Triangulations and Resultants

Christos Konaxis

Supervisor: Ioannis Z. Emiris

$\mu \prod \lambda \forall$

July 2006

Outline

Outline

Outline

Outline

Compute the Newton Polytope of the Resultant

Without Computing the Resultant!

- A system of equations, $f_1, f_2 \in \mathbb{C}[x]$

$$\begin{aligned} f_1 = a + bx^3 &= 0 \\ f_2 = cx^5 + d &= 0 \end{aligned}$$

- The Newton polytope $N(\mathcal{R})$ is the segment defined by the points:

$$v_1 = (5, 0, 3, 0),\ v_2 = (0, 5, 0, 3).$$

- The (Sylvester) resultant of f_1, f_2 is the polynomial

$$\mathcal{R} = b^5 d^3 - a^5 c^3 \in \mathbb{Z}[a, b, c, d]$$

Compute the Newton Polytope of the Resultant

Without Computing the Resultant!

- A system of equations, $f_1, f_2 \in \mathbb{C}[x]$

$$\begin{aligned} f_1 = a + bx^3 &= 0 \\ f_2 = cx^5 + d &= 0 \end{aligned}$$

- The Newton polytope $N(\mathcal{R})$ is the segment defined by the points:

$$v_1 = (5, 0, 3, 0),\ v_2 = (0, 5, 0, 3).$$

- The (Sylvester) resultant of f_1, f_2 is the polynomial

$$\mathcal{R} = b^5 d^3 - a^5 c^3 \in \mathbb{Z}[a, b, c, d]$$

Compute the Newton Polytope of the Resultant

Without Computing the Resultant!

- A system of equations, $f_1, f_2 \in \mathbb{C}[x]$

$$\begin{aligned} f_1 = a + bx^3 &= 0 \\ f_2 = cx^5 + d &= 0 \end{aligned}$$

- The Newton polytope $N(\mathcal{R})$ is the segment defined by the points:

$$v_1 = (5, 0, 3, 0),\ v_2 = (0, 5, 0, 3).$$

- The (Sylvester) resultant of f_1, f_2 is the polynomial

$$\mathcal{R} = b^5 d^3 - a^5 c^3 \in \mathbb{Z}[a, b, c, d]$$

Outline

Why Toric (Sparse) Elimination Theory?

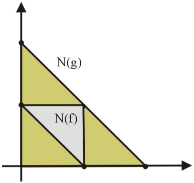
- Real life examples: equations are often sparse.
- Exploits the structure of polynomials.
- Considers only affine roots.
- Toric resultant matrices are (usually) smaller than projective resultant matrices.
  They can be defined even if the projective resultant matrix vanishes identically.
- Toric resultants eliminate all variables at once.
- Applications: polynomial system solving, variable elimination, implicitization etc.

Newton Polytopes

Definition

- The support A of a (Laurent) polynomial $f = \sum c_\alpha x^\alpha \in \mathbb{C}[x^{\pm 1}]$ is the set of the exponents α, of its monomials with nonzero coefficient.
- The Newton polytope $N(f)$ of a polynomial f is the convex hull of its support A.

Newton polytopes model the sparseness of a polynomial.



$$f(x,y) = a_1x + a_2y + a_3xy,$$
$$g(x,y) = b_0+b_1x+b_2y+b_3x^2+b_4xy+b_5y^2$$

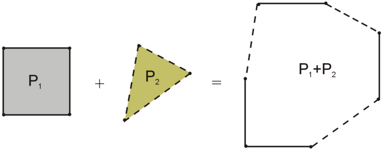
Minkowski Sum

Definition

The Minkowski sum of two convex polytopes P_1 and P_2 is the convex polytope

$$P = P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1, p_2 \in P_2\}$$

Minkowski addition of polytopes $N(f) + N(g)$ corresponds to polynomial multiplication $f \cdot g$.



