Single-Lifting Macaulay-Type Formulae of Generalized Unmixed Sparse Resultants
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Abstract

Resultants are defined in the toric (or sparse) context in order to exploit the structure of the polynomials as expressed by their Newton polytopes. Since determinantal formulae are not always possible, the most efficient general method for computing resultants is by rational formulae. This is made possible by Macaulay’s famous determinantal formula in the dense homogeneous case, extended by D’Andrea to the toric case. However, the latter requires a lifting of the Newton polytopes, defined recursively on the dimension. Our main contribution is a single lifting function of the Newton polytopes, which avoids recursion, and yields a simpler method for computing Macaulay-type formulae of toric resultants, in the case of generalized unmixed systems, where all Newton polytopes are scaled copies of each other. In the mixed subdivision used to construct the matrices, our algorithm defines significantly fewer cells than D’Andrea’s, though the formulae are same in both cases. We fully study a bivariate example and sketch how our approach extends to mixed systems of up to four polynomials, and those whose Newton polytopes have a sufficiently different face structure.

Keywords Toric resultant, Macaulay formula, Minkowski sum, mixed subdivision, generalized unmixed system


1 Introduction

There are a few symbolic methods for algebraic variable elimination, including Gröbner (or standard) bases, and resultants. Both have exponential complexity in the number of variables, which is expected since the problem is NP-hard; but the latter are preferable in certain situations because they eliminate many variables at one step and can handle symbolic coefficients. Resultants also seem more efficient for solving certain classes of zero-dimensional algebraic systems. In particular, they reduce system solving to linear algebra, via matrix formulae, or to solving univariate polynomials, via the rational univariate representation of all common roots. The resultant generalizes the determinant of the coefficient matrix in the linear case, and the discriminant of a multivariate polynomial. For more information, see [CLO05, DE05, Stu02].

The toric (or sparse) resultant captures the structure of the polynomials by combinatorial means and constitutes the cornerstone of toric elimination theory [GKZ94, Stu02], [CLO05, chap.7], [DE05, chap.7]. It is an important tool in deriving new, tighter complexity bounds for system solving, Hilbert’s Nullstellensatz, and related problems. These bounds depend on the polynomials’ Newton polytopes and their mixed volumes, instead of total degree, which is the only parameter in classical elimination theory. In particular, if d bounds the total degree of each polynomial, the projective resultant has complexity roughly $d^{O(n)}$, whereas the toric resultant is computed in time roughly proportional to the number of integer lattice points in the Minkowski sum of the Newton polytopes.

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The resultant is defined for an overconstrained system of \( n + 1 \) polynomials in \( n \) variables over some coefficient ring \( K \). It is the unique, up to sign, integer polynomial over \( K \) which vanishes precisely when the system has a root in some variety \( X \). There are two main cases:

- The projective, or classical, resultant expresses solvability of a system of dense polynomials \( f_i \in K[x_1, \ldots, x_n] \) in the projective space over the algebraic closure \( \overline{K} \) of \( K \).
- The toric (or sparse) resultant expresses solvability of a system of Laurent polynomials \( f_i \in K[x_1^{\pm1}, \ldots, x_n^{\pm1}] \) over the toric projective variety \( X \) defined by the supports of \( f_i \), in which the torus \( (\overline{K})^n \) is a dense subset.

A resultant is most efficiently expressed by a matrix formula: this is a generically nonsingular matrix, whose determinant is a multiple of the resultant with degree with respect to the coefficients of one polynomial equal to the corresponding degree of the resultant. For \( n = 1 \) there are matrix formulae named after Sylvester and Bézout, whose determinant equals the resultant. Unfortunately, such determinantal formulae do not generally exist for \( n > 1 \), except for specific cases, e.g. [DD01, DE03b, EM09, Khe03, KSG04, SZ94]. Macaulay’s seminal result [Mac02] expresses the extraneous factor as a minor of the matrix formula, for projective resultants of (dense) homogeneous systems, thus yielding the most efficient general method for computing such resultants. There exists a method which, given a Macaulay-type formula of the resultant, constructs a determinant which equals the resultant [KK08].

Matrix formulae for the toric resultant were first constructed in [CE93]. The construction relies on a lifting of the given polynomial supports, which defines a mixed subdivision of their Minkowski sum into mixed and non-mixed cells, then applies a perturbation \( \delta \) so as to define the integer points that index the matrix. The algorithm was extended in [CE00, CP93, Stu94]. In the case of dense systems, the matrix coincides with Macaulay’s numerator matrix. As a corollary of this construction, one obtains a limited version of a toric effective Nullstellensatz [CE00, Sec.8].

Extending the Macaulay formula to toric resultants had been conjectured in [CE00, CLO05, Emi94, GKZ94, Stu94]; it was a major open problem in elimination theory. We cite [Stu94, p.219], where \( P_{\omega,\delta} \) is the extraneous factor, and \( \omega \) denotes the lifting: “It is an important open problem to find a more explicit formula for \( P_{\omega,\delta} \) in the general toric case. Does there exist such a formula in terms of some smaller resultants? This problem is closely related to the following empirical observation. For suitable choice of \( \delta \) and \( \omega \), the matrix \( M_{\delta,\omega} \) seems to have a block structure which allows to extract the resultant from a proper submatrix. This leads to faster algorithms for computing the sparse mixed resultant.”

D’Andrea’s fundamental result [D’A02] answers the conjecture by a recursive definition of a Macaulay-type formula, see Section 3. But this approach does not offer a global lifting, in order to address the stronger original Conjecture 1. Let \( M \) be a matrix formula, also known as Newton matrix, and \( M^{(nm)} \) its submatrix indexed by points in non-mixed cells of the mixed subdivision.

**Conjecture 1.** [Emi94, Conj.3.1.19] [CE00, Conj.13.1] There exist perturbation vector \( \delta \) and \( n + 1 \) lifting functions for which the determinant of matrix \( M^{(nm)} \) divides exactly the determinant of Newton matrix \( M \) and, hence, the toric resultant of the given polynomial system is \( \det M / \det M^{(nm)} \).

Our main contribution is to give an affirmative answer to this stronger conjecture by presenting a single lifting which constructs Macaulay-type formulae for generalized unmixed systems, i.e. when all Newton polytopes are scaled copies of each other. We state our main result, to be proven in Section 4:

**Theorem 2.** Algorithm B of Section 2 constructs a Macaulay-type formula for the toric resultant of an overconstrained generalized unmixed algebraic system, by means of the lifting function of Definition 6.

Our method is generalized, in Section 6, to certain mixed systems: those with \( n \leq 3 \), as well as reduced systems, defined in [Zha98] to possess sufficiently different Newton polytopes. Most of these
cases have been studied: reduced systems were settled in [D’A01], and bivariate systems \((n = 2)\) in [DE03a], by directly establishing the extraneous factor. We expect that our approach would eventually make the single-lifting algorithm applicable to the fully general case.

Using a unique lifting function essentially means that we consider a deformed system, defined by adding a new variable \(t\) so that each input monomial \(x^a\) gets multiplied by \(t^b\), where \(b \in \mathbb{Z}\). Such deformations capture the system’s behavior at toric infinity, hence lie at the heart of most theorems in toric elimination (e.g. sparse homotopies, toric resultants, the toric Nullstellensatz [Ber75, CE00, CLO05, GKZ94, HS95, Stu94]). Having a unique deformed system in defining the Macaulay-type formula might allow for further applications of this formula. Such combinatorial methods constitute one of the two main approaches for studying toric resultants, e.g. [CE00, CLO05, DD01, Min03, Stu94], the other relying on Koszul complexes and their generalizations, e.g. [DE03b, EM09, Khe03].

D’Andrea’s [D’A02] recursive construction requires one to associate integer points with cells of every dimension from \(n\) to 1. Our method constructs the matrix formula directly, without recursion, by examining only \(n\)-dimensional cells. These are more numerous than the \(n\)-dimensional cells in [D’A02] but our algorithm defines significantly fewer cells totally. The disadvantage of our method is to consider extra points besides the input supports. Our single lifting algorithm is conceptually simpler and also easier to implement; see [GLW99], where the authors argue for the advantages of a single lifting over a recursive one in the context of polyhedral homotopy methods for solving algebraic systems. Existing public-domain Maple implementations cover only the original Canny-Emiris method [CE00], either standalone\(^1\) or as part of library Multires\(^2\).

The rest of the paper is structured as follows. The next section introduces some necessary notions, and defines the single lifting that produces Macaulay-type formulae. Section 3 recalls the recursive algorithm of [D’A02], and Section 4 proves the equivalence of the two constructions. Section 5 studies and defines the single lifting that produces Macaulay-type formulae. Section 3 recalls the recursive algorithm of [D’A02], and Section 4 proves the equivalence of the two constructions. Section 5 studies and defines the single lifting that produces Macaulay-type formulae.

2 Single lifting construction

This section describes our approach to defining Macaulay-formulae. For any polytopes or point sets \(A, B\), let \(\langle A \rangle\) denote the affine span (or hull) of \(A\) over \(\mathbb{R}\) and \(\langle A, B \rangle\) the affine span of \(A \cup B\) over \(\mathbb{R}\). Let \(f_0, \ldots, f_n\) be polynomials with supports \(A_0, \ldots, A_n \subset \mathbb{Z}^n\) and Newton polytopes

\[
Q_0, \ldots, Q_n \subset \mathbb{R}^n, Q_i = \text{CH}(A_i),
\]

where \(\text{CH}(\cdot)\) denotes convex hull.

Our lifting shall induce a regular and fine (or tight) mixed subdivision of the Minkowski sum \(\sum_{i=0}^n Q_i\) [CLO05, GKZ94]. Regularity implies the subdivision is in bijective correspondence with the face structure of the upper (or lower) hull of the Minkowski sum of \(Q_0, \ldots, Q_n\) after they are lifted to \(\mathbb{R}^{n+1}\). Each cell in \(\mathbb{R}^n\) is written uniquely as the Minkowski sum of faces \(F_i\) of the \(Q_i\). A fine subdivision is characterized by an equality between cell dimension and the sum of the faces’ dimensions. We focus on cells of maximal dimension \(n\), and call them maximal or, simply, cells. We distinguish them as mixed and non-mixed: the former are the Minkowski sum of \(n\) edges and a vertex. Mixed cells are \(i\)-mixed if this vertex lies in \(A_i\). The type of a cell is either \(i\)-mixed or non-mixed.

Let \(Z\) be the integer lattice generated by \(\sum_{i=0}^n A_i\). The Minkowski sum \(\sum_{i=0}^n Q_i\) is perturbed by a vector \(\delta \in \mathbb{Q}^n\), which is sufficiently small with respect to \(Z\), and in sufficiently generic position with respect to the \(Q_i\). The lattice points in \(E = Z \cap (\sum_{i=0}^n Q_i + \delta)\) are associated to a unique maximal cell of the subdivision, and this allows us to construct a matrix formula \(M\) whose rows and columns are indexed by these points. In particular, polynomial \(x^{p-a_i} f_i\) fills in the row indexed by the lattice point \(p\) in Definition 3.

\(^1\)http://www.di.uoa.gr/~emiris/soft_alg.html

\(^2\)http://www-sop.inria.fr/galaad/logiciels/multires.html
Definition 3. Let \( p \in \mathcal{E} \) lie in a cell \( F_0 + \cdots + F_n + \delta \) of the perturbed mixed subdivision, where \( F_i \) is a face of \( Q_i \). The row content (RC) of \( p \) is \((i, j)\), if \( i \in \{0, \ldots , n\} \) is the largest integer such that \( F_i \) equals a vertex \( a_{ij} \in A_i \).

Our method is based on the matrix construction algorithm of [CE00, Emi94], see also [CP93, Stu94] for generalizations. For completeness, we recall the basic steps:

1. Pick (affine) liftings \( H_i : \mathbb{Z}^n \rightarrow \mathbb{R} : A_i \rightarrow \mathbb{Q}, i = 0, \ldots , n \).
2. Construct a regular fine mixed subdivision of the Minkowski sum \( \sum_{i=0}^{n} Q_i \) using liftings \( H_i \).
3. Perturb the Minkowski sum \( \sum_{i=0}^{n} Q_i \) by a sufficiently small \( \delta \in \mathbb{Q}^n \), so that integer points in \( \sum_{i=0}^{n} Q_i + \delta \) belong to a unique cell of the subdivision, and assign row content to these points by Definition 3.
4. Construct resultant matrix \( M \) with rows and columns indexed by the previous integer points.

