Online Competitive Auctions

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Joint work with George Pierrakos
Digital goods auctions

**Digital good auction**
- We want to sell a digital good (with no replication cost)
- $n$ bidders who have a private valuation for the good
- Objective: Maximize the profit

**Types of auctions**
- Offline: All bidders are present
- Online: Bidders appear online
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How to model uncertainty?

Models

Adversarial  The input is designed by a powerful adversary who knows the algorithm and tailors the set of bids to defeat it

Stochastic  There is a known or unknown probability distribution.
- Independent bids: Each bid is selected independently from the others
- Correlated bids: The probability distribution is for all bids and not for each one separately

Random-order (online)  The adversary selects the set of bids and they are presented in a random order, as in the secretary problem
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Auction Definitions

**Definition**

An auction is **truthful** if and only if the price offered to a bidder is independent of his bid.

**Some auctions**

**DOP (offline)**
To every bidder offer the optimal single price of the other bidders.
### Some auctions

**RSOP (offline)**
- Partition the bidders randomly into two sets
- Find the optimal single price for each set and offer it to the bidders of the other set

**SCS (offline)**
Similar to RSOP but try to extract the profit of each set instead of offering its optimal price

**BPSF (online)**
To every bidder offer the optimal single price for the revealed bids (the online version of DOP)
How to evaluate an auction?

Notation: Let $b_1 > b_2 > \cdots > b_n$ be the bids

Compare a mechanism against?

- Sum of all bids: $\sum_i b_i$ (unrealistic)
- Optimal single-price profit: $\max_i i \cdot b_i$ (problem: highest bid impossible to get)
- A reasonable benchmark:

$$F^{(2)} = \max_{i \geq 2} i \cdot b_i$$

The optimal profit of

- a single-price auction
- which sells the good to at least 2 bidders

This is the benchmark we adopt.

We call an algorithm $\rho$-competitive if its profit is at least $F^{(2)}/\rho$. 
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Worst-case (known) distribution

- Suppose that the bids are drawn from a known probability distribution
- We can then design the auction with the best competitive ratio
- How high can it be?
- For which distribution?

Yao’s lemma / minmax property

The competitive ratio of the worst-case distribution provides a (usually tight) lower bound for randomized algorithms in the worst-case input.
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Independent vs correlated distributions

We will only consider i.i.d.’s or simply i.d’s
The equal-revenue distribution

The equal-revenue cumulative distributions are of the form

\[ F_c(x) = \begin{cases} 
0 & x < c \\
1 - \frac{c}{x} & x \geq c 
\end{cases} \]

It has profit \( x(1 - F_c(x)) = c \) independent of the price offered.
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The worst-case independent distribution

**Theorem**

Among the independent distributions, the equal-revenue distributions have maximum competitive ratio.

**Proof.**

- Let $F$ be a cumulative distribution with competitive ratio $\rho$
- The optimal pricing mechanism selects price $p$ which maximizes $p(1 - F(p))$
- Let $c$ be its profit
- Then for every $x$: $x(1 - F(x)) \leq c$, or equivalently, $F(x) \geq 1 - c/x$.
- Thus, $F(x)$ dominates the equal-revenue distribution $F_c(x)$. 


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Lemma

Let \( F_1, F_2 \) be two cumulative distributions with \( F_1(x) \leq F_2(x) \) for every \( x \). Let also \( G : \mathbb{R}^n \to \mathbb{R} \) be a function which is non-decreasing in all its variables. Then

\[
E_{b \in F_1^n}[G(b)] \geq E_{b \in F_2^n}[G(b)]
\]
The proof of the lemma

- For a single variable the proof depends on the following property of integrals
  \[ \int_0^\infty F'(x)G(x) \, dx = \int_0^\infty (1 - F(x))G'(x) \, dx + G(0) \]

- For many variables, we can apply this inductively

- The independence of variables is crucial for the induction

- The benchmark \( F^{(2)}(b) \) is non-decreasing in each bid

- Therefore the equal-revenue distributions have maximum competitive ratio
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\[
\frac{n}{1 - \sum_{i=2}^{n} \left( \frac{-1}{n} \right)^{i-1} \cdot \frac{i}{i-1} \cdot \binom{n-1}{i-1}}
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The competitive ratio ranges from 2 (when \( n = 2 \)) to 2.42 (when \( n \to \infty \))

**Conjecture**

The optimal offline competitive ratio is 2.42
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Questions for competitive auctions

Optimal competitive ratio for the **adversarial offline** case?