Mixed Volume

Definition

The mixed volume $MV(P_1, \ldots, P_n)$, $P_i \subset \mathbb{R}^n$ is the unique real function st.:

1. It is multilinear wrt Minkowski addition and scalar multiplication:
$$MV(P_1, \ldots, \lambda P_k + \mu P'_k, \ldots, P_n) = \lambda MV(P_1, \ldots, P_k, \ldots, P_n) + \mu MV(P_1, \ldots, P_k, \ldots, P_n)$$
2. $MV(P_1, \ldots, P_n) = n! \cdot Vol(P)$, if $P_1 = \ldots = P_n = P$.

Computation of Mixed Volume is done using mixed subdivisions.

Mixed Subdivisions

Definition

Let $P = P_0 + \ldots + P_n \subset \mathbb{R}^n$, be a n-dimensional convex polytope.
A tight mixed subdivision of P, is a collection of n-dimensional convex polytopes R, called cells, st.:

- they form a polyhedral complex that partitions P and
- every cell R is a Minkowski sum of faces of the polytopes P_i:

$$R = F_0 + \cdots + F_n, \quad dim(R) = dim(F_0) + \cdots + dim(F_n) = n,$$

Definition

A cell R is called $i - mixed$ if it is a Minkowski sum of n edges $E_j \subset P_j$ and one vertex $v_i \in P_i$,

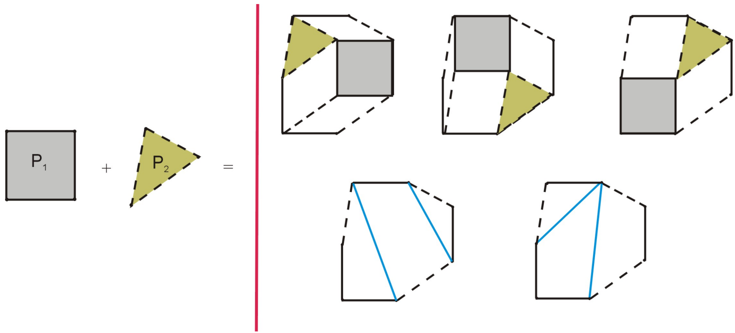
$$R = E_0 + \cdots + v_i + \cdots + E_n.$$

Construction of a Regular Tight Mixed Subdivision

For the convex polytopes $P_0, \ldots, P_n \subset \mathbb{R}^n$, we construct a regular tight mixed subdivision of $P = P_0 + \ldots + P_n$:

1. We choose affine liftings $\omega_i : P_i \to \mathbb{R}$ and define the lifted polytopes
$$\hat{P}_i := \{(p_i, \omega_i(p_i)) \mid p_i \in P_i\}.$$
2. We form the Minkowski sum $\hat{P} = \sum_{i=0}^{n} \hat{P}_i$.
3. We project the lower-hull of $\hat{P}$ onto P. The lower-hull facets induce a regular mixed subdivision of P.
   If ω_i are generic, the induced regular mixed subdivision is tight.

Examples of Mixed (and not mixed) Subdivisions



Computation of (Partial) Mixed Volumes

Theorem

If $P_0, \ldots, P_n \subset \mathbb{R}^n$, are convex polytopes and S is a mixed subdivision of the Minkowski sum $P = \sum_{i=0}^{n} P_i$, then

$$MV_{-i}(P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n) = \sum_R Vol(R),$$

where the sum is over all i-mixed cells R of S.

An Application of Mixed Volumes: Bernstein Bound

Theorem (Bézout)

The number of isolated roots in $\mathbb{C}^n$ *of the polynomial system* $f_1 = \ldots = f_n = 0$, $f_i \in \mathbb{C}[x_1, \ldots, x_n]$, *is at most* $d_1 \ldots d_n$, *where* $d_i = degree(f_i)$. *Moreover, if we count roots at infinity with multiplicities, or the* f_i *are generic, then the bound is exact (in* $\mathbb{P}^n$*).*

Theorem (Bernstein,Kushnirenko, Khovanskii)

The number of roots in $(\mathbb{C}^*)^n$ *of the polynomial system* $f_1 = \ldots = f_n = 0$, $f_i \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, *is at most* $MV(P_1, \ldots, P_n)$. *If the* f_i *are generic then the bound is exact.*

Definition of the Toric Resultant

Definition

Let $f_0, \ldots, f_n \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, be $n+1$ Laurent polynomials in n variables with symbolic coefficients $c_{i,j}$.
The toric or sparse resultant $\mathcal{R}$ of the f_i is the unique (up to sign) irreducible polynomial in $\mathbb{Z}[c_{i,j}]$ which vanishes iff the f_i have a common root in $(\mathbb{C}^*)^n$.

