Online Competitive Auctions

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Joint work with George Pierrakos
Digital goods auctions

Digital good auction
- We want to sell a digital good (with no replication cost)
- $n$ bidders who have a private valuation for the good
- Objective: Maximize the profit

Types of auctions
- Offline: All bidders are present
- Online: Bidders appear online
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## Models

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Truthfulness

An auction is **truthful** if and only if the price offered to a bidder is independent of his bid.

Some auctions

**DOP (offline)** To every bidder offer the optimal *single price* of the other bidders.
Some truthful offline auctions

Some auctions

RSOP (offline)
- Partition the bidders randomly into two sets
- Find the optimal single price for each set and offer it to the bidders of the other set

SCS (offline) Similar to RSOP but try to extract the profit of each set instead of offering its optimal price

BPSF (online) To every bidder offer the optimal single price for the revealed bids (the online version of DOP)
How to evaluate an auction?

Notation: Let $b_1 > b_2 > \cdots > b_n$ be the bids

Compare a mechanism against?

- Sum of all bids: $\sum_i b_i$ (unrealistic)
- Optimal single-price profit: $\max_i i \cdot b_i$ (problem: highest bid impossible to get)
- A reasonable benchmark:

$$F^{(2)} = \max_{i \geq 2} i \cdot b_i$$

The optimal profit of

- a single-price auction
- which sells the good to at least 2 bidders

This is the benchmark we adopt

We call an algorithm $\rho$-competitive if its profit is at least $F^{(2)}/\rho$
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Questions for competitive auctions

Optimal competitive ratio for the adversarial offline case?

- Symmetric deterministic: unbounded
- Randomized: $\in [2.42, 3.24]
- RSOP is 4.64-competitive
- Conjecture: RSOP is 4-competitive

(Goldberg-Hartline-Karlin-Saks-Wright, Hartline-McGrew)

Optimal competitive ratio for the stochastic case?

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Suppose that the bids are drawn from a known probability distribution. We can then design the auction with the best competitive ratio. How high can it be? For which distribution?

Yao’s lemma / minmax property

The competitive ratio of the worst-case distribution provides a (usually tight) lower bound for randomized algorithms in the worst-case input.
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Independent vs correlated distributions

We will only consider i.i.d.’s or simply i.d’s
The equal-revenue distribution

The equal-revenue cumulative distributions are of the form

\[ F_c(x) = \begin{cases} 
0 & x < c \\
1 - \frac{c}{x} & x \geq c 
\end{cases} \]

It has profit \( x(1 - F_c(x)) = c \) independent of the price offered.
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The worst-case independent distribution

**Theorem**

Among the independent distributions, the equal-revenue distributions have maximum competitive ratio.

**Proof.**

- Let $F$ be a cumulative distribution with competitive ratio $\rho$.
- The optimal pricing mechanism selects price $p$ which maximizes $p(1 - F(p))$.
- Let $c$ be its profit.
- Then for every $x$: $x(1 - F(x)) \leq c$, or equivalently, $F(x) \geq 1 - c/x$.
- Thus, $F(x)$ dominates the equal-revenue distribution $F_c(x)$. 
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Lemma

Let $F_1, F_2$ be two cumulative distributions with $F_1(x) \leq F_2(x)$ for every $x$. Let also $G : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function which is non-decreasing in all its variables. Then

$$E_{b \in F_1^n}[G(b)] \geq E_{b \in F_2^n}[G(b)]$$
The proof of the lemma

For a single variable the proof depends on the following property of integrals

\[ \int_{0}^{\infty} F'(x)G(x) \, dx = \int_{0}^{\infty} (1 - F(x))G'(x) \, dx + G(0) \]

For many variables, we can apply this inductively.

The independence of variables is crucial for the induction.

The benchmark \( F^{(2)}(b) \) is non-decreasing in each bid.

Therefore the equal-revenue distributions have maximum competitive ratio.
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\[ \int_0^\infty F'(x)G(x) \, dx = \int_0^\infty (1 - F(x))G'(x) \, dx + G(0) \]

- For many variables, we can apply this inductively

- The independence of variables is crucial for the induction

- The benchmark \( F^{(2)}(b) \) is non-decreasing in each bid
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The competitive ratio of independent distributions

- [GHKSW06] has shown that if $b_1, \ldots, b_n$ are drawn from the equal-revenue distribution $F_1$, the expected value of $F^{(2)}$ is

$$n \cdot \left( