The 2-Evader Problem

Elias Koutsoupias* elias@cs.ucla.edu

Christos Papadimitriou† christos@cs.ucsd.edu

Abstract

It is shown that the work function algorithm for the 2-evader problem has competitive ratio $m-2$ for all metric spaces with m points. This settles the k-server conjecture for metric spaces with $k+2$ points.

Keywords: On-line algorithms, analysis of algorithms, server problem

1 Introduction

The *ℓ-evader problem* is defined on a finite metric space $\mathcal{M}$. On the points of $\mathcal{M}$ there are ℓ evaders that are free to move from point to point and they respond to a series of *ejections*. An ejection from a point forces each evader residing at this point to move to some other point. The objective of the evaders is to respond to a finite sequence of ejections by traveling a total distance that is as small as possible. We are interested in the *on-line* version of the problem when the ejection sequence is not known to the evaders in advance, but ejections are revealed (to all evaders) one by one.

One can define different variants of the ℓ-evader problem by assigning to each point p a capacity $c(p)$ such that *no more than $c(p)$ evaders can reside at p at the same time.* When every point has capacity one, the problem relates to the well-known *k-server problem* [MMS90, KP95]. In particular, when the metric space $\mathcal{M}$ has m points, the capacity-one variant is equivalent to the

*Computer Science Department, University of California, Los Angeles, CA 90095.

†Computer Science and Engineering, University of California, San Diego, La Jolla, CA 92093.

$(m-\ell)$-server problem: Servers occupy precisely the points not occupied by evaders, and an ejection of the evaders is a request for the servers. (This is a valid transformation only when two servers do not reside at the same point, but this condition is clearly satisfied by any *lazy* k-server algorithm, that is, one that never moves a server unless necessary —assuming that the servers are initially at distinct points. It was pointed out in [MMS90] that for any non-lazy algorithm there exists a lazy algorithm with the same or better performance). Consequently, any result for the ℓ-evader problem with all capacities equal to one translates immediately to a result for the k-server problem and vice-versa. *In this paper we shall restrict our attention to the unit capacity case of the ℓ-evader problem.*

The standard framework of analyzing on-line algorithms is *competitive analysis* [ST85]: The quality of an algorithm is measured by its competitive ratio —the smaller the competitive ratio the better the algorithm. Specifically for the ℓ-evader problem, the competitive ratio of an algorithm is the worst case ratio (over all ejection sequences) of the total distance traversed by the evaders, divided by the optimum off-line solution achievable if the ejection sequence were known in advance. We usually think that an on-line algorithm competes against a powerful *adversary* who chooses the ejection sequence and also responds to it. In this setting, the competitive ratio is the ratio of the cost of the on-line algorithm over the cost of the adversary.

There are very few results in the literature concerning directly the ℓ-evader problem. On the other hand, any result for the k-server problem can be translated to a result for the $(m-k)$-evader problem. In particular, it is known [MMS90] that the k-server problem has competitive ratio at least k on any metric space with at least $k+1$ points. This translates to a lower bound of $m-\ell$ for the competitive ratio of the ℓ-evader problem on any metric with m points. If we concentrate only on finite metric spaces, the well-known k-server conjecture simply states that this lower bound is achievable. Since the k-server conjecture holds for metric spaces with $k+1$ points, the 1-evader problem (also called the pursuit-evade game [BKRS92]) has competitive ratio $m-1$. For the 2-evader problem, in the general case the best known upper bound of the competitive ratio is $2m-5$, the counterpart of the upper bound of $2k-1$ of the competitive ratio for the k-server problem [KP95, Kou94]. For certain special metric spaces such as uniform metric spaces and tree metric spaces it is known that the lower bound of $m-2$ is tight [ST85, CL91].

In this paper we generalize this to any metric space: *the 2-evader problem* *has competitive ratio* $m - 2$. Via the correspondence outlined above, this implies that the k-server conjecture holds for metric spaces with $k+2$ points (it was known since [MMS90] that it holds in $k+1$ points). The algorithm we employ is the *work function algorithm;* it is widely believed that the same algorithm has competitive ratio $m - \ell$ for any number ℓ of evaders. Our proof of the competitiveness of the work function algorithm uses the *duality property* introduced in [KP95]; the argument then rests on a novel potential that we introduce, based on *minimum spanning trees.* The complexity of the potential in this relatively simple case is indicative of why the k-server conjecture is still unsettled.