Properties of the Toric Resultant

- Suppose that the supports $A_0, \ldots, A_n$ of the Laurent polynomials f_i generate $\mathbb{Z}^n$.
- The toric resultant $\mathcal{R}$ is a homogenous polynomial in the symbolic coefficients of each f_i, of degree equal to the partial mixed volume

$$MV_{-i} := MV(P_0, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n).$$

- Reduces to: the projective resultant for dense polynomials, the Sylvester resultant for two univariate polynomials and to the determinant of a system of linear equations.
- Construction of the resultant matrix uses mixed subdivisions. Generalization of Macauley's construction [D' Andrea '01]: There exists a matrix M st. $\mathcal{R} = det(M)/det(M')$, M': submatrix of M.

The Extreme Monomials of the Toric Resultant

Definition

Let ω be a generic lifting function. A monomial $init_\omega(\mathcal{R})$ of the toric resultant $\mathcal{R}$ is an extreme monomial corresponding to ω iff its exponent vector is a vertex of the Newton polytope $N(\mathcal{R})$ with normal vector ω.

Computation of the Extreme Monomials

Let $P_i = N(f_i),\ i = 0, \ldots, n,$ be n-dimensional Newton polytopes.

Theorem (Sturmfels)

For every generic lifting function ω, we obtain an extreme monomial of $\mathcal{R}$, of the form

$$init_\omega(\mathcal{R}) = c \cdot \prod_{i=0}^{n} \prod_{R} c_{i,v_i}^{\mathrm{Vol}(R)},$$

where the second product is over all i-mixed cells of the regular tight mixed subdivision of $P = \sum_{i=0}^{n} P_i$, induced by ω and c_{i,v_i} is the coefficient of the monomial of f_i corresponding to the vertex v_i. The constant c is +1 or -1.

Mixed Cell Configurations

- Two regular tight mixed subdivisions of P are equivalent if the have the same mixed cells. We will call the equivalence classes mixed cell configurations.
- Sturmfels theorem establishes an one to one and onto correspondence between the mixed cell configurations of the Minkowski sum P and the extreme monomials of $\mathcal{R}$.
- To compute the Newton polytope of the toric resultant, we have to compute all mixed cell configurations of P.

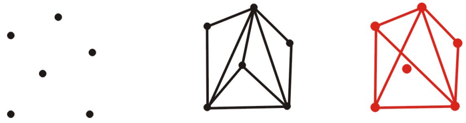
Outline

Triangulations of Point Sets

Definition

A triangulation $\mathcal{T}$ of a (finite) point set $A \subset \mathbb{R}^n$ is a collection of n-dimensional simplices $T_i \subset P = conv(A)$, called the cells of $\mathcal{T}$, st.:

- The cells partition P.
- Every pair of cells intersect at a common facet (possibly empty).



A point set A, a triangulation of P and a partition that is not a triangulation.

Regular Triangulations of Point Sets

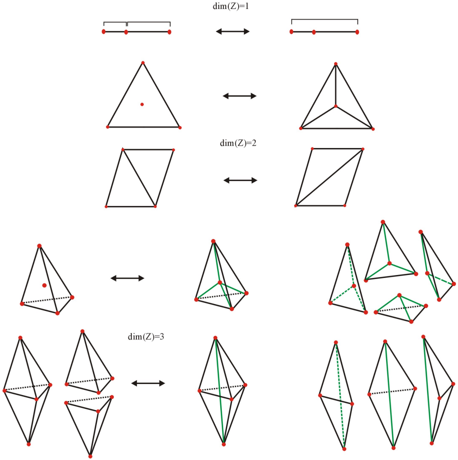
Definition

A triangulation $\mathcal{T}$ is called regular if there exists a generic lifting function ω such that $\mathcal{T}$ is obtained by the projection onto P of the lower facets of the set $\hat{A} := \{(a, \omega(a)) \mid a \in A\}$.
The vector w with coordinates the values $\omega(a)$, is called the weight vector of the triangulation.