Below, we modify step 1 of this algorithm to use the lifting function of Definition 6, and shall extend the last step to produce additionally the denominator matrix. We shall refer to the modified algorithm as Alg. B.

The main idea of both our and D’Andrea’s methods is that one point, say \( b_{01} \in Q_0 \), is lifted significantly higher. Then, the 0-summand of all maximal cells is either \( b_{01} \) or a face not containing it. In D’Andrea’s case, facets not containing \( b_{01} \) correspond to different subsystems where the algorithm recurses (each time on the integer lattice specified by that subsystem). In designing a unique lifting, the issue is that points appearing in two of these subsystems may be lifted differently in different recursions. To overcome this, we introduce several points \( c_{ijs} \), each lying in a suitable face of \( Q_i \) indexed by \( s \), very close (with respect to \( \mathbb{Z} \)) to every \( b_{ij} \), which is lifted very high at recursion \( i \) by D’Andrea’s method. This captures the multiple roles \( b_{ij} \) may assume in every recursion step.

Algorithm B. Our algorithm uses \( \mathcal{E} \) to index the rows (and columns) of the numerator matrix of our Macaulay-type formula. We now focus on generalized unmixed systems, where

\[
Q_i = k_i Q \subset \mathbb{R}^n,
\]

for some \( n \)-dimensional lattice polytope \( Q \) and \( k_i \in \mathbb{N}^* \) \( i = 0, \ldots , n \). Then, the denominator shall be indexed by points lying in non-mixed cells.

Definition 4. For \( i = 0, \ldots , n - 2 \), consider any \((n - i)\)-dimensional face \( F_s^{(i)} \subset Q \), where \( s \) ranges over all such faces. Take any vertex \( b_{ij} \in F_s^{(i)} \), for any valid \( j \). Let \( \delta_{ijs} \in \mathbb{Q}^n \) denote a perturbation vector such that:

1. \( b_{ij} + \delta_{ijs} \) lie in the relative interior of \( k_i F_s^{(i)} \),
2. it is sufficiently small compared to lattice \( \mathbb{Z} \), and \( \| \delta_{ijs} \| \ll \| \delta \| \), where \( \| \cdot \| \) is the Euclidean norm and \( \delta \) as above, and
3. it is sufficiently generic to avoid all edges in the mixed subdivision of \( \sum_{i=0}^{n} Q_i \).

Condition 1 of Definition 4 implies that \( \delta_{ijs} \) also lies in the relative interior of \( k_i F_s^{(i)} \). We shall use the perturbation vectors of Definition 4 to define additional points not contained in the input supports.

Definition 5. We define points \( c_{ijs} \in Q_i \cap \mathbb{Q}^n \), for \( i = 0, \ldots , n - 2 \). Firstly, set \( c_{011} := b_{01} + \delta_{011} \in Q_0 \cap \mathbb{Q}^n \) where \( \delta_{011} \) satisfies Definition 4. Now let \( \{c_{ijs} \in k_i F_s^{(i)} \} \) be the set of points defined in \( Q_i \), where \( s \) ranges over all \((n - i)\)-dimensional faces \( F_s^{(i)} \subset Q \) and \( j \) over the set of indices of points in \( Q_i \). Then, let \( F_u^{(i+1)} \) be a facet of \( F_s^{(i)} \) such that:
1. \( k_i F_u^{(i+1)} \) does not contain any of the \( b_{ij} \)’s corresponding to the already defined \( c_{ij} \)’s, and

2. \( k_{i+1} F_u^{(i+1)} \) does not contain any of the already defined \( c_{(i+1)j} \)’s.

For each such facet choose a vertex \( b_{(i+1)j} \in A_{i+1} \), for some \( j \), and a suitable perturbation vector \( \delta_{(i+1)ju} \) satisfying Definition 4, and set \( c_{(i+1)ju} := b_{(i+1)j} + \delta_{(i+1)ju} \in Q_{i+1} \cap \mathbb{Q}^n \).

The previous definition implies a many-to-one mapping from the set of \( c_{ij} \)’s to that of \( b_{ij} \)’s; it reduces to a bijection when restricted to a fixed face \( k_i F_u^{(i)} \subset Q_i \) containing \( b_{ij} \). Condition 1 of Definition 4 implies that \( c_{ij} \) does not lie on a face of dimension \( < n - i \) and lies in the interior of \((n-i)\)-dimensional \( F_u^{(i)} \). We can reduce the number of the \( c_{ij} \)’s in Alg. B, but this would complicate the subsequent proofs.

For an application of Definition 5 when \( n = 2 \) see Figure 1 where \( Q \) is the unit square, and also Figure 7 where \( Q \) is a pentagon. In both examples, for illustration purposes, we define points \( c_{ij} \) also on edges of polytope \( Q_1 \). See also Figure 2 where \( Q \) is the unit cube.

![Figure 1: Two scenarios of an application of Def. 5 for 3 unit squares. Facets are numbered clockwise starting from the left vertical edge](image)

**Definition 6.** Let \( h_0 \gg h_1 \gg \ldots \gg h_{n-1} \gg 1 \). Alg. B uses sufficiently random linear functions \( H_i, i = 0, \ldots, n \), such that:

\[
1 \gg H_i(a_{ij}) > 0, \quad \text{and} \quad H_i \gg H_t, \quad i < t,
\]

where \( a_{ij} \in A_i \) and \( i, t = 0, \ldots, n, \ j = 1, \ldots, |A_i| \). Alg. B defines global lifting \( \beta \) as follows:

1. \( c_{ij} \mapsto h_i, \ c_{ij} \in k_i F_u^{(i)} \subset Q_i, \ i = 0, \ldots, n - 1 \); this is called primary lifting.

2. \( a_{ij} \mapsto H_i(a_{ij}), \ a_{ij} \in A_i, \ i = 0, \ldots, n \).

Let \( F^\beta \) denote face \( F \) lifted under \( \beta \). Now \( c_{ij}^\beta, \) for all valid \( j, s \), is much higher, respectively lower, than any \( c_{ij}^\beta \) for \( i > t \), respectively \( i < t \). The \( \beta \)-induced subdivision contains edges with one or two vertices among the \( c_{ij} \), and edges from the \( Q_i \). The vertex set of the upper hull of \( Q_i^\beta \) contains some or all of the \( c_{ij}^\beta \) and the lifted vertices of \( Q_i \).
When all $Q_i$ are simplices, as in the classical dense case, it suffices to apply a primary lifting to one point of every $Q_i$ as in Definition 5. Thus our scheme generalizes the approach by Macaulay [Mac02].

Figure 3 shows the mixed subdivisions of three unit squares and their Minkowski sum, induced by lifting $\beta$. Here, the perturbation vectors are not sufficiently small compared to $\mathbb{Z}^2$ for illustration purposes.

The matrix formula constructed by Alg. B is indexed by all lattice points in $\mathcal{E}$. To decide the content of each row, every point is associated to a unique (maximal) cell of the mixed subdivision according to Definition 3. The $t$-mixed cells contain lattice points as follows:

$$p \in \sum_{i=0}^t k_i E_0 + \cdots + k_{t-1} E_{t-1} + c_{t} + k_{t+1} E_{t+1} + \cdots + k_n E_n \cap \mathbb{Z},$$
for edges $E_i \subset Q$ spanning $\mathbb{R}^n$. This gives unique writing

$$p = p_0 + \cdots + p_{t-1} + (b_{ij} + \delta_{ij}) + p_{t+1} + \cdots + p_n, \quad p_i \in A_i \cap E_i.$$ 

Hence, the row indexed by $p$, as with matrix constructions in [CE00, D’A02], contains a multiple of $f_t(x)$:

$$x^{p_0 + \cdots + p_{t-1} + p_{t+1} + \cdots + p_n} f_t(x),$$

and the diagonal element is the coefficient of the monomial with exponent $b_{ij}$ in $f_t(x)$. Similarly, for the rows corresponding to lattice points in non-mixed cells.

Let us sketch the asymptotic complexity of our algorithm. Alg. B, implemented by the direct approach of [CE00], comprises of two main steps. First, the computation of the vertices of each $Q_i$ which is typically dominated. Second, we compute RC for all $p \in \mathcal{E}$, which includes the matrix construction. Both steps can be reduced to linear programming with $C$ constraints in $V$ variables, and coefficient bitsize $B$. If we use a poly-time algorithm such as Karmarkar’s [Kar84], the bit complexity is $O^*(|E|^2 B^2)$, where $B$ depends on the bitsize of the input coordinates and of $\delta, \delta_{ij}s$. It is related to the probability that the chosen perturbations are not sufficiently generic; see [CE00] for the full analysis.

Let $m$ be the maximum number of vertices of the $Q_i$, $r$ the total number of $c_{ij}s$, and let $O^*(\cdot)$ indicate that we ignore polylog factors. The linear programs have complexity $O^*(r^2 B^2) = O^*(m^2 B^2)$ because $r$ is bounded by the total number $O(m^{n/2})$ of faces in $Q$, which is quite pessimistic. In an output sensitive manner, $r = O(|\mathcal{E}|)$, because the addition of every $c_{ij}$ is made in order to handle at least one distinct point in $\mathcal{E}$. Hence, the complexity of constructing the Macaulay-type formula is $O^*(|\mathcal{E}|^3 B^2)$. This holds for matrices in sparse and dense representation. For generalized unmixed systems, one can use $|\mathcal{E}| = O(k^a e^a D)$ from [CE00, thm.3.10], where $k = \max_i \{|k_i|\}$, $D$ is the total degree of the toric resultant as a polynomial in the input coefficients, and $e$ the basis of natural logarithms.

A better implementation finds RC for one point in a maximal cell, then enumerates all points in this cell in time proportional to their cardinality multiplied by a polynomial in $m, n, B$ [Emi02, thm.16]. The neighbours of these points which lie outside the cell will yield new cells, so as to explore the entire Minkowski sum; detecting new cells does not increase the overall complexity. If $S \leq |\mathcal{E}|$ is the number of maximal cells containing at least one lattice point, Alg. B has complexity $O^*(S r^2 B^2 + |\mathcal{E}|) = O^*(S |\mathcal{E}|^2 B^2)$, where typically, $S \ll |\mathcal{E}|$. This may be compared to the complexity of Alg. A at the end of the next section.

3 Recursive construction

This section discusses D’Andrea’s recursive construction of a Macaulay-type formula [D’A02]. There are certain free parameters in the algorithm which we specify so as to obtain a version very similar to our approach.

At the input of the 0-step the algorithm may use an additional polytope $mQ$, for any $m \in \mathbb{R}$, which we omit by setting $m = 0$. We describe the $t$-th recursive step, for $t = 0, 1, \ldots, n - 1$.

Algorithm A. The input are polytopes

$$l_0 P^{(t)}, \ldots, l_{t-1} P^{(t)}, k_t P^{(t)}, \ldots, k_n P^{(t)} \subset \mathbb{R}^{n-t}, \quad l_i \in [0, k_i] \cap \mathbb{Q},$$

the integer lattice $L^{(t)}$ spanned by $\sum_{i=0}^{t} A_i \cap k_i P^{(t)}$, and perturbation vector $\delta_t \in \mathbb{Q}^{n-t}$. Here, $k_{t} P^{(t)}, \ i \geq t$, is an $(n-t)$-dimensional face of $k_t Q$, thus $P^{(t)} \cap Q = Q$. Also, $P^{(t)}$ is a facet of $P^{(t-1)}$, and $l_i P^{(t)}, \ i < t$, is homothetic to $k_i P^{(t)}$. These constructions shall be specified at the Recursion Phase. Also, $L^{(0)} = Z$ and $\delta_0 = \delta$. 