- Symmetric deterministic: unbounded
- Randomized: $\in [2.42, 3.24]$
- RSOP is 4.64-competitive
- Conjecture: RSOP is 4-competitive

(Goldberg-Hartline-Karlin-Saks-Wright, Hartline-McGrew)

Optimal competitive ratio for the **stochastic** case?

- Again $\in [2.42, 3.24]$
- Why the same? Because of Yao’s lemma
- Theorem: For bid-independent distributions the answer is 2.42
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- Symmetric deterministic: unbounded
- Randomized: $\in [2.42, 3.24]$
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Assumptions

- Unknown bids $b_1 > b_2 > \cdots > b_n$
- They arrive in order $b_{\pi_1}, \ldots, b_{\pi_n}$, where $\pi$ is a random permutation
- For each bid we offer a take-it-or-leave price
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Natural (?) pricing algorithms

Pricing algorithms
- MIN, MEAN, MEDIAN: unbounded competitive ratio
- Why? Consider bids 1, 1, 0, 0, ..., 0

Theorem

The algorithm (MAX) which offers the maximum revealed bid has competitive ratio \( k/(H_k - 1) \), where \( F^{(2)} = kb_k \).

Proof.

The exact (!) profit of MAX is

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\frac{1}{2}b_2 + \cdots + \frac{1}{n}b_n
\]

The ratio \( k/(H_k - 1) \) is not bad for small values of \( k \) (it is less than 4 for \( k \leq 5 \)).
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### Transforming an offline mechanism to online

#### How to transform an offline algorithm to online

- Simply run the offline algorithm for the set of revealed bids and the current (unrevealed bid)
- For example, the online version of DOP is the BPSF auction
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#### Theorem

The competitive ratio of the online algorithm is at most $k/(k - 1) \leq 2$ times greater than the offline competitive ratio, where $F^{(2)} = kb_k$. 
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The competitive ratio of the online algorithm is at most $k / (k - 1) \leq 2$ times greater than the offline competitive ratio, where $F^{(2)} = kb_k$. 
Proof

- Let $\rho$ be the offline competitive ratio
- Let $F(2(b_1, \ldots, b_n)) = k \cdot b_k$
- Expected online profit at step $t = \frac{1}{t} \cdot$ expected offline profit of the first $t$ bids
- With probability $\binom{t}{m} \binom{n-t}{k-m}/\binom{n}{k}$ the first $t$ bids have $m$ of the high $k$ bids
- offline profit $\geq \frac{1}{\rho} \cdot m \cdot b_k$, when $m \geq 2$
- Putting everything together

$$\text{online profit} \geq \sum_{t=2}^{n} \sum_{m=2}^{\min\{t,k\}} \frac{\binom{t}{m} \binom{n-t}{k-m}}{\binom{n}{k}} \cdot \frac{1}{t} \cdot \frac{1}{\rho} \cdot mb_k$$

$$= \frac{k-1}{\rho} b_k = \frac{k-1}{k} \cdot \frac{1}{\rho} \cdot F(2)$$
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- Let $\rho$ be the offline competitive ratio
  - Let $F(2)(b_1, \ldots, b_n) = k \cdot b_k$
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  - With probability $\binom{t}{m} \frac{(n-t)}{(k-m)} / \binom{n}{k}$ the first $t$ bids have $m$ of the high $k$ bids
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Consequences

**Theorem**

*The online competitive ratio is between 4 and 6.48*

**Why?**

- The lower bound comes from specific cases: 2 distinct bids or
  \[ b = (2 + \epsilon, 2 - \epsilon, 1) \]
- For the upper bound, take the offline auction of
  Hartline-McGrew with competitive ratio 3.24 and transform it
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**Conjecture**