Circuits of a Triangulation

- To compute all regular triangulations of a point set A, we start with one and we transform it locally.
- For the local transformations we use circuits.
- A circuit Z is a minimal affinely dependent subset of A.
- Every subset of a circuit Z is a simplex of some dimension.
- Every circuit has exactly two triangulations $\mathcal{T}_+, \mathcal{T}_-$.

Examples of Circuits

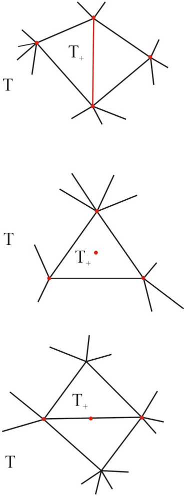


Circuits of small dimension and the corresponding triangulations.

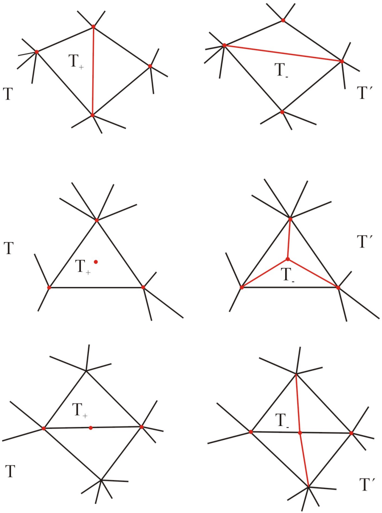
Computing Triangulations Using Circuits

- Not all circuits are suitable for transformation of a triangulation $\mathcal{T}$. For a suitable circuit Z, we say that $\mathcal{T}$ is supported on Z.
- If $\mathcal{T}$ is supported on Z, the transformation consists of changing the current triangulation of Z (say $\mathcal{T}_+$), to the other ($\mathcal{T}_-$).
- This operation is called a (bistellar) flip over Z.
- The new triangulation $\mathcal{T}'$ may not be regular. A flip is followed by a regularity check.

Examples of Bistellar Flips



Examples of Bistellar Flips



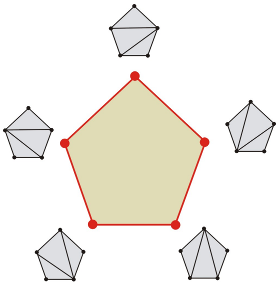
The Secondary Polytope of a Point Set

- For a triangulation $\mathcal{T}$ of a point set A we define the volume vector:

$$\phi_{\mathcal{T}} = (\varphi_1, \ldots, \varphi_{|A|}), \quad \varphi_i = \sum_{\sigma \in \mathcal{T},\ a_i \in \sigma} \mathrm{Vol}(\sigma),$$

  where φ_i is the sum of the volumes of all cells σ having point a_i as its vertex.
- The secondary polytope $\Sigma(A)$ is the convex hull of the volume vectors of all triangulations of A.
- The dimension of the secondary polytope is $|A| - n - 1$.
- The vertices of the secondary polytope are in bijection with the regular triangulations of A. Edges correspond to bistellar flips.

Examples of Secondary Polytopes



Secondary polytopes of a pentagon and a quadrilateral.