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Construction Phase: Vertex \( b_{tj} \in k_tP(t) \cap A_t \) is lifted to 1. We require that \( b_{tj} = c_{tjs} - \delta_{tjs}, \) where \( s \) is determined by the face \( k_tP(t) \). All other vertices of all input polytopes are lifted to 0. This is the primary lifting which partitions the Minkowski sum of the input polytopes into a primary cell
\[
l_0P(t) + \cdots + l_{t-1}P(t) + b_{tj} + k_{t+1}P(t) + \cdots + k_nP(t) + \delta_t,
\]
of dimension \( n - t \), and several secondary cells. Each secondary cell is defined by an inner normal \( v \in \mathbb{Q}^{n-1} \) to a facet of \( k_tP(t) \) not containing \( b_{tj} \).

Polytopes \( \sum_{i=0}^{t-1} l_iP(t), k_{t+1}P(t), \ldots, k_nP(t) \) are lifted by applying the restriction of \( \beta \) on them. We consider \( \beta \) fixed throughout the algorithm. The upper hull of the Minkowski sum of the lifted polytopes induces a mixed subdivision of \( \sum_{i=0}^{t-1} P(t) + k_{t+1}P(t) + \cdots + k_nP(t) \), which is then perturbed by \( \delta_t \). The lattice points \( p \) of \( L(t) \) contained in the perturbed subdivision are assigned RC by Definition 3. This also assigns RC to points \( p + b_{tj} \) contained in the intersection of \( 1 \) with \( L(t) \). Let us take care of the \( c_{tjs} \). If point \( p \) lies in
\[
(F + F_{t+1} + \cdots + F_n + \delta_t) \cap L(t),
\]
where \( F_i \subset k_iQ_i, \ i > t, \ F \subset \sum_{i=0}^{t-1} l_iP(t), \) having \( \text{RC}(p) = (h, j) \), where \( F_h = c_{hjs} = b_{hj} + \delta_{hjs} \), then the corresponding matrix row is filled in by \( x^p - b_{tj} f_i \).

Face \( F \subset \sum_{i=0}^{t-1} P(t) \) in \( 2 \), can be written as \( F = l_0F_0 + \cdots + l_{t-1}F_{t-1} \), where \( F_i \subset P(t) \) for \( i < t \). Moreover, every cell in \( 1 \) is the Minkowski sum of \( b_{tj} \) and the cell in \( 2 \).

Mixed cells of type 0 are defined here as in Section 2. A \( t \)-mixed cell with respect to Alg. A, for \( t > 0 \), shall have \( n - t \) linear summands from polytopes \( k_{t+1}P(t), \ldots, k_nP(t) \) and a zero-dimensional summand from polytope \( \sum_{i=0}^{t-1} l_iP(t) \). This summand can be written as \( l_0p_0 + \cdots + l_{t-1}p_{t-1}, \) where \( p_i \in P(t) \), for \( i = 0, \ldots, t - 1 \) and \( l_ip_i \) stands for a scalar multiple of \( p_i \), seen as a vector. This leads to:

Lemma 7. The maximal cells at step \( t \) of Alg. A are, for some \( j \) and \( l_i \in [0, k_i] \), of the form:
\[
l_0F_0 + \cdots + l_{t-1}F_{t-1} + b_{tj} + k_{t+1}F_{t+1} + \cdots + k_nF_n + \delta_t,
\]
where \( F_i \) is the projection of a face of the upper hull of \( P(t) \) lifted by \( \beta \), and
\[
\text{dim}((F_0, \ldots, F_{t-1}, F_{t+1}, F_n)) = n - t.
\]
Specifically, the \( t \)-mixed cells in Alg. A are:
\[
l_0p_0 + \cdots + l_{t-1}p_{t-1} + b_{tj} + k_{t+1}E_{t+1} + \cdots + k_nE_n + \delta_t,
\]
where \( E_{t+1}, \ldots, E_n \) are projections of edges on the upper hull of \( P(t) \) lifted by \( \beta \), \( \text{dim}((E_{t+1}, \ldots, E_n)) = n - t \), and points \( p_i \in P(t) \), for \( i = 0, \ldots, t - 1 \).

Example 8. Consider the three pentagons of Example 22. In the 0 step of the recursion, \( b_{t0} \) is lifted to 1, while all other vertices of all polygons are lifted to 0. Then, the primary cell is subdivided using lifting \( \beta \). The primary and secondary cells are shown in Figure 4, left, in white and grey color respectively (also in Figure 7). To illustrate Lemma 7, consider cells 1,2 and 3 of the primary cell. They can be written as

Cell 1: \( b_{t0} + \text{CH}(c_{122}, c_{143}, c_{154}) + b_{21} \), non-mixed.

Cell 2: \( b_{t0} + (c_{122}, c_{154}) + (b_{21}, b_{21}) \), 1-mixed.

Cell 3: \( b_{t0} + \text{CH}(c_{122}, b_{11}, c_{154}) + b_{21} \), non-mixed.

Now, consider the recursion step of Alg. A at the secondary cell of step 0 with respect to vector (1,0) shown in Figure 4, right. In this cell the algorithm recurses on a segment containing points \((0, 4), (0, 5), (0, 6), (0, 7)\). This segment is partitioned into new primary and secondary cells and the new primary cell is subdivided again using \( \beta \). The cells are:
Secondary cell: $\frac{20}{39}b_{03} + (b_{12}, b_{13}) + b_{23}$, 2-mixed.

Cell 4: $\frac{20}{39}(b_{02}, b_{03}) + b_{11} + b_{22}$, non-mixed.

Cell 5: $\frac{20}{39}b_{02} + b_{11} + (b_{22}, b_{23})$, 1-mixed.

For details see Example 22.

Recursion Phase: When $t = n - 1$, the algorithm terminates, since it has reached the Sylvester case. Otherwise, it recurses: let $P^{(t+1)}$ be the facet of $P^{(t)}$ supported by $v$. The (perturbed) secondary cell corresponding to $v$ is

$$F_v = l_0P^{(t+1)} + \cdots + l_{t-1}P^{(t+1)} + \text{CH}(b_{tj}, k_tP^{(t+1)})$$

$$+ k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_l.$$

(5)

Its associated diameter is

$$d_v = b_{tj} \cdot v - \min_{p \in \text{CH}(b_{tj}, k_tP^{(t+1)})} \{p \cdot v\} \in \mathbb{N}^s,$$

where $\cdot$ stands for inner product. We define two sublattices of $L^{(t)}$: $L_+^{(t)}$ is spanned by $\sum_{i=t+1}^n A_i \cap k_tP^{(t+1)}$ and $L_v^{(t)}$ is the sublattice orthogonal to $v$. They have the same dimension, so we define the (finite) index $\text{ind}_v = [L_v^{(t)} : L_+^{(t)}]$, equal to the quotient of the volumes of their base cells. Let $q$ range over the ind$_v$ coset representatives for $L_+^{(t)}$ in $L_v^{(t)}$.

Let $l_t \in [0, k_t]$ take $d_v$ distinct values corresponding to different values of $p \cdot v$ for all $p \in (\text{CH}(b_{tj}, k_tP^{(t+1)}) + \delta_t) \cap L^{(t)}$. Note that $l_tP^{(t+1)}$ is homothetic to $k_tP^{(t+1)}$. Let $\delta'_t \in \mathbb{Q}^{n-t}$ be a translation vector such that $l_tP^{(t+1)} + \delta'_t$ contains at least one point in $(\text{CH}(b_{tj}, k_tP^{(t+1)}) + \delta_t) \cap L^{(t)}$.

In particular, $l_tP^{(t+1)} + \delta'_t$ equals $k_tP^{(t+1)}$ if and only if $l_t = k_t$, and vertex $b_{tj}$, if and only if $l_t = 0$, otherwise it equals $(\text{CH}(b_{tj}, k_tP^{(t+1)}) + \delta_t) \cap H$, where $H$ is a hyperplane parallel to a supporting hyperplane of $k_tP^{(t+1)}$; see [D’A02, lem.3.3]. By abuse of notation, in the rest of this paper we shall denote $H$, and the supporting hyperplanes of faces $k_tP^{(t+1)}$ and $b_{tj}$ of the previous convex hull, as $\langle l_tP^{(t+1)} \rangle$.

Points in $(F_v + \delta_t) \cap L^{(t)}$ are partitioned into $d_v$ subsets (one per value of $l_t$), called slices, of the form

$$l_0P^{(t+1)} + \cdots + l_{t-1}P^{(t+1)} + (l_tP^{(t+1)} + \delta'_t) + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_t \cap L^{(t)},$$

(6)
which can be rearranged as

\[ l_0 P^{t+1} + \cdots + l_t P^{t+1} + k_{t+1} P^{t+1} + \cdots + k_n P^{t+1} + \delta_\lambda \cap L^{(t)}, \]  

(7)

where \( \delta_\lambda = \delta t + \beta \). Moreover, \( \delta_\lambda \) can be decomposed as \( \delta_\lambda = \delta_\lambda \cap L^{(t)} + \delta_\lambda \cap L^{(t)} \), where \( \delta_\lambda \cap L^{(t)} \in L^{(t)} \). Now, every point in (7) corresponds to a point in

\[ l_0 P^{t+1} + \cdots + l_t P^{t+1} + k_{t+1} P^{t+1} + \cdots + k_n P^{t+1} + \delta_\lambda \cap L^{(t)} \cap (q + L^{(t)}), \]

for some coset representative \( q \). Set \( \delta_{t+1} := \delta_\lambda - q \), \( L^{(t+1)} := L^{(t)} \), and observe that point \( p \) belongs to (7) if and only if point

\[ p' := p - \delta_\lambda - q \]  

(8)

belongs to

\[ l_0 P^{t+1} + \cdots + l_t P^{t+1} + k_{t+1} P^{t+1} + \cdots + k_n P^{t+1} + \delta_{t+1} \cap L^{(t+1)}. \]  

(9)

We call this set a piece; \( \delta_{t+1} \) carries the information to define the piece from the input polytopes and \( L^{(t+1)} \). The algorithm recurses on each of the \( \delta_\lambda \) such pieces. The set

\[ l_0 P^{t+1}, \ldots, l_t P^{t+1}, k_{t+1} P^{t+1}, \ldots, k_n P^{t+1}, \delta_{t+1} \]

over \( L^{(t+1)} \) is exactly like the original input, only one dimension lower. This completes the algorithm.

Remark 9. Since every point \( p' \) in a piece corresponds bijectively to a point \( p \) in a slice via the monomial bijection (8), we shall consider a piece as a subset of a slice and omit the translation.

At the end of the recursion, \( RC \) is defined on \( E \). Alg. A defines a partition of \( E \) in the form of a collection of mixed subdivisions of primary cells (of decreasing dimension). The edges of the cells of this partition, coming from polytope \( Q_i \), are defined by any point in \( A_i \) or among the \( c_{ij} \), for all valid \( j, s \), and may be multiplied by a rational number in \( (0, k_i] \).

D’Andrea’s algorithm uses at every construction step the matrix construction algorithm of [CE00], so its complexity is dominated by \( O(|E|n) \) linear programs, since every \( p \in E \) may require \( O(n) \) of them for its image under \( RC \) to be determined. Each linear program has bit complexity \( O(n^{2-2}m^2B^2) \), by Karmarkar’s algorithm, where \( m \) is the maximum number of vertices of the \( Q_i \), and \( B \) is the bitsize of the input coordinates. This process essentially decides in which slice of which secondary cell lies \( p \).

Although this subdivision contains much more cells than Alg. B, the asymptotic analysis indicates that the latter is competitive for large \( n \); see the end of section 2 for comparing with Alg. A.

4 Equivalence of constructions

This section demonstrates that both approaches define the same Macaulay-formula. Intuitively, the single-lifting algorithm (Alg. B) has an overall effect very similar to that of Alg. A, since they both use \( \beta \). The former partitions \( E \) into sets of points in \( n \)-dimensional cells and assigns \( RC \), whereas Alg. A partitions \( E \) into subsets which, at step \( t \), lie on the intersection of a \( (n-t) \)-dimensional hyperplane with an \( n \)-dimensional cell of \( \beta \). Note that the intersection itself, as a subset of \( \mathbb{R}^{n-t} \), does not coincide with the cell of Alg. A. However, their set difference is of infinitesimal volume and thus contains no lattice points. Although both algorithms use \( \beta \) to subdivide their input polytopes, they do so in a distinct fashion: Alg. B applies \( \beta \) to every \( Q_i \), whereas Alg. A does so recursively to a different set of polytopes at every step.