Conceivably, the techniques we employ here can be useful for the general case (actually, it was this result that eventually led us to the $2k-1$ upper bound for the k-server problem [KP95]). Although we could present our result, algorithm, and proof in terms of the k-server problem, we have chosen to present them in terms of the ℓ-evader problem because we find this presentation more natural and intuitive (and furthermore, it was in this form that we first discovered it).

2 The work function algorithm

Let $\mathcal{M}$ be a finite metric space and let $d(a,b)$ denote the distance between points a and b of $\mathcal{M}$; d is symmetric, non-negative, and satisfies the triangle inequality. In the 2-evader problem two evaders reside on the points of $\mathcal{M}$ and they are free to move from point to point, but they cannot occupy the same point simultaneously. When the two evaders occupy the points a and b we say that they are in *configuration* $\{a,b\}$.

We assume that the evaders (on-line or off-line) are initially at a fixed configuration. An ejection sequence ρ is an element of $\mathcal{M}^*$. For each such sequence, let $w_\rho(a,b)$ denote the optimal cost for two evaders to respond to the sequence of ejections ρ and end up at the configuration $\{a,b\}$. In other words, $w_\rho(a,b)$ is the cost of the adversary that after the sequence of ejections ρ is at the configuration $\{a,b\}$. A symmetric function w from $\mathcal{M}^2$ to the positive reals is called *a work function* if there is a $\rho \in \mathcal{M}^*$ such that $w(a,b) = w_\rho(a,b)$ for all distinct a and b. Here are some important properties of the work functions (for a proof see [KP95, Kou94]):

Lemma 1 *Let w be a work function and let w' be the resulting work function after one more ejection r. Then the following are true for all x, y:*

a. $w'(x,y) = \begin{cases} w(x,y), & \text{if } r \neq x,y \\ \min_{x' \neq x,y} \{w(x',y) + d(x,x')\}, & \text{if } r = x. \end{cases}$

b. $w'(x,y) \geq w(x,y)$.

c. For all y': $w(x,y) + d(y,y') \geq w(x,y')$.

d. [Duality] *If $\{x,y\}$ minimizes the expression $w(x,y) + d(r,x) + d(r,y)$ then $\{x,y\}$ minimizes also the same expression for w', $w'(x,y)+d(r,x)+d(r,y)$, and maximizes the expression $w'(x,y) - w(x,y)$.*

The configuration $\{x,y\}$ that minimizes $w(x,y)+d(r,x)+d(r,y)$, as in the duality property of the above lemma, is called a *minimizer of r with respect to w*; the maximum of the other expression that appears in the duality property, $\max_{\{x,y\}}\{w'(x,y) - w(x,y)\}$, is called the *extended cost* for the ejection r.

We shall next describe the work function algorithm. Assume that the two on-line evaders are at points a and b. If the next ejection is on some point different from a and b the work function algorithm simply does nothing; otherwise, we can assume without loss of generality that the next ejection is at a. Then the evader residing at a moves to a new point a' which is different from b and minimizes the expression $w(a',b) + d(a,a')$.

The following lemma allows us to deduce the competitiveness of the work function algorithm from a condition about work functions (see [KP95, Kou94] or [CL92] for a proof).

Lemma 2 *If the total extended cost, i.e. the sum of the extended costs for all ejections, is bounded above by $c+1$ times the total optimum (off-line) cost plus a constant, then the work function algorithm is c-competitive.*

3 The potential

Consider a work function w and define an edge-weighted graph G_w with nodes the points of the metric space; the weight of G_w on an edge $\{x,y\}$ is defined as $w(x,y) + d(x,y)$. We will use the following potential of w: Φ_w is simply the weight of a minimum spanning tree of G_w.

The following lemma is at the heart of our proof of competitiveness.

Lemma 3 *Let w be a work function and let w' be the resulting work function after one more ejection r. Then there exists a minimizer $\{a, b\}$ of r with respect to w such that $\Phi_{w'} - \Phi_w \geq w'(a, b) - w(a, b)$.*

Proof. First we show that there exists a minimizer of r with respect to w that contains r: From the properties of the work function (Lemma 1.c), for all x and y: $w(x, y) + d(r, x) \geq w(r, y)$ and consequently $w(x, y) + d(r, x) + d(r, y) \geq w(r, y) + d(r, r) + d(r, y)$. From this we can conclude that if $\{x, y\}$ is a minimizer then $\{r, y\}$ is also a minimizer.