Outline

The Cayley Embedding

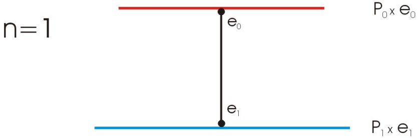
Definition

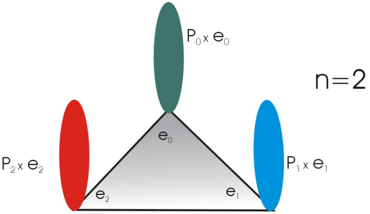
Given polytopes $P_0, \ldots, P_n$, the Cayley embedding κ introduces a new polytope

$$C := \kappa(P_0, P_1, \ldots, P_n) = conv\left(\bigcup_{i=0}^{n}(P_i \times \{e_i\})\right) \subset \mathbb{R}^{2n+1},$$

where e_i are an affine basis of $\mathbb{R}^n$.
The dimension of the polytope C is $d := 2n$.

Intuition


n=1
$P_0 \times e_0$
e_0
e_1
$P_1 \times e_1$

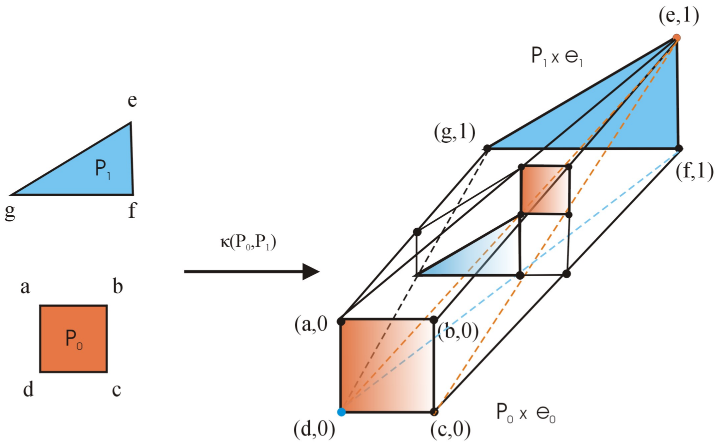

n=2
$P_0 \times e_0$
e_0
$P_2 \times e_2$
e_2
e_1
$P_1 \times e_1$

The Cayley Trick

Theorem (The Cayley Trick)

There is a bijection between the tight regular mixed subdivisions of the Minkowski sum $P = P_0 + \cdots + P_n$ and the regular triangulations of the polytope $C = \kappa(P_0, P_1, \ldots, P_n)$.

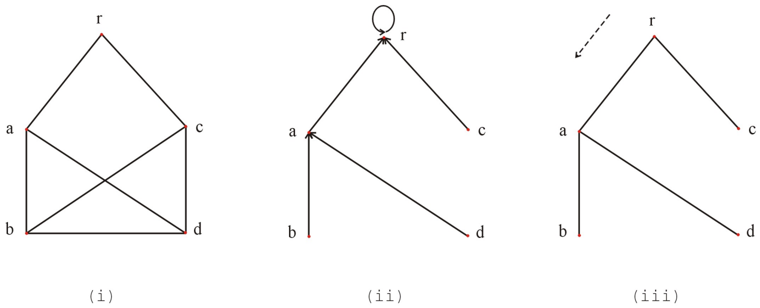
An Example of the Cayley Trick



Enumeration Using Reverse Search

- Reverse search is a technique introduced by Avis and Fukuda which allows the enumeration of large discrete objects with low memory usage.
- Runs in time proportional to the size of the objects to be enumerated.
- In addition to the usual adjacency relation between the objects, parent - children relation is required to save memory.
- Defines a tree structure underlying the graph of adjacency relation.

An Example of Enumeration Using Reverse Search



The adjacency relation (i), parent-children relation (ii) and the reverse search tree (iii).

The Algorithm [Imai, Masada, Takeuchi, Imai]

- Enumerates all regular triangulations of a point set.
- Variation of reverse search: parent-children relation defined by a total order.
- Total order by lexicographic ordering of volume vectors.
- Two triangulations are adjacent iff one can be transformed from the other via a bistellar flip.