In the rest of the paper, for simplicity, we shall omit the translation vectors \( \delta_t \). Moreover, unless otherwise stated, we shall treat every slice and piece as a polytope and not as the set of points in the intersection of this polytope with an appropriate lattice. In particular, we shall be interested
only on the form of a slice or piece as a Minkowski sum of polytopes. The existence of a translation vector, for this polytope to contain integer points in the considered lattice, shall be implied.

We now establish the correspondence between the two algorithms for \( t = 0 \), then generalize to \( t > 0 \). We introduce the notation \( \text{pr.cell}^{(A)}_i \), \( \text{sec.cell}^{(A)}_i \), where \( i \) indicates the recursion step of Alg. A and \( X \in \{A, B\} \) indicates the algorithm under consideration. At step 0 of Alg. A, \( b_{01} \) is lifted to 1, while every other vertex of all input polytopes to 0; this creates a primary cell

\[
\text{pr.cell}^{(A)}_0 := b_{01} + k_1 Q + \cdots + k_n Q,
\]

and several secondary cells of the form

\[
\text{sec.cell}^{(A)}_0 := \text{CH}(b_{01}, k_0 P^{(1)}) + k_1 P^{(1)} + \cdots + k_n P^{(1)},
\]

each corresponding to a facet \( P^{(1)} \) of \( Q \) not containing \( b_{01} \). In Alg. B, \( c_{011} \) plays the role of \( b_{01} \) and this leads to a group of cells covering the corresponding primary cell

\[
\text{pr.cell}^{(B)}_0 := c_{011} + k_1 Q + \cdots + k_n Q,
\]

and several groups of cells, each group covering

\[
\text{sec.cell}^{(B)}_0 := \text{CH}(c_{011}, k_0 P^{(1)}) + k_1 P^{(1)} + \cdots + k_n P^{(1)},
\]

which is a typical \( n \)-dimensional secondary cell with respect to Alg. B. Not all cells in \( \text{sec.cell}^{(B)}_0 \) may have \( k_i P^{(1)} \) as a summand. Those who do not, have a summand where some or all of the vertices of \( k_i P^{(1)} \) are replaced by the corresponding additional points \( c_{ij} \) slab from Definition 5.

**Remark 10.** All cells within \( \text{pr.cell}^{(A)}_0 \) and \( \text{pr.cell}^{(B)}_0 \) differ only at their first summand; the former are of the form \( b_{01} + F_1 + \cdots + F_n \), whereas the latter are \( c_{011} + F_1 + \cdots + F_n \), where \( F_i \) is a face of \( Q_1 \), since \( \beta \) is used by both algorithms to subdivide \( Q_1 + \cdots + Q_n \), and \( c_{011} = b_{01} + \delta_{011} \).

**Lemma 11.** \( \text{pr.cell}^{(A)}_0 \cap E = \text{pr.cell}^{(B)}_0 \cap E \), and points in this set are assigned the same RC under both algorithms.

**Proof.** Recall that \( \delta_0 = \delta \) and consider the subdivision of \( \sum_{i=0}^n Q_i \) induced by \( \beta \) and compare \( \text{pr.cell}^{(A)}_0 + \delta \) and \( c_{011} + \sum_{i=0}^n Q_i + \delta = b_{01} + \delta_{011} + \sum_{i=0}^n Q_i + \delta \). These polytopes differ by \( \delta_{011} \), which is very small. Moreover, by the choice of \( \delta \), the boundary of \( \text{pr.cell}^{(A)}_0 + \delta \) has no points in \( Z \). Since, by Definition 4, \( \|\delta\| \gg \|\delta_{011}\| \), the two polytopes contain the same \( Z \)-points. This settles the first claim. The second claim follows from Remark 10 and the fact that the two subdivisions may only differ in cells having vertex \( b_{01} \) instead of \( c_{011} \). Since \( c_{011} - b_{01} = \delta_{011} \) is very small compared to \( Z \), even these cells contain the same \( Z \)-points.

**Example 12.** Let us return to our running Example 22. It holds that \( \text{pr.cell}^{(A)}_0 \cap E = \text{pr.cell}^{(B)}_0 \cap E \). Now, consider points \( (8, 1) \), \( (7, 2) \) and \( (4, 4) \), see Figures 7,8. They belong to cells of \( \text{pr.cell}^{(A)}_0 \) and \( \text{pr.cell}^{(B)}_0 \) as in the following table:

<table>
<thead>
<tr>
<th>point</th>
<th>cell in ( \text{pr.cell}^{(A)}_0 )</th>
<th>cell in ( \text{pr.cell}^{(B)}_0 )</th>
<th>type</th>
<th>RC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8, 1)</td>
<td>( b_{01} + c_{154} + \text{CH}(b_{22}, b_{23}, b_{25}) )</td>
<td>( c_{011} + c_{154} + \text{CH}(b_{22}, b_{23}, b_{25}) )</td>
<td>non-mixed</td>
<td>(1, 5)</td>
</tr>
<tr>
<td>(7, 2)</td>
<td>( b_{01} + (c_{143}, c_{154}) + (b_{23}, b_{24}) )</td>
<td>( c_{011} + (c_{143}, c_{154}) + (b_{23}, b_{24}) )</td>
<td>0-mixed</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>( b_{01} + (c_{143}, c_{154}) + (b_{22}, b_{23}) )</td>
<td>( c_{011} + (c_{143}, c_{154}) + (b_{22}, b_{23}) )</td>
<td>0-mixed</td>
<td>(0, 1)</td>
</tr>
</tbody>
</table>

Note that, for simplicity, we have omitted the global perturbation vector \( \delta \).
Each $sec.cell_{0}^{(A)}$ is divided by Alg. A into slices

$$l_{0}P^{(1)} + k_{1}P^{(1)} + \cdots + k_{n}P^{(1)},$$

one for each value of $l_{0} \in [0, k_{0}]$. Each slice is partitioned into pieces on which Alg. A recurses producing $(n-1)$-dimensional primary cell

$$pr.cell_{1}^{(A)} := l_{0}P^{(1)} + b_{1j} + k_{2}P^{(1)} + \cdots + k_{n}P^{(1)},$$

and secondary cells

$$sec.cell_{1}^{(A)} := l_{0}P^{(2)} + CH(b_{1j}, k_{1}P^{(2)}) + k_{2}P^{(2)} + \cdots + k_{n}P^{(2)}.$$  

Every piece of a given slice lies on lattice $L^{(1)}$ and can be thought of as the intersection of a translation of that slice, regarded as a polytope, with $L^{(1)}$. Recall that, by Remark 9, we shall consider a piece as subset of a slice.

Similarly to Alg. A, we can partition the corresponding $sec.cell_{0}^{(B)}$ into slices:

$$l'_{0}P^{(1)} + k_{1}P^{(1)} + \cdots + k_{n}P^{(1)},$$

by intersecting $CH(c_{011}, k_{0}P^{(1)})$ with a hyperplane parallel to (a supporting hyperplane of) $k_{0}P^{(1)}$. Recall that we denote this hyperplane as $(l'_{0}P^{(1)})$.

**Remark 13.** Observe that each slice of $sec.cell_{0}^{(B)}$ (resp. $sec.cell_{0}^{(A)}$) parameterized by $l'_{0}$ (resp. $l_{0}$), is homothetic to a facet of this secondary cell, supported by $(k'_{0}P^{(1)})$ (resp. $(k_{0}P^{(1)})$). Moreover, this homothecy is defined by homothecy only on the first summand $k_{0}P^{(1)}$ of this facet.

**Example 14.** To illustrate Remark 13, consider in our running Example 22 the secondary cell with respect to Alg. A

$$F_{v_{3}} = CH(b_{01}, k_{0}F_{v_{3}}) + k_{1}F_{v_{3}} + k_{2}F_{v_{3}} + \delta,$$

defined by the facet $F_{v_{3}} = ((3, 0), (1, 2))$ of $Q$ supported by $v_{3} = (-1, -1)$, and its slice

$$(l_{0}F_{v_{3}} + \delta') + k_{1}F_{v_{3}} + k_{2}F_{v_{3}},$$

where $l_{0} = \frac{32}{60}$ and $\delta' = \left(\frac{7}{15}, 0\right)$. This slice contains the integer points $(11, 0), (10, 1), (9, 2), (8, 3), (7, 4), (6, 5), (5, 6), (4, 7)$ and is the dashed segment in Figure 5. It is homothetic to the facet

$$k_{0}F_{v_{3}} + k_{1}F_{v_{3}} + k_{2}F_{v_{3}} + \delta$$

of $F_{v_{3}}$ and the homothecy is defined by the homothecy $l_{0}F_{v_{3}} + \delta'$ of the 0-summand $k_{0}F_{v_{3}}$ of the facet, see Figure 5. The second slice of $F_{v_{3}}$ is

$$\left(\frac{1}{30}F_{v_{3}} + \frac{29}{30}, 0\right) + k_{1}F_{v_{3}} + k_{2}F_{v_{3}} + \delta$$

and contains integer points $(10, 0), (9, 1), (8, 2), (7, 3), (6, 4), (5, 5), (4, 6)$. It is homothetic to the facet $(13)$ of $F_{v_{3}}$ and the homothecy is defined by the homothecy $\frac{1}{30}F_{v_{3}} + \left(\frac{29}{30}, 0\right)$ of the 0-summand $k_{0}F_{v_{3}}$ of the facet, see Figure 5 (dotted segment).

Hyperplanes $(l'_{0}P^{(1)})$ and $(l_{0}P^{(1)})$ are identical; they differ only on the homothecy on $k_{0}P^{(1)}$ expressed by $l'_{0}$ and $l_{0}$ respectively. Obviously, $l'_{0} \approx l_{0}$ because $c_{011} \approx b_{01}$. Note that we omit the translation vector so that the slice lies in $sec.cell_{0}^{(B)}$. Thus, corresponding slices contain the same points in the lattice $L^{(0)} = Z$. This, moreover, leads to the following extension of Lemma 11.

**Lemma 15.** Every maximal cell of the subdivision induced by $\beta$ on $pr.cell_{1}^{(A)}$ corresponds to the intersection of a unique maximal cell of the same type in $sec.cell_{0}^{(B)}$, with a slice defined by hyperplane $(l'_{0}P^{(1)})$, for some $l'_{0}$. The cells contain the same points in $L^{(1)}$, with the same image under $RC$. 

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Proof. Any maximal cell in $pr.cell_1^{(A)}$ has the form $l_0 F_0 + b_{ij} + k_2 F_2 + \cdots + k_n F_n$, where faces $F_i \subset P^{(1)}$, $i = 0, 2, \ldots, n$, have dimensions adding up to $n - 1$. Recall $pr.cell_1^{(A)}$ lies on a slice of $sec.cell_0^{(A)}$ parameterized by the value of $l_0$ hence, when $b$ is employed, it gives rise to the same subdivision in every such primary cell. By construction, subspace $\langle b_{01}, F_0 \rangle$ is orthogonal and complementary to $\langle P^{(1)} \rangle$.

In $k_1 P^{(1)}$, point $c_{1js}$ is lifted sufficiently higher than any other, so there exist maximal cells in $sec.cell_0^{(B)}$ that has it as summand. The other summands are induced by $\beta$ on $CH(c_{011}, k_0 P^{(1)})$, $k_2 P^{(1)}$, \ldots, $k_n P^{(1)}$. These $n$-dimensional cells of Alg. B correspond, when intersected with the slice parameterized by $\langle l'_0 P^{(1)} \rangle$, to $(n - 1)$-dimensional cells in $pr.cell_1^{(A)}$. It is straightforward to show that, for $l'_0 \in [0, k_0]$ and any $\beta$-induced cell in this Minkowski sum, its intersection with the slice defined by $\langle l'_0 P^{(1)} \rangle$ is a $\beta$-induced cell in $l'_0 P^{(1)} + k_2 P^{(1)} + \cdots + k_n P^{(1)}$.