Consider now a minimizer $\{r, b\}$ of r with respect to w. This is also a minimizer of r with respect to w': $w'(r, b) + d(r, b) = \min_y\{w'(r, y) + d(r, y)\}$. This means that the edge $\{r, b\}$ of $G_{w'}$ has minimum weight among the edges incident to r. But this is precisely the property that enables us to conclude that there exists some minimum spanning tree of $G_{w'}$ that contains the minimizer $\{r, b\}$ (this is a fundamental property of matroids). Let T be such a minimum spanning tree; its weight is, by definition, the potential $\Phi_{w'}$ of w':

$$\Phi_{w'} = \sum_{\{x,y\} \in T} \{w'(x, y) + d(x, y)\} \tag{1}$$

Consider now the weight of the same tree T in the graph G_w. Its weight, of course, cannot be lower than the weight of the minimum spanning tree in G_w and consequently:

$$\Phi_w \leq \sum_{\{x,y\} \in T} \{w(x, y) + d(x, y)\} \tag{2}$$

From (1) and (2) we get that

$$\Phi_{w'} - \Phi_w \geq \sum_{\{x,y\} \in T} \{w'(x, y) - w(x, y)\}$$

and since in general we have $w'(x, y) \geq w(x, y)$ (Lemma 1.b) we can drop all terms but $w'(r, b) - w(r, b)$ from the right hand side. This completes the proof of the lemma because $\{r, b\}$ is a minimizer of r with respect to w. □

Notice that in the above lemma the extended cost for the ejection r is equal to $w'(a, b) - w(a, b)$; this is so because of the duality property and the fact that $\{a, b\}$ is a minimizer of r with respect to w.

We are now ready to prove the main result.

Theorem 1 *The competitive ratio of the work function algorithm for the 2-evader problem on a metric space with m points is $m-2$.*

Proof. According to Lemma 3 the difference in potential for responding to an ejection r is at least equal to the extended cost. By considering the sequence of ejections we get that

$$\Phi_{w_\rho} - \Phi_{w_e} \geq \text{total extended cost for } \rho.$$

By the definition of the work function the optimal off-line cost is at least $w_\rho(x, y)$, for some x and y. Also every other value of the work function w_ρ differs from $w_\rho(x, y)$ by a constant (sum of two distances). Therefore, Φ_{w_ρ} which is the sum of $m-1$ values of the work function w_ρ and $m-1$ distances is equal, within an additive constant, to $(m-1)$ times the optimal off-line cost. Similarly, the initial potential Φ_{w_e} is equal to a constant. We conclude that the total extended cost for ρ is bounded from above by $m-1$ times the optimal cost plus a constant that is independent of ρ. Applying Lemma 2 completes the proof of the theorem. □

4 Open Problems

The technique we used for the 2-evader problem can conceivably be applied to the general case of the ℓ-evader problem. Recall that the potential we used is the weight of a minimum spanning tree of a graph that depends on the current work function. For the ℓ-evader problem the graph becomes a hypergraph. The difficult part is *to generalize appropriately the notion of minimum spanning tree to weighted hypergraphs.* The *quasiconvexity property* of work functions [KP95, Kou94], that is used to prove the Duality Lemma (Lemma 1.d), may prove very useful for such a generalization.

Finally, the general ℓ-evader problem, in which points of the metric space can accommodate more than one evader, is a fundamental on-line problem and, in our opinion, its solution would greatly advance our understanding of on-line computation.

References

[BKRS92] A. Blum, H. Karloff, Y. Rabani, and M. Saks. A decomposition theorem and lower bounds for randomized server problems. In *Proc. 33rd IEEE Symposium on Foundations of Computer Science*, pages 197–207, 1992.

[CL91] M. Chrobak and L. L. Larmore. An optimal online algorithm for k servers on trees. *SIAM Journal on Computing*, 20:144–148, 1991.

[CL92] M. Chrobak and L. L. Larmore. The server problem and on-line games. In *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, volume 7, pages 11–64, 1992.

[Kou94] E. Koutsoupias. *On-line Algorithms and the k-server conjecture.* PhD thesis, University of California, San Diego, 1994.

[KP95] E. Koutsoupias and C. H. Papadimitriou. On the k-server conjecture. *Journal of the ACM*, 45(5):971–983, 1995. Also in *Proc. 25th Annual ACM Symposium on Theory of Computing*, pages 507–511, 1994.

[MMS90] M. Manasse, L. A. McGeoch, and D. Sleator. Competitive algorithms for server problems. *Journal of Algorithms*, 11:208–230, 1990. Also in *Proc. 20th Annual ACM Symposium on Theory of Computing*, pages 322–333, 1988.

[ST85] D. Sleator and R. E. Tarjan. Amortized efficiency of list update and paging rules. *Communications of the ACM*, 28:202–208, 1985.