The Algorithm (cont'd)

- Time complexity: $O(d^2 s^2 LP(n-d-1,s)|R|)$,
  $d =$ dimension,
  $s = O(m^{\lfloor \frac{d+1}{2} \rfloor}) =$ #of any dimensional simplices in a triangulation,
  $|R| =$ #of regular triangulations, $m = |A|$.
- Time complexity dominated by $LP(n-d-1,s)$.
- Space complexity: $O(ds)$.
- If the points are in general position both space and time complexities can be improved.

Outline

1. Toric Elimination Theory
   - Geometry
   - Toric resultants
2. Triangulations
   - Definitions
   - Computing Triangulations
   - Secondary Polytopes
3. Enumeration Algorithms
   - The Cayley trick
   - The Reverse Search Algorithm
   - Enumeration of Regular Triangulations
4. Enumeration of Mixed Cell Configurations
   - Points in General position
   - Points Not in General Position

Modification of the Algorithm

- Enumerate only the mixed cell configurations.
- Equivalently: enumerate only some of the vertices of the secondary polytope.
- Let $M_1 \neq M_2$, mixed cell configurations.

$$M_1 \ni \mathcal{T}_1 \xrightarrow{flip_Z} \mathcal{T}_2 \in M_2.$$

Which are the circuits Z that make the above scheme work?

The Points are in General Position

- General position assumption: every $d + 1$ points have a convex hull of dimension d. Not three points collinear, four points coplanar etc.
- Every circuit is d-dimensional.
  Consists of $d + 2$ points forming at most d simplices.
- Lemma: Every cell of $\mathcal{T}$ is the image (via κ) of a cell of S.
- Corollary: A cell $T = (T_0, \ldots, T_n)$ is full dimensional iff $\forall i \; T_i \neq \emptyset$.

Characterization of Circuits

- A circuit $Z \subset \mathcal{T}$ involves a mixed cell $R \equiv \kappa(R)$ if

$$R \notin flip_Z(\mathcal{T})$$

- A flip on a circuit Z involving a mixed cell leads to a new mixed cell configuration. (Provided that $\mathcal{T}$ is supported on Z).
- Which circuits involve mixed cells?
- Those that have at least one simplex of the form $\kappa(R)$, R a mixed cell of S.
- A suitable circuit contains a simplex of the form

$$I = \kappa(E_0, \ldots, v_i, \ldots, E_n)$$

Characterization of Circuits

- A circuit $Z \subset \mathcal{T}$ involves a mixed cell $R \equiv \kappa(R)$ if

$$R \notin flip_Z(\mathcal{T})$$

- A flip on a circuit Z involving a mixed cell leads to a new mixed cell configuration. (Provided that $\mathcal{T}$ is supported on Z).
- Which circuits involve mixed cells?
- Those that have at least one simplex of the form $\kappa(R)$, R a mixed cell of S.
- A suitable circuit contains a simplex of the form

$$I = \kappa(E_0, \ldots, v_i, \ldots, E_n)$$

Characterization of Circuits

- A circuit $Z \subset \mathcal{T}$ involves a mixed cell $R \equiv \kappa(R)$ if

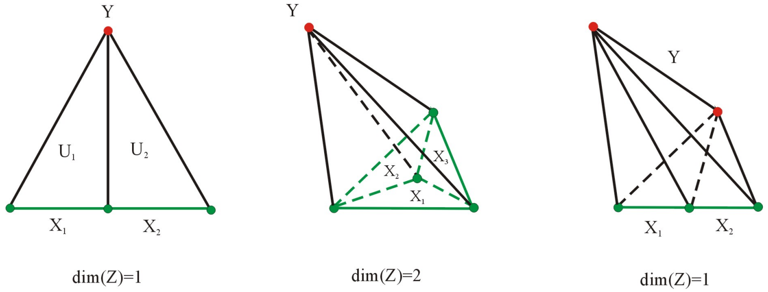
$$R \notin flip_Z(\mathcal{T})$$

- A flip on a circuit Z involving a mixed cell leads to a new mixed cell configuration. (Provided that $\mathcal{T}$ is supported on Z).
- Which circuits involve mixed cells?
- Those that have at least one simplex of the form $\kappa(R)$, R a mixed cell of S.
- A suitable circuit contains a simplex of the form

$$I = \kappa(E_0, \ldots, v_i, \ldots, E_n)$$

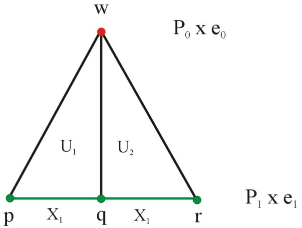
What if Points are Not in General Position?