There exists $l'_0 \approx l_0$ that establishes the Lemma, because $\beta$ is applied to $(n - 1)$-dimensional Minkowski sums which are almost identical, and the effect of $b_{ij}$ and $c_{1js}$ is the same in what concerns the lattice points in corresponding cells, following the proof of Lemma 11.

Example 16. We shall return to our running example to illustrate Lemma 15. Consider the slice

$$(l_0 F_{v_3} + \delta') + k_1 F_{v_3} + k_2 F_{v_3} + \delta$$

of the secondary cell with respect to Alg. A

$$sec.cell_0^{(A)} = CH(b_{01}, k_0 F_{v_3}) + k_1 F_{v_3} + k_2 F_{v_3} + \delta,$$

where $l_0 = \frac{32}{60}$, $\delta' = (\frac{7}{15}, 0)$, $\delta = (-\frac{1}{30}, -\frac{1}{30})$, see also equation (27). This slice is obtained by intersecting $CH(b_{01}, b_{04}, b_{05})$ with the hyperplane $\langle l_0 F_{v_3} \rangle := \langle \frac{32}{60} F_{v_3} + (\frac{7}{15}, 0) \rangle$, and contains integer points $(11, 0), (10, 1), (9, 2), (8, 3), (7, 4), (6, 5), (5, 6), (4, 7)$ in $L$. The corresponding slice of $sec.cell_0^{(B)}$ is obtained by intersecting $CH(c_{011}, b_{04}, b_{05})$ with the hyperplane $\langle l'_0 F_{v_3} \rangle := \langle \frac{639}{1199} F_{v_3} + (\frac{1274}{2725}, \frac{28}{89925}) \rangle$, see Figure 6 (dotted segment). It contains the same points in $L$.

Slice (15) of $sec.cell_0^{(A)}$ contains two pieces in $L^{(1)} := L_+ = \langle (9, 0), (7, 2) \rangle \cong 2\mathbb{Z}$:

$$piece_0 := \frac{32}{60} F_{v_3} + k_1 F_{v_3} + k_2 F_{v_3} + (\frac{17}{30}, \frac{31}{30}),$$

$$piece_1 := \frac{32}{60} F_{v_3} + k_1 F_{v_3} + k_2 F_{v_3} + (\frac{13}{30}, \frac{61}{30}).$$
Figure 6: Example 16: The two pieces of the secondary cell w.r.t. \((-1,-1)\) of Alg. A and the correspondence between their cells and the cells of the similar secondary cell w.r.t. Alg. B

Piece (16) is partitioned into a primary cell \(\frac{32}{60}F_{v_3} + b_{15} + k_2F_{v_3} + (-\frac{17}{30}, -\frac{31}{30})\) and a secondary cell \(\frac{32}{60}b_{04} + k_1F_{v_3} + b_{24} + (-\frac{17}{30}, -\frac{31}{30})\). Then, lifting \(\beta\) induces a mixed subdivision on the primary cell consisting of the cells

\[
\sigma_1 = \frac{32}{60}F_{v_3} + b_{15} + b_{25} + (-\frac{17}{30}, -\frac{31}{30}) \text{ and } \sigma_2 = \frac{32}{60}b_{04} + b_{15} + k_2F_{v_3} + (-\frac{17}{30}, -\frac{31}{30}).
\]

Cell \(\sigma_1\) is non-mixed and contains point \((9,0)\) \( \in L_+\), which translates to point \((10,1)\) \( \in L\). This cell corresponds to the intersection of the slice of \(sec.cell_0^{(B)}\), defined by hyperplane \(\langle l'_0F_{v_3} \rangle\), with its non-mixed cell \(\text{CH}(c_{011}, b_{04}, b_{05}) + c_{15} + b_{25} + \delta\). Cell \(\sigma_2\) is 1-mixed and contains the point \((7,2)\) \( \in L_+\) which translates to the point \((8,3)\) \( \in L\). This cell corresponds to the intersection of the slice of \(sec.cell_0^{(B)}\), defined by hyperplane \(\langle l'_0F_{v_3} \rangle\), with the 1-mixed cell with respect to Alg. B \((c_{011}, b_{04}) + c_{154} + (b_{24} + b_{25}) + \delta\), see Figure 6, (left).

The second piece (17) is partitioned into a primary cell \(\frac{32}{60}F_{v_3} + b_{15} + k_2F_{v_3} + (\frac{13}{60}, -\frac{61}{30})\) and a secondary cell \(\frac{32}{60}b_{04} + k_1F_{v_3} + b_{24} + (\frac{13}{60}, -\frac{61}{30})\). Lifting \(\beta\) induces a mixed subdivision on the primary cell consisting of the cells

\[
\sigma'_1 = \frac{32}{60}F_{v_3} + b_{15} + b_{25} + (\frac{13}{60}, -\frac{61}{30}) \text{ and } \sigma'_2 = \frac{32}{60}b_{04} + b_{15} + k_2F_{v_3} + (\frac{13}{60}, -\frac{61}{30}).
\]

The former is non-mixed and contains point \((11,-2)\) \( \in L_+\) corresponding to \((11,0)\) \( \in L\). It corresponds to the intersection of the slice cell of \(sec.cell_0^{(B)}\), defined by hyperplane \(\langle l'_0F_{v_3} \rangle\), with its non-mixed cell \(\text{CH}(c_{011}, b_{04}, b_{05}) + c_{154} + b_{25} + \delta\). Cell \(\sigma'_2\) is 1-mixed and contains the integer point \((9,0)\) \( \in L_+\) corresponding to point \((9,2)\) \( \in L\). It corresponds to the intersection of the slice defined by hyperplane \(\langle l'_0F_{v_3} \rangle\) with the 1-mixed cell of \(sec.cell_0^{(B)}\) \((c_{011}, b_{04}) + c_{154} + (b_{24} + b_{25}) + \delta\), see Figure 6, (right).

In each \(sec.cell_0^{(B)}\) we distinguish 2 types of cells: cells in

\[
pr.cell_1^{(B)} := \text{CH}(c_{011}, k_0P^{(1)}) + c_{1js} + k_2P^{(1)} + \ldots + knP^{(1)},
\]

which, by Lemma 15, contains exactly the integer points in all primary cells of Alg. A of the form (10) (for each slice/coset), and for each facet \(P^{(2)}\) of \(P^{(1)}\), cells in

\[
sec.cell_1^{(B)} := \text{CH}(c_{011}, k_0P^{(2)}) + \text{CH}(c_{1js}, k_1P^{(2)}) + k_2P^{(2)} + \ldots + knP^{(2)}.
\]

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Note that both \(pr.cell_1^{(B)}\) and \(sec.cell_1^{(B)}\) are \(n\)-dimensional, whereas \(pr.cell_1^{(A)}\) and \(sec.cell_1^{(A)}\) are \((n-1)\)-dimensional.

**Remark 17.** Every maximal cell in \(sec.cell_1^{(B)}\) must have summands \(F_0 = \text{CH}(c_{011}, G_0), F_1 = \text{CH}(c_{11j}, G_1)\), for some \(G_0 \subset k_0 P^{(2)}\) and \(G_1 \subset k_1 P^{(2)}\).

A similar argument as in Lemma 15, implies that (19) contains exactly the integer points in the union of all secondary cells (11) defined over the various values of \(l_0 \in [0, k_0]\), for a given \(j\). The recursion steps of Alg. A, for \(t \geq 2\) are defined over a chain of facets \(P^{(2)} \supset P^{(3)} \supset \cdots \supset P^{(n-1)}\). Hence, every \(pr.cell_t^{(A)}\), for \(t > 1\), contains integer points in \(sec.cell_1^{(B)} \cap Z\). Therefore, we generalize the correspondence between the two algorithms by focusing on \(sec.cell_1^{(B)}\).

**Lemma 18.** (Main) Every maximal cell of the subdivision induced by \(\beta\) on \(pr.cell_t^{(A)}\), for \(t \geq 2\), corresponds to the intersection of hyperplane \(\langle l_{t-1}^{(t)}, P^{(t)} \rangle\), for some \(l_{t-1}^{(t)} \approx l_{t-1} \in [0, k_{t-1}] \cap \mathbb{Q}\), with a unique maximal cell in \(sec.cell_1^{(B)}\), of the same type. The cells contain the same points in lattice \(L^{(t)}\) with the same image under \(RC\).

**Proof.** Primary cells of step \(t\) lie on \((n-t)\)-dimensional slices of the \((n-t+1)\)-dimensional \(sec.cell_t^{(A)}\), parameterized by the value of \(l_{t-1} \in [0, k_{t-1}]\):

\[
\sum_{i=0}^{n} l_i P^{(t)} + k_t P^{(t)} + \cdots + k_n P^{(t)}. \tag{20}
\]

Similarly to Remark 13, let \(l_0, \ldots, l_{t-1}, l_t \in [0, k_t] \cap \mathbb{Q}\), define the homotheties on the first \(t\) summands of (20) and the corresponding hyperplanes \(\langle l_0 P^{(t)}, \ldots, l_{t-1} P^{(t)} \rangle\). Note, that \(pr.cell_t^{(A)}\) is a subset of (20) and is subdivided by \(\beta\) into maximal cells of the form (3).

Intersecting \(sec.cell_1^{(B)}\) with the above hyperplanes, yields a \((n-t)\)-dimensional subset:

\[
\sum_{i=0}^{n} l_i^{(t)} P^{(t)} + k_t P^{(t)} + \cdots + k_n P^{(t)}. \tag{21}
\]

This subset can also be obtained by directly intersecting \(sec.cell_1^{(B)}\) with \(\langle l_{t-1} P^{(t)} \rangle\). Now, \(l_i^{(t)} \approx l_i\), for \(i = 0, 1, \ldots, t-1\) because \(c_{ij} \approx b_{ij}\). For \(i = 0, \ldots, t-1\), each \(l_i^{(t)}\) defines a hyperplane \(\langle l_i P^{(t)} \rangle\) identical to \(\langle l_i P^{(t)} \rangle\), except on the homothecy on the \(i\)-th summand. Hence, (21) is very similar to (20) in the sense that they contain the same integer points in \(L^{(t)}\) and their volumes differ infinitesimally.

By Definition 5 there exist \(n\)-dimensional cells in \(sec.cell_1^{(B)}\) which have \(c_{ij}\) as a summand. The intersection of each of these cells with (21) shall also have \(c_{ij}\) as a summand, because this is the only point lifted highest in \(P^{(t)}\). These cells correspond to the primary cell with respect to Alg. A of the slice (20). Moreover, this intersection is a \(\beta\)-induced cell in (21):

\[
\sum_{i=0}^{n} l_i^{(t)} F_0 + \cdots + l_{t-1}^{(t)} F_{t-1} + c_{ij} + k_{t+1} F_{t+1} + \cdots + k_n F_n, \tag{22}
\]

which contains the same integer points as (3). Since \(\beta\) is applied on \((n-t)\)-dimensional polytopes which are almost identical, both (3) and (22) are of the same type.

**Corollary 19.** Using the notation of Lemma 7, in particular for \(t\)-mixed cells of Alg. A in the form of (4), a \(t\)-mixed cell of Alg. B is of the form:

\[
k_0 E_0 + \cdots + k_{t-1} E_{t-1} + c_{ij} + k_{t+1} E_{t+1} + \cdots + k_n E_n + \delta_t \cap L,
\]

where \(E_i\) is the projection of an edge of \(Q^3\),

(a) \(\langle E_0, \ldots, E_{t-1} \rangle\) is a \(t\)-dimensional space complementary to \(\langle P^{(t)} \rangle\), and for \(i < t\), \(k_i E_i = (c_{ij}, k_i P_i)\), where \(p_i \in P^{(t)}\) in Lemma 7, and

(b) edges \(E_{t+1}, \ldots, E_n\) are the same as in (4) at Lemma 7.
Proof. For $t = 0$, the Corollary follows from Remark 10.

