 + t + 1)/(t^2 + 1)$, $y = (t^4 + t^3 - 1)/(t^2 + 1)$ yields implicit equation $\phi = 59 - 21x + 110y + 52y^2 - 13x^2 - 48xy + 5x^3 - 5x^2y - x^4 + 8y^3 - 2x^2y^2 + 2x^3y - 12xy^2$. The Newton polytope of ϕ has vertices $(0,3), (2,2), (4,0), (0,0)$. The supports of f_0, f_1 are $A_0 = \{0^+, 1^-, 2^+, 3^-\}$, $A_1 = \{0^+, 2^+, 3^-, 4^-\}$ where the notation is under the first selection. The selection under the second criterion gives $A_0 = \{0^-, 1^-, 2^+, 3^-\}$, $A_1 = \{0^-, 2^+, 3^-, 4^-\}$. Our method yields the vertices $(4,2), (2,3), (4,0), (0,0), (3,0)$ and $(0,3)$. The intersection of the polygon defined by these vertices, with the polygon predicted by degree bounds rule out vertices $(4,2)$ and $(2,3)$ and introduces vertex $(1,3)$.

5.3 Parametric surfaces

This section considers polynomially parameterized surfaces. We use an example to illustrate the problem of fully describing the Newton polytope of the implicit equation. Let $A, B, D \in \mathbb{Z}^2$ be the Newton polytopes of the three polynomials, each containing $(0,0)$. The are embedded, by applying κ, into $\mathbb{R}^4$, with the embedded points being denoted by $(a_i, 0, 0), (b_i, 1, 0), (d_i, 0, 1)$, respectively.

Lemma 5.16. *Consider some mixed subdivision Δ of B and D in $\mathbb{R}^2$. The triangulation containing all simplices in $\mathbb{R}^4$ defined by $(a_0, 0, 0)$ and any mixed cell of Δ achieves the maximum value of x, namely $MV(B, D)$. A symmetric result holds by switching between z and x or y.*

Proof. The power of x is

$$\sum_F \mathrm{Vol}_4(\mathrm{CH}(a_0, F)) = \sum_F \mathrm{Vol}_3(F),$$

where $F = (b_i, b_j) + (d_i, d_j)$ ranges over all mixed cells of Δ, i.e. it is the Minkowski sum of edges from B, D respectively, and $\mathrm{CH}(p, F)$ is the simplex with vertices a_0 and those of F. □

Lemma 5.17. *If there is a constant term in at least one P_i, then there is a vertex $(0,0,0)$ in the implicit polytope, generically.*

Proof. If there is a constant term in at least one P_i then, by expanding the term which maximizes the power of one variable, we obtain a constant term. □

The question is, for the maximum degree of x, what are the possible degrees of y, z. By our example below, we see that it is possible to have implicit vertices which (a) maximize one variable and minimize (namely to 0) the other two, and (b) maximize one variable, minimize (namely to 0) another, whereas the third achieves a value smaller than its maximum.

We show by example that it may be hard to be describe the vertices *a priori*, since this depends on the geometry of the problem. We concentrate on the case of all A, B, D being straight-line segments, such that every two of them are linearly independent. The Newton polytope vertices are denoted by $\{a_0 = 0, a_1\}, \{b_0 = 0, b_1\}, \{d_0 = 0, d_1\}$.

5.3.1 An example

More specifically, we shall analyze a (sparse) example from [3]; the surface is drawn in Figure 10. The parametric expressions are:

$$x = s\ t,\ y = s\ t^2,\ z = s^2. \tag{5}$$

The corresponding polynomials are $f_0 = c_{00} - c_{01} st,\ f_1 = c_{10} - c_{11} st^2,\ f_2 =$

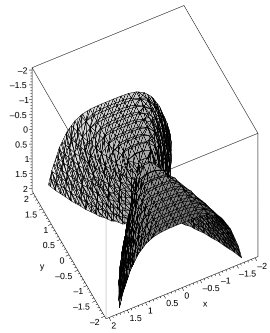


Figure 10: The surface parameterized by (5).