We have circuits of arbitrary dimension.



The Form of a k-dimensional Circuit

Every cell X of a triangulation $\mathcal{T}_{\pm}$ of a k-dimensional circuit Z is a k-face of a cell $U \subset \mathcal{T}$.



Z can be written as: $Z = (\emptyset, \{p, q\} \cup \{r\})$ or $Z = (\emptyset, \{q, r\} \cup \{p\})$.

The Form of a k-dimensional Circuit (cont'd)

Lemma

If $\mathcal{T}$ is supported on Z and X is a cell of the triangulation of Z induced by $\mathcal{T}$, then there exists a cell $U = (U_0, \ldots, U_n) \subset \mathcal{T}$, such that X is a k-face of U and

$$Z = (Z_0, \ldots, Z_r \cup \{c\}, \ldots, Z_n), \qquad Z_i \subseteq U_i,\ c \in P_r \setminus vert(U_r)$$

Characterization of Circuits

Theorem

Let $Z = (Z_0, \ldots, Z_n)$ a circuit of $\mathcal{T}$ involving a mixed cell $R = (E_0, \ldots, v_s, \ldots, E_n)$. Then there exist $0 \leq r \leq n$ and $c \in P_r$ st.:

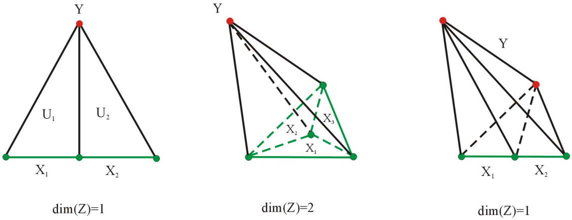
$$Z_i = E_i \quad \text{or} \quad Z_i = \emptyset, \qquad \text{if } i \neq r$$

$$\text{and}$$

$$Z_r = E_r \cup \{c\} \quad \text{or} \quad Z_r = \{v_r\} \cup \{c\},\ v_r \in E_r, \quad \text{if } i = r.$$

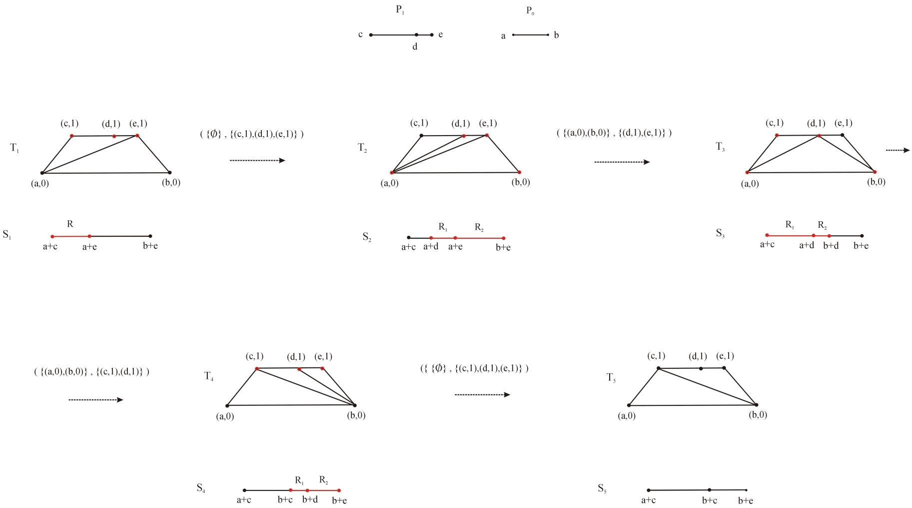
Characterization of Circuits (cont'd)

Suitable circuits are of the form $Z = (Z_0, \ldots, Z_n)$, where
$|Z_i| \in \{0, 2\}\ \forall i$ (even circuits), or
$|Z_i| \in \{0, 2\}\ \forall i \neq r$ and $|Z_r| = 3$ (odd circuits).