All 1-mixed cells with respect to Alg. B lie in (18), since every maximal cell in it has $c_{1js}$ as a summand. By Lemma 15, edges $k_2E_2, \ldots, k_nE_n$ span the $(n - 1)$-dimensional space $\langle P^{(1)} \rangle$. Hence, edge $k_0E_0$ has to be of the form $(c_{011}, k_0p_0)$, where $p_0 \in P^{(1)}$, by Lemma 15, is as in Lemma 7.(4).

Similarly, Lemma 18 implies that for $t > 1$, the last $(n - t)$ edges of any $t$-mixed cell with respect to Alg. B span the $(n - t)$-dimensional space $\langle P^{(t)} \rangle$, because $\beta$ induces the same subdivision on the last $n - t$ summands of (20) and (21). For the cell to be maximal, $\langle k_0E_0, \ldots, k_{t-1}E_{t-1} \rangle$ must be a $t$-dimensional space complementary to $\langle P^{(t)} \rangle$. By construction (see proof of Lemma 18), each $k_iE_i$, for $i < t$, is an edge in $\text{CH}(c_{ij}s, k_ip_i)$ of the form $(c_{ij}s, k_ip_i)$, where $p_i \in P^{(t)}$ is as in Lemma 7.(4).

We now consider non-mixed cells, by extending Corollary 19:

**Corollary 20.** Consider any non-mixed cell of Alg. A, which has the form of (3) in Lemma 7. It corresponds to cell:

$$\text{CH}(c_{011}, k_0F_0) + \cdots + \text{CH}(c_{(t-1)js}, k_{t-1}F_{t-1}) + c_{ts} + k_{t+1}F_{t+1} + \cdots + k_nF_n,$$

which is a non-mixed cell defined by $\beta$, where

(a) the $F_0, \ldots, F_{t-1}$ are projections of faces in $Q^\beta$, for $i < t$, and

$$\langle \text{CH}(c_{011}, k_0F_0), \ldots, \text{CH}(c_{(t-1)js}, k_{t-1}F_{t-1}) \rangle$$

is a $t$-dimensional space complementary to $\langle F_{t+1}, \ldots, F_n \rangle$,

(b) $F_0, \ldots, F_{t-1}, F_{t+1}, \ldots, F_n$ are the same in both cells.

For an illustration of Corollaries 20, 19, see Table 1 in our running Example 22. We have shown that each row of the constructed matrices, indexed by points of $\mathcal{E}$ lying in a mixed or non-mixed cell, is identical for both algorithms, where $\mathcal{E}$ is the same pointset for both algorithms.

**Theorem 21.** The Macaulay-type formula for the toric resultant of generalized unmixed systems constructed by Alg. B and that constructed by Alg. A, implementing D’Andrea’s approach [D’A02], are identical.

As a consequence of Theorem 21 and [D’A02, Thm. 3.8], follows Theorem 2.

5 A bivariate example

This section details the following example.

![Input polygons of Exam. 22 and their subdivisions induced by the lifting of Def. 6](image)

**Example 22.** Let $n = 2$, $Q$ be the pentagon with vertices $\{(1, 0), (0, 1), (0, 2), (1, 2), (3, 0)\}$, $k_0 = k_2 = 1$, $k_1 = 2$. The input polygons are $Q_i = k_iQ$, $i = 0, 1, 2$ and the input supports are $A_0 = A_2 = \ldots$
{(1, 0), (0, 1), (0, 2), (1, 2), (3, 0)}, and \(A_1 = \{(2, 0), (0, 2), (0, 4), (2, 4), (6, 0)\}\). The lattice generated by \(\sum_{i=0}^2 A_i\) is \(\mathbb{Z}^2\). The normals to the facets of \(Q\) not containing vertex \((1, 0)\) are \(v_1 = (-1, 0), v_2 = (0, -1), v_3 = (-1, -1)\). Let \(\delta = (-1/30, -1/30)\) be the global perturbation vector. See Figure 7.

\[
\begin{array}{c}
(1, 0) \quad (0, 1) \quad (0, 2) \\
(0, 2) \quad (1, 2) \quad (3, 0)
\end{array}
\]

Alg. B: We fix vertices of the input polygons in order to define the additional points required by Definition 6. Let \(b_{01} := (1, 0) \in Q_0, b_{12} := (0, 2), b_{14} := (2, 4), b_{15} := (6, 0) \in Q_1\), and perturbation vectors \(\delta_{011} = (\frac{1}{1000}, \frac{1}{1500}), \delta_{122} = (0, \frac{1}{2000}), \delta_{143} = (-\frac{1}{3000}, 0), \delta_{154} = (-\frac{1}{2000}, \frac{1}{2000})\). In the subdivision of \(\sum_{i=0}^2 Q_i\), consider the integer points and their cells (Figure 9):

<table>
<thead>
<tr>
<th>point</th>
<th>cell in secondary cell w.r.t. (v_2) under Alg. B</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 7), (2, 7))</td>
<td>((c_{011}, (0, 2)) + ((0, 4), c_{143}) + (0, 2) + \delta)</td>
<td>2-mixed</td>
</tr>
<tr>
<td>((3, 7))</td>
<td>((c_{011}, (0, 2)) + c_{143} + ((0, 2), (1, 2)) + \delta)</td>
<td>1-mixed</td>
</tr>
</tbody>
</table>

where summands come from \(Q_0, Q_1, Q_2\) respectively. These cells together with cell

\[
\sigma = \text{CH}(c_{011}, (0, 2), (1, 2)) + c_{143} + (1, 2) + \delta,
\]

and some infinitesimal cells which do not contain any integer points, correspond to the secondary cell with respect to \(v_2\) of Alg. A, which contains the same integer points. Points \((1, 7), (2, 7), (3, 7)\) correspond (via an appropriate translation) to points of a piece of the secondary cell on which Alg. A recurses. Cell \(\sigma\) does not contain any integer points because of the choice of \(\delta_{ijk}, \delta\).

Now, consider the points corresponding to a piece of the secondary cell with respect to \(v_3\), of Alg. A, and their cells in the subdivision induced by \(\beta\) under Alg. B:

<table>
<thead>
<tr>
<th>point</th>
<th>cell in secondary cell w.r.t. (v_3) under Alg. B</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>((4, 7), (5, 6), (6, 5), (7, 4))</td>
<td>((c_{011}, (1, 2)) + (c_{154}, c_{143}) + (1, 2) + \delta)</td>
<td>2-mixed</td>
</tr>
<tr>
<td>((8, 3), (9, 2))</td>
<td>((c_{011}, (1, 2)) + c_{154} + ((3, 0), (1, 2)) + \delta)</td>
<td>1-mixed</td>
</tr>
<tr>
<td>((10, 1), (11, 0))</td>
<td>(\text{CH}(c_{011}, (3, 0), (1, 2)) + c_{154} + (3, 0) + \delta)</td>
<td>non-mixed</td>
</tr>
</tbody>
</table>

Consider the piece of the secondary cell with respect to \(v_1\), of Alg. A. Points in it lie in the following cells of Alg. B:

<table>
<thead>
<tr>
<th>point</th>
<th>cell in secondary cell w.r.t. (v_1) under Alg. B</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 4))</td>
<td>((c_{011}, (0, 1)) + c_{122} + ((0, 1), (0, 2)) + \delta)</td>
<td>1-mixed</td>
</tr>
<tr>
<td>((0, 5))</td>
<td>(\text{CH}(c_{011}, (0, 1), (0, 2)) + c_{122} + (0, 2) + \delta)</td>
<td>non-mixed</td>
</tr>
<tr>
<td>((0, 6), (0, 7))</td>
<td>((c_{011}, (0, 3)) + (c_{122}, (0, 4)) + (0, 2) + \delta)</td>
<td>2-mixed</td>
</tr>
</tbody>
</table>
**Alg. A:** \( b_{01} \) is lifted to 1, all other vertices of all polygons are lifted to 0. This partitions \( Q_0 + Q_1 + Q_2 \) into a primary cell \( b_{01} + Q_1 + Q_2 \) and 3 secondary cells corresponding to \( v_1, v_2, v_3, \) normals to the facets of \( Q_0 \) not containing \( b_{01} \). The \( Q_1, Q_2 \) are lifted using \( \beta \), which subdivides the primary cell (Figure 8). This subdivision “coincides” with the restriction in \( c_{01} + Q_1 + Q_2 \) of the subdivision by \( \beta \), except that the latter uses \( c_{01} \) whereas the former uses \( b_{01} \), i.e. the integer points in both subdivisions are the same and are assigned the same RC.

- We study the Recursion Phase on secondary cell:

\[ F_{v_1} = \text{CH}(b_{01}, k_0 F_{v_1}) + k_1 F_{v_1} + k_2 F_{v_1}, \]

defined by facet \( F_{v_1} = ((0, 1), (0, 2)) \subset Q \) supported by \( v_1 \), see Figure 10. Now,

\[ A_{1v_1} = \{(0, 2), (0, 4)\}, \quad A_{2v_1} = \{(0, 1), (0, 2)\}, \]

and the lattice generated by \( A_{1v_1} + A_{2v_1} \) is \( L_+ := \langle (0, 3), (0, 4) \rangle \cong L_{v_1} \cong \mathbb{Z} \). The index of \( L_+ \) in \( L_{v_1} \) is \( \text{ind}_{v_1} = 1 \) and the coset representative for \( L_+ \) in \( L_{v_1} \) is \( q_0 = (0, 0) \). The \( v_1 \)-lattice diameter is

\[ d_{v_1} := b_{01} \cdot v_1 - \min_{p \in \text{CH}(b_{01}, k_0 F_{v_1})} p \cdot v_1 = 1. \]

Hence, there is one slice corresponding to one piece. We describe the recursion step on this piece. It contains points corresponding to \((0, 4), (0, 5), (0, 6), (0, 7)\) lying on the slice of \( F_{v_1} + \delta \) of the form

\[ (\lambda k_0 F_{v_1} + \delta') + k_1 F_{v_1} + k_2 F_{v_1} + \lambda F_{v_1} + \delta. \]

To define the piece, following notation in [D’A02], the scalar multiple of \( F_{v_1} \) is \( \tilde{\lambda} F_{v_1} = \frac{29}{30} F_{v_1} \) and the translation vector is \( \delta' := (\frac{1}{30}, 0) \). Since we do not use an initial additional polytope, \( \lambda = 0 \) and \( \lambda_{v_1} := \lambda + \tilde{\lambda} = \frac{29}{30} \).

Let \( \delta_{v_1} := \delta + \delta' = (0, -\frac{1}{30}) \), and \( \delta_{v_1} = \delta_{v_1} + \delta_{v_1} \), where \( \delta_{v_1} = (0, 0) \in \mathbb{Q} v_1 \) and \( \delta_{v_1} = (0, -\frac{1}{30}) \in L_+ \otimes \mathbb{Q} \), hence \( \delta_{0v_1} := \delta_{v_1} - q_0 = (0, -\frac{1}{30}) \). So, the slice of \( F_{v_1} + \delta \) is

\[ k_1 F_{v_1} + k_2 F_{v_1} + \lambda_{v_1} k_0 F_{v_1} + \delta_{v_1}, \tag{23} \]

and the corresponding piece in \( L_+ \) is

\[ k_1 F_{v_1} + k_2 F_{v_1} + \lambda_{v_1} k_0 F_{v_1} + \delta_{0v_1}. \tag{24} \]
The bijection between points in (23) and (24) is
\[ p = \tilde{p} + \delta'^{v_1} + q_0 = \tilde{p}, \]
where \( p \in (23) \) and \( \tilde{p} \in (24) \). After re-indexing, the input of the recursion step is:
- the polygons \( \mathcal{Q}_0 := k_1 F_{v_1}, \mathcal{Q}_1 := k_2 F_{v_1} \), and \( \mathcal{Q}_2 := \frac{b_0}{k_0} F_{v_1} \) which is the additional polytope,
- the lattice \( L^{(1)} := L_+ = \{(0,3),(0,4)\} \) and
- the perturbation vector \( \delta_0 := \delta_{v_1} = (0,-\frac{1}{30}) \).