$c_{20} - c_{21} s^2$, with supports

$$A = \{(0,0), (1,1)\}, B = \{(0,0), (1,2)\}, C = \{(0,0), (2,0)\}.$$

There are two possible mixed subdivisions, each containing exactly three maximal cells, all of which are mixed; see Figure 11. By degree arguments

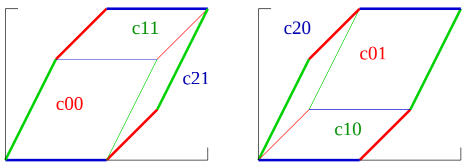


Figure 11: Mixed cells in the subdivisions, with vertex summands shown.

the total implicit degree is bounded by 4 and variables x, y, z have degree bounded by $4, 2, 2$ respectively. The implicit equation is $x^4 - y^2 z = 0$.

Let us now follow the approach of this paper to the same problem and see how the two implicit terms could be obtained.

In applying lemma 5.16, there is a single facet F of H of the form $F = (b_i, b_j) + (d_i, d_j)$, namely $F = (b_0, b_1) + (d_0, d_1)$. The simplex $\mathrm{CH}(a_0, F)$ has normalized volume $2 \cdot 2 = 4$, which yields the power of x in the specific monomial as by lemma 5.16.

Now we show that there is no b_0-mixed cell in the corresponding mixed subdivision by showing that there is no 4-simplex $(a_0, a_1) + b_0 + (d_0, d_1)$. If there were, it would intersect $\mathrm{CH}(a_0, F)$ in the 3-simplex defined by a_0, b_0, d_0, d_1; in other words, the corresponding hyperplane would separate b_1, a_1. This implies that the following Orientation determinant (see e.g. [7]), with rows corresponding to a_0, b_0, d_0, d_1, should have opposite signs when the last row expressed b_1 or a_1:

$$\begin{array}{r} a_0 \\ b_0 \\ d_0 \\ d_1 \\ b_1 \text{ or } a_1 \end{array} \left| \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 1 & 2 \text{ or } 1 & 1 \text{ or } 0 & 0 \end{array} \right| . = 2 \cdot 2 \text{ or } 2 \cdot 1,$$

hence the hypothesis that a b_0-mixed cell exists does not hold. The same method shows there is no d_0-mixed cell either, therefore we conclude correctly that one implicit monomial is x^4.

We now use b_0 instead of a_0 in lemma 5.16. There is a single relevant simplex $\mathrm{CH}(b_0, (a_0, a_1) + (d_0, d_1)$, with normalized volume 2, which is the power of y. We can show that there is no a_0-mixed cell in the mixed subdivision as above.

But is there a d_0-mixed cell corresponding to simplex $\mathrm{CH}(d_0, (a_0, a_1) + (b_0, b_1)$? By using Orientation matrices we show this simplex exists in the triangulation; in other words, the hyperplane of a_0, a_1, b_0, d_0 separates d_1, b_1:

$$\begin{array}{r} a_0 \\ a_1 \\ b_0 \\ d_0 \\ d_1 \text{ or } b_1 \end{array} \left| \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 2 \text{ or } 1 & 0 \text{ or } 2 & 0 \text{ or } 1 & 1 \text{ or } 0 \end{array} \right| = \left| \begin{array}{cc} 1 & 1 \\ 2 & 0 \end{array} \right| = -2 \text{ or } \left| \begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right| = 1.$$

The d_0-mixed cell has normalized volume 1, hence the implicit monomial is $y^2 z$.

Acknowledgment. All authors are supported by the General Secretariat of Research and Technology of Greece through PENED 2003 programme, contract nr. 70/03/8473, co-funded by the European Social Fund (75%) and national resources (25%).

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