First and third circuits are odd. Second circuit does not involve a mixed cell (a subset Z_i of Z has cardinality 4).

An Example


P1
c
d
e
P0
a
b
T1
(c,1)
(d,1)
(e,1)
(a,0)
(b,0)
({∅} , {(c,1),(d,1),(e,1)})
T2
({(a,0),(b,0)} , {(d,1),(e,1)})
T3
S1
R
a+c
a+e
b+e
S2
R1
R2
a+c
a+d
a+e
b+e
S3
a+c
a+d
b+d
b+e
({(a,0),(b,0)} , {(c,1),(d,1)})
T4
({ {∅} , {(c,1),(d,1),(e,1)})
T5
S4
a+c
b+c
b+d
b+e
S5
a+c
b+c
b+e

An Application to Implicitization [IPSOS]

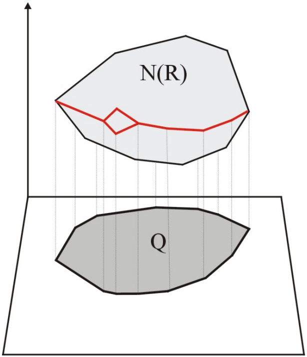
Input: $x_i = \frac{P_i(t)}{Q(t)}, \; i = 0, \dots, n, \quad gcd(P_i(t), Q(t)) = 1.$
Output: A superset of support of the implicit equation.

1. Define the polynomials $f_i = x_i Q(t) - P_i(t)$ and look at them as polynomials in t: $f_i = \sum c_{ij} t^{a_{ij}} \in \mathbb{C}[t]$, $c_{i,j}$ generic coefficients.
2. Compute the extreme monomials of the resultant of f_i using modified algorithm of Imai et. al. Then compute a superset of the support of the resultant.
3. Transform the support from a set of monomials of the form $\prod c_{ij}^{e_{ij}}$, to a set of monomials in the variables x_i.

Future work

Let $M_1 \neq M_2$, mixed cell configurations corresponding to vertices on the silhouette of $N(\mathcal{R})$.

$$M_1 \ni \mathcal{T}_1 \xrightarrow{flip_Z} \mathcal{T}_2 \in M_2.$$



Summary

- Computing $N(\mathcal{R})$ of polynomials f_i with Newton polytopes P_i
  $\Leftrightarrow$ Computing all mixed cell configurations of $P = P_0 + \cdots + P_n$.
- For every family of polytopes $P_0, \ldots, P_n$ there is polytope C st.: computing all tight regular mixed subdivisions of $P = \sum P_i$
  $\Leftrightarrow$ computing all regular triangulations of C.
- We can enumerate all mixed cell configurations efficiently using reverse search and flips over suitable circuits.
- Application to implicitization.

THANK YOU!

References

I.M. Gel'fand, M.M. Kapranov, A.V. Zelevinsky
Discriminants, Resultants and Multidimensional Determinants.
Birkhauser, 1994.

H. Imai, T. Masada, F. Takeuchi, K. Imai.
Enumerating Triangulations in General Dimensions.
IJCGA, 12(6):455–480, 2002.

I.Z. Emiris, I.S. Kotsireas
Implicitization with Polynomial Support Optimized for Sparseness.
Proc. Intern. Conf. Comput. Science & Appl., LNCS 2669, 397–406, 2003.

Bernd Sturmfels.
On the Newton Polytope of the Resultant.
J. Algebraic Comb., 3 (2): 207–236, 1994.

T. Michiels, J. Verschelde.
Enumerating Regular Mixed-Cell Configurations.
Discrete Comput. Geom., 21 (4): 569–579, 1999.