In order to be compatible with \( \beta \), we choose \( b_{01} = b_{12} = (0,2) \) and apply the primary lifting. This partitions \( \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2 + \delta_0 \) into a primary \( b_{01} + \mathcal{Q}_1 + \mathcal{Q}_2 + \delta_0 \) and a secondary cell \( \mathcal{Q}_0 + (0,2) + \frac{b_0}{k_0}(0,2) + \delta_0 \). Lifting \( \beta \) induces a mixed subdivision on the primary cell consisting of the cells \( b_{01} + (0,1) + \mathcal{Q}_2 + \delta_0 \) and \( b_{01} + \mathcal{Q}_1 + \frac{b_0}{k_0}(0,1) + \delta_0 \). The former is non-mixed and contains point \( (0,5) \), corresponding to the same point on the slice, which is also non-mixed under Alg. B. The latter cell is \( \overline{0} \)-mixed, hence 1-mixed and contains point \( (0,4) \), corresponding to the same point on the slice, which is also 1-mixed under Alg. B. The secondary cell \( \mathcal{Q}_0 + (0,2) + \frac{b_0}{k_0}(0,2) + \delta_0 \) is \( \overline{1} \)-mixed, hence 2-mixed and contains the integer points \( (0,6),(0,7) \) corresponding to the same points on the slice. They are also 2-mixed under Alg. B.

- We apply recursion on secondary cell:

\[ \mathcal{F}_{v_2} = \text{CH}(b_{01}, k_0 F_{v_2}) + k_1 F_{v_2} + k_2 F_{v_2}, \]

defined by the facet \( F_{v_2} = \{(0,2),(1,2)\} \) of \( Q \) supported by \( v_2 \), see Figure 11. Now,
\[ A_{1v_2} = \{(0,4),(2,4)\}, \quad A_{2v_2} = \{(0,2),(1,2)\} \]

and the lattice generated by \( A_{1v_2} + A_{2v_2} \) is \( L_{+} := \langle (0,6),(1,6) \rangle \cong L_{v_2} \cong \mathbb{Z} \). The index of \( L_{+} \) in \( L_{v_2} \) is \( \text{ind}_{v_2} = 1 \) and the coset representative for \( L_{+} \) in \( L_{v_2} \) is \( q_0 = (0,0) \). The \( v_2 \)-lattice diameter is
\[ d_{v_2} := b_{01} \cdot v_2 - \min_{p \in \text{CH}(b_{01}, k_0 F_{v_2})} p \cdot v_2 = 2. \]

Hence, there are two slices, each containing one piece, and the algorithm recurses on each such piece.

We analyze the recursion step on the piece of the shifted secondary cell \( \mathcal{F}_{v_2} + \delta \), which contains the integer points corresponding to the points \( (1,7),(2,7),(3,7) \) lying on a slice of the shifted secondary cell \( \mathcal{F}_{v_2} + \delta \) of the form
\[ (\tilde{\lambda} k_0 F_{v_2} + \delta') + k_1 F_{v_2} + k_2 F_{v_2} + \lambda F_{v_2} + \delta. \]

Figure 11: Example 22: A slice of the secondary cell \( \mathcal{F}_{v_2} \) w.r.t. vector \( v_2 = (0,-1) \) containing points \( (1,7),(2,7),(3,7) \) (dotted segment, left subfigure), the corresponding piece and its mixed subdivision w.r.t. Alg. A. The arrows show the correspondence between points on the slice and points on the piece. Also depicted is the mixed subdivision of the corresponding secondary cell w.r.t. Alg. B (right subfigure).

To define this piece we have that \( F_{v_2} \) is \( \tilde{\lambda} F_{v_2} = \frac{31}{60} F_{v_2} \) and the translation vector \( \delta' = (\frac{29}{60},0) \). Now \( \lambda = 0 \) and hence \( \lambda_{v_2} := \lambda + \tilde{\lambda} = \frac{31}{60} \). Let \( \delta_{\lambda} := \delta + \delta' = (\frac{29}{60},-\frac{1}{60}) \). Then, \( \delta_{\lambda} \) can be
written as \( \delta_{\lambda} = \delta_{\lambda}^{\nu_2} + \delta_{\lambda}^{\nu_3}, \) where \( \delta_{\lambda}^{\nu_2} = (0, 1) \in \mathbb{Q}v_2 \) and \( \delta_{\lambda}^{\nu_3} = \left( \frac{9}{20}, -\frac{31}{30} \right) \in L_+ \otimes \mathbb{Q}, \) hence \( \delta_{0v_2} := \delta_{\lambda}^{\nu_2} - q_0 = \left( \frac{9}{20}, -\frac{1}{3} \right). \)

So, the slice of \( \mathcal{F}_{v_2} + \delta \) is
\[
k_1 F_{v_2} + k_2 F_{v_2} + \lambda v_2 k_0 F_{v_2} + \delta_{\lambda},
\]

and the corresponding piece in \( L_+ \) is
\[
k_1 F_{v_2} + k_2 F_{v_2} + \lambda v_2 k_0 F_{v_2} + \delta_{0v_2}.
\]

The bijection between points in (25) and points in (26) is
\[
p = \tilde{p} + \delta_{\lambda}^{\nu_2} + q = \tilde{p} + (0, 1),
\]

where \( p \in (25) \) and \( \tilde{p} \in (26). \)

After re-indexing, the input of the recursion step is:
- the polygons \( Q_0 := k_1 F_{v_2}, \ Q_1 := k_2 F_{v_2}, \) and \( Q_2 := \frac{33}{60} k_0 F_{v_3} \) which is the additional polytope,
- the lattice \( L^{(1)} := L_+ = \langle (0, 6), (1, 6) \rangle \) and
- the perturbation vector \( \delta := \delta_{0v_2} = \left( \frac{9}{20}, -\frac{31}{30} \right). \)

To be compatible with \( \beta, \) we choose \( \beta_{01} = b_{14} = (2, 4) \) and apply the primary lifting; this partitions the Minkowski sum \( Q_0 + Q_1 + Q_2 + \delta \) into a primary \( \overline{Q}_0 + \overline{Q}_1 + \overline{Q}_2 + \delta \) and a secondary cell \( Q_0 + (0, 2) + \frac{33}{60} (0, 2) + \delta. \) Lifting \( \beta \) induces a mixed subdivision of the primary cell consisting of the cells \( \overline{Q}_0 + (1, 2) + \overline{Q}_2 + \delta \) and \( \overline{Q}_0 + \overline{Q}_1 + \frac{33}{60} (0, 2) + \delta. \) The latter is \( 0 \)-mixed, hence 1-mixed and contains the integer point \( (3, 6) \) corresponding to point \( (3, 7) \) on the slice which is also 1-mixed under Alg. B. The former is non-mixed and does not contain any integer points.

The secondary cell \( \overline{Q}_0 + (0, 2) + \frac{33}{60} (0, 2) + \delta \) is \( 1 \)-mixed, hence 2-mixed and contains the integer points \( (1, 6), (2, 6) \) corresponding to the points \( (1, 7), (2, 7) \) of the slice respectively; they are also 2-mixed under Alg. B.

- The last secondary cell is \( \mathcal{F}_{v_3} = \text{CH}(b_{01}, k_0 F_{v_3}) + k_1 F_{v_3} + k_2 F_{v_3}, \)

defined by the facet \( F_{v_3} = \langle (3, 0), (1, 2) \rangle \) of \( Q \) supported by \( v_3 = (-1, -1)., \) see also Figure 6 and Example 16. Now,
\[
A_{1v_3} = \{(0, 0), (2, 4)\}, \quad A_{2v_3} = \{(3, 0), (1, 2)\},
\]

the lattice generated by \( A_{1v_3} + A_{2v_3} \) is \( L_+ := \langle (9, 0), (7, 2) \rangle \cong 2\mathbb{Z} \) and \( L_{v_3} \cong \mathbb{Z}. \) The index of \( L_+ \) in \( L_{v_3} \) is \( \text{ind}_{v_3} = 2 \) and the cosets representatives for \( L_+ \) in \( L_{v_3} \) are \( q_0 = (0, 0) \) and \( q_1 = (-1, 1). \) The \( v_3 \)-lattice diameter is
\[
d_{v_3} := b_{01} \cdot v_3 - \min_{p \in \text{CH}(b_{01}, k_0 F_{v_3})} p \cdot v_3 = 2.
\]

Hence there are two slices, each corresponding to two pieces, and the algorithm recurses on each such piece.

We analyze the recursion step on the two pieces that contain integer points corresponding to points \( (11, 0), (10, 1), (9, 2), (8, 3), (7, 4), (6, 5), (5, 6), (4, 7) \) lying on a slice of the shifted secondary cell \( \mathcal{F}_{v_3} + \delta \) of the form
\[
(\lambda k_0 F_{v_3} + \delta') + k_1 F_{v_3} + k_2 F_{v_3} + \lambda v_3 + \delta. \]

To define these pieces, we have that the scalar multiple of \( F_{v_3} \) is \( \lambda F_{v_3} = \frac{32}{60} F_{v_3} \) and the translation vector is \( \delta' := \frac{13}{15}, 0 \). Now, \( \lambda = 0 \) and hence \( \lambda v_3 := \lambda + \frac{13}{15} \). Let \( \lambda := \delta + \delta' = \left( \frac{13}{15}, -\frac{1}{3} \right). \)

Then, \( \delta_{\lambda} \) can be written as \( \delta_{\lambda} = \delta_{\lambda}^{\nu_3} + \delta_{\lambda}^{\nu_2}, \) where \( \delta_{\lambda}^{\nu_3} = (1, 1) \in \mathbb{Q}v_3 \) and \( \delta_{\lambda}^{\nu_2} = \left( \frac{32}{60}, -\frac{1}{3} \right) \in L_+ \otimes \mathbb{Q}, \) hence \( \delta_{0v_3} := \delta_{\lambda}^{\nu_3} - q_0 = \left( \frac{32}{60}, -\frac{1}{3} \right) \) and \( \delta_{1v_3} := \delta_{\lambda}^{\nu_3} - q_1 = \left( \frac{32}{60}, -\frac{1}{3} \right). \)

So, the slice of \( \mathcal{F}_{v_3} + \delta \) is
\[
k_1 F_{v_3} + k_2 F_{v_3} + \lambda v_3 k_0 F_{v_2} + \delta_{\lambda},
\]

(27)
and the corresponding pieces in $L_+$ are

$$k_1 F_{v_3} + k_2 F_{v_3} + \lambda_{v_3} k_0 F_{v_3} + \delta_{0v_3},$$  
\hspace{1cm}
(28)

$$k_1 F_{v_3} + k_2 F_{v_3} + \lambda_{v_3} k_0 F_{v_3} + \delta_{1v_3},$$  
\hspace{1cm}
(29)

The correspondences between points in the slice and points in the pieces are

$$p = \bar{p} + \delta_{0}^\alpha + q_0 = \bar{p} + (1, 1),$$

where $p \in (27)$ and $\bar{p} \in (28)$, and

$$p = \bar{p} + \delta_{0}^\alpha + q_1 = \bar{p} + (0, 2),$$

where $p \in (27)$ and $\bar{p} \in (29)$.

After re-indexing, the input of the recursion step is:

- the polygon $\mathcal{Q}_0 := k_1 F_{v_3}$, $\mathcal{Q}_1 := k_2 F_{v_3}$, and $\mathcal{Q}_2 := \frac{32}{60} k_0 F_{v_3}$ which is the additional polytope,
- the lattice $L^{(1)} := L_+ = \langle (9,0), (7,2) \rangle$ and
- the perturbation vectors $\bar{\delta}_0 := \delta_{0v_3} = \left( -\frac{17}{32}, \frac{31}{32} \right)$ and $\bar{\delta}_1 := \delta_{1v_3} = \left( \frac{13}{32}, -\frac{61}{32} \right)$.

As $\beta$ indicates, we choose $b_{01} = b_{15} = (6,0)$ and apply the primary lifting.