*The online competitive ratio is 4. Stronger: BPSF has competitive ratio 4.*
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MAX has competitive ratio $\frac{k}{H_{k-1}} \leq 4$ for $k \leq 5$

Online-SCS has competitive ratio $\frac{k}{k-1} \left( \frac{1}{2} - \left( \frac{k-1}{[k-1]} \right) \cdot 2^{-k} \right)^{-1}$, which is less than 4 for $k \geq 5$.

If we know $k$, we can achieve 4-competitiveness.
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"Almost" 4-competitive

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Deterministic vs randomized

Offline auctions
No offline symmetric deterministic auction has bounded competitive ratio [GHKSW06]

Online auctions
- Order seems to matter!
- BPSF has bounded competitive ratio (open!)
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Define $y(S) = \max\{1 \cdot b_{j_1}, 2 \cdot b_{j_2}, \ldots, r \cdot b_{j_r}\}$ the optimal single price profit of $S$

Define $z(S)$ the profit from offering the optimal single price of $S$ to the other side

$z(S) = (j_i - i)b_{j_i}$, where $i = \text{argmax} \ y(S)$

RSOP $= \sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot 2^{-(n-1)}$

BPSF $= \sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot \left(\frac{n-1}{|S|}\right)^{-1} \cdot n^{-1}$
RSOP and BPSF

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$z(S) = (j_i - i) b_{j_i}$, where $i = \text{argmax } y(S)$.

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\text{RSOP} = \sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot 2^{-(n-1)}
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\text{BPSF} = \sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot \left(\frac{n-1}{|S|}\right)^{-1} \cdot n^{-1}
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RSOP and BPSF

Let $S = \{b_{j_1} > b_{j_2} > \cdots > b_{j_r}\}$, a subset of bids.

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Conjectures

**Conjecture**

*RSOP is 4-competitive. Equivalently, for every set of bids $b$:*

$$\sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot 2^{-(n-1)} \geq y(b_2, b_2, b_3, \ldots, b_n)$$

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*BPSF is 4-competitive. Equivalently, for every set of bids $b$:*

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A coupling argument

\[
\sum_{S \in \{b_2, \ldots, b_n\}} y(S) \geq \sum_{b_2 \in S} b_2 \in S y(S) = 2^n - i \sum_{j=0}^{i-2} \binom{i-2}{j} \cdot (j + 1) \cdot b_i = 2^{n-3} \cdot i \cdot b_i
\]

Lemma

\[
\sum_{S \in \{b_2, \ldots, b_n\}} y(S) \geq 2^{n-3} \cdot F^{(2)}
\]
Relations between $z$ and $y$

Conjecture

$$\sum_{S \in \{b_2, \ldots, b_n\}} z(S) \geq \sum_{S \in \{b_2, \ldots, b_n\}, \ b_2 \in S} y(S)$$

This will show that RSOP is 4-competitive

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$$\sum_{S \in \{b_3, \ldots, b_n\}} z(S) \geq \sum_{S \in \{b_3, \ldots, b_n\}} y(S)$$

The second conjecture implies the first because

$$z(b_{j_1}, \ldots, b_{j_r}) \geq y(b_{j_1}, \ldots, b_{j_r}) - y(b_{j_2}, \ldots, b_{j_r})$$
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Open problems

- Prove or disprove that the worst-case distribution is bid-independent
- Prove that BPSF is 4-competitive
- Prove that RSOP is 4-competitive
- Typical question:
  - Consider a set of positive numbers $b_1 > b_2 > \cdots > b_n$
  - Let $b_{j_1}, b_{j_2}, \ldots, b_{j_r}$ be a random subset. Then for every $i = 1, \ldots, r$:
    \[
    E[(j_i - i + 2) \cdot b_{j_i}] \geq E[i \cdot b_{j_i}]
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Open problems

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Thank you!