For the first piece, the lifting partitions the Minkowski sum $\mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2 + \delta_0$ into a primary $b_{01} + \mathcal{Q}_1 + \mathcal{Q}_2 + \delta_0$ and a secondary cell $\mathcal{Q}_0 + (1,2) + \frac{32}{60} (1,2) + \delta_0$. Lifting $\beta$ induces a mixed subdivision on the primary cell consisting of the cells $b_{01} + (3,0) + \mathcal{Q}_2 + \delta_0$ and $b_{01} + \mathcal{Q}_1 + \frac{32}{60} (1,2) + \delta_0$. The former is non-mixed and contains point $(9,0)$, which corresponds to $(10,1)$ on the slice which is also non-mixed under Alg. B. The latter is $\beta$-mixed, hence 1-mixed and contains the point $(7,2)$ corresponding to the point $(8,3)$ in the slice which is also 1-mixed under Alg. B.

The secondary cell $\mathcal{Q}_0 + (1,2) + \frac{32}{60} (1,2) + \delta_0$ is $\beta$-mixed, hence 2-mixed and contains the integer points $(3,6), (5,4)$ corresponding to the points $(4,7), (6,5)$ of the slice respectively which are also 2-mixed under Alg. B.

For the second piece, the lifting partitions the Minkowski sum $\mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2 + \delta_1$ into a primary $b_{01} + \mathcal{Q}_1 + \mathcal{Q}_2 + \delta_1$ and a secondary cell $\mathcal{Q}_0 + (1,2) + \frac{32}{60} (1,2) + \delta_1$. Lifting $\beta$ induces a mixed subdivision on the primary cell consisting of the cells $b_{01} + (3,0) + \mathcal{Q}_2 + \delta_1$ and $b_{01} + \mathcal{Q}_1 + \frac{32}{60} (1,2) + \delta_1$. The former is non-mixed and contains point $(11,-2)$ corresponding to $(11,0)$ on the slice which is also non-mixed under Alg. B, whereas the latter cell is $\bar{\beta}$-mixed, hence 1-mixed and contains the integer point $(9,0)$ corresponding to point $(9,2)$ on the slice which is also 1-mixed under Alg. B.

The secondary cell $\mathcal{Q}_0 + (1,2) + \frac{32}{60} (1,2) + \delta_1$ is $\bar{\beta}$-mixed, hence 2-mixed and contains the integer points $(7,2), (5,4)$ corresponding to the points $(7,4), (5,6)$ of the slice respectively. These are also 2-mixed under Alg. B.

The second slice of $F_{v_3} + \delta$ is $(\frac{1}{30} F_{v_3} + (\frac{29}{30},0)) + k_1 F_{v_3} + k_2 F_{v_3} + (-\frac{1}{30}, -\frac{1}{30})$, and contains integer points $(10,0), (9,1), (8,2), (7,3), (6,4), (5,5), (4,6)$.

Table 1 illustrates corollaries 19 and 20, where the summands come from $\mathcal{Q}_0, \mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively. Recall that $c_{011} := (1,0) + \delta_{011}$, $c_{143} := (2,4) + \delta_{143}$ and $c_{154} := (6,0) + \delta_{154}$.

<table>
<thead>
<tr>
<th>Cell w.r.t. Alg. A</th>
<th>Corresponding cell w.r.t. Alg. B</th>
<th>Type of cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(1,2) + (6,0) + (3,0), (1,2)) + \delta_{0v_3}$</td>
<td>$(c_{011}, (1,2)) + c_{154} + ((3,0), (1,2)) + \delta$</td>
<td>1-mixed non-mixed</td>
</tr>
<tr>
<td>$\lambda((3,0), (1,2)) + (6,0) + (3,0) + \delta_{0v_3}$</td>
<td>$\text{CH}(c_{011}, (1,2), (3,0)) + c_{154} + (3,0) + \delta$</td>
<td>1-mixed non-mixed</td>
</tr>
<tr>
<td>$\lambda((3,0), (1,2)) + (6,0) + (3,0) + \delta_{1v_3}$</td>
<td>$\text{CH}(c_{011}, (1,2), (3,0)) + c_{154} + (3,0) + \delta$</td>
<td>1-mixed non-mixed</td>
</tr>
<tr>
<td>$\lambda(0,2) + (2,4) + ((0,2), (1,2)) + \delta_{0v_2}$</td>
<td>$(c_{011}, (0,2)) + c_{143} + ((0,2), (1,2)) + \delta$</td>
<td>1-mixed non-mixed</td>
</tr>
<tr>
<td>$\lambda((0,2), (1,2)) + (2,4) + (1,2) + \delta_{0v_2}$</td>
<td>$\text{CH}(c_{011}, (1,2), (0,2)) + c_{143} + (1,2) + \delta$</td>
<td>1-mixed non-mixed</td>
</tr>
</tbody>
</table>
6 Further work

Let us conclude with some preliminary results on mixed algebraic systems. In studying systems with different Newton polytopes, we need the following:

**Definition 23.** The set of polytopes $Q_1, \ldots, Q_h \subset \mathbb{R}^n$, s.t. $\dim((Q_1, \ldots, Q_h)) = h - 1$, is *essential* if every subset of cardinality $j$, $1 \leq j < h$ spans a space of dimension $\geq j$.

The toric resultant is well defined only for essential sets of Newton polytopes. An essential set defines a Minkowski sum of dimension $h - 1$ but the converse is not always true.

Alg. A admits one main modification in the mixed case: At the Recursion Phase, the faces $F_i \subset Q_i$ supported by vector $v$ are not always the same. Let the input be $n + 1$ polytopes; we describe the 0-th iteration for simplicity. Consider the $n$-dimensional secondary cell:

$$ \text{CH}(b_{01}, F_0) + F_1 + \cdots + F_n \subset \mathbb{R}^n,$$

where $F_i \subset \mathbb{R}^{n-1}$. Without loss of generality, let $\{F_1, \ldots, F_k\}$ be an essential subset and let $L_+(k)$ be the integer lattice it defines. The algorithm recurses on lattice $L_+(k)$ and polytope set (representing a piece)

$$ \text{CH}(b_{01}, F_0) \cap L_+(k), F_1, \ldots, F_k, F_{k+1} \cap L_+(k), \ldots, F_n \cap L_+(k),$$

where $L_+(k)$ ranges over all possible homothetic copies of $L_+(k)$ defined by the different cosets of $L_+(k)$ in its saturation, and the different slices that can be defined as intersections with $\text{CH}(b_{01}, F_0)$. Alg. A distinguishes two cases, according to whether there is one or more essential subsets of $\{F_1, \ldots, F_n\}$. In the former case, $v$ and the corresponding secondary cell are called *admissible*. For non-admissible cells, all integer points are considered as non-mixed, i.e. treated as if they lied in non-mixed cells. For admissible cells, integer $d_{F_0}$ is defined [D’A02, Sec.4] (cf. [Min03]), and $d_{F_0}$-pieces of the form (30) are (arbitrarily) selected. Lattice points labeled as mixed in these pieces by the recursive application of Alg. A are labeled as mixed overall, the rest are non-mixed.

Before sketching the extension of our algorithm to the mixed case, let us consider some special cases. Reduced systems are such that, for any vector $v \in \mathbb{R}^n$, there is some $i \in \{1, \ldots, n\}$ so that the face supported by $v$ in $Q_i$ is a vertex [D’A01]. For us, it suffices that this holds for any vector $v$ associated with secondary cells at the 0-th recursion step of Alg. A. For such systems, as well as for arbitrary systems of three bivariate polynomials ($n = 2$), the lifting function (31) produces a Macaulay-type formula [DE03a].

$$ l_0 : \quad A_0 \rightarrow \begin{cases} \{0, 1\} \\ 1, \\ 0, \quad \text{if } j \neq 0, \end{cases} \quad l_i : \quad A_i \rightarrow \mathbb{R} \quad \text{(} i \geq 1 \text{)} \quad b_{01} \rightarrow 1, \quad p \rightarrow 0, \quad \text{if } p \notin \cup_{v \neq A_i} A_{i,v} \quad (31) \quad p \rightarrow r_p \quad \text{otherwise.} $$

Here, $A_{i,v} := A_i \cap Q_{i,v}$, where $Q_{i,v}$ is the face of $Q_i$ supported by $v$, and $r_p$ is a positive random number satisfying $0 < r_p \ll 1$. It is not difficult to see that our lifting $\beta$ has an overall effect similar to that of lifting (31), therefore it also produces a Macaulay-type formula for the previous systems. For bivariate systems, the idea of the proof is subsumed by that for $n = 3$ at the end of this section.

For extending Alg. B to the mixed case, we must modify it so that Definition 5 applies to different polytopes and also up to $i = n - 1$. We sketch a proof that it produces the same matrix as Alg. A, by extending the correlation between maximal cells, established in the unmixed case. Our proof might extend to $n > 3$, but seems complicated; we hope that a more elegant approach is possible.

In non-admissible secondary cells of Alg. A, for any $n$, we show that both algorithms behave in the same way, namely that the corresponding lattice points lie in non-mixed cells of Alg. B. We demonstrate the contrapositive by focusing on a mixed cell of Alg. B and a corresponding secondary cell of Alg. A, following Lemma 18.

**Lemma 24.** Every $t$-mixed cell by Alg. B, when intersected with a $(n - t)$-dimensional hyperplane as in Lemma 18, is contained in an admissible secondary cell of step $t - 1$ of Alg. A.
Proof. Any $t$-mixed cell of Alg. B is of the form $E_0 + \cdots + E_{t-1} + a_{tj} + E_{t+1} + \cdots + E_n$, where $a_{tj}$ is either a vertex of $Q_i$ or some $c_{tjs}$ in the interior of an $(n-t)$-dimensional face, and edges $E_{t+1}, \ldots, E_n$ span an $(n-t)$-dimensional space. This cell is intersected by a $(n-t)$-dimensional hyperplane, similarly to Lemma 18. The intersection is contained in a $t$-primary cell of Alg. A with $t$-summand $b_{tij}$; it lies in a piece of $(t-1)$-secondary cell

\[ F_0 + \cdots + F_{t-2} + \text{CH}(b_{(t-1)jk}, F_{t-1}) + F_t + \cdots + F_n, \]

where the $F_i$ are faces of the $Q_i$, $i = 1, \ldots, n$, supported by the same vector, with $\dim F_i \leq n - t$. We claim $\{F_1, \ldots, F_n\}$ contains a unique essential set, with cardinality $r+1$, spanning an $r$-dimensional space, which is defined as follows: $F_t$ and $r \leq n-t$ faces, denoted, without loss of generality, $F_{t+1}, \ldots, F_{t+r}$, where $r$ is minimal so that $\dim H = r$, for $H = \langle F_1, \ldots, F_{t+r} \rangle$.

By hypothesis, $\dim(F_{t+1}, \ldots, F_n) = n-t$, since a subspace is spanned by the $E_i$ and has same dimension. So subsets indexed in $\{t+1, \ldots, n\}$ span a space of dimension at least equal to their cardinality. In addition, none of the $F_{i}, i > t + r$ is contained in $H$. So every subset indexed in $\{t, \ldots, n\}$ containing $\{t\} \cup J$, for $J \subset \{t + r + 1, \ldots, n\}$, will be of cardinality $\leq r + |J|$ and span a space of dimension $r + |J|$. Hence there are no other essential subsets. □

For $n = 3$, all admissible secondary cells have $d_F$ pieces, since there is no extra artificial polytope in the input of Alg. A. We distinguish cases on the dimension $k-1$ of the space generated by the essential set $\{F_1, \ldots, F_k\}$, $1 \leq k \leq 3$, on which the recursion of Alg. A occurs:

1. If $k - 1$ is 0 or 1, the recursion is either trivial (occurs on a vertex), or corresponds to the Sylvester case.

2. If $k - 1 = 2$ and $\dim F_1 = 1$, $i = 1, 2, 3$, the two algorithms behave similarly, since Definition 5 defines points $c_{tjs}$ in the edges of $Q_2$ and Lemma 18 applies. Notice that $\dim Q_2 \geq 1$; otherwise the $Q_i$’s would not form an essential set.

3. If $k - 1 = 2$, then $\dim F_i \in \{1, 2\}$ for $i = 1, 2, 3$ and at least one face is two-dimensional. If $\dim F_1 = 2$, then Lemma 18 applies. Otherwise, $\dim F_1 = 1$ and $\dim F_2 \geq 1$. Irrespective of $\dim F_2$, the $c_{tjs}$’s play the role of distinguished points and Lemma 18 applies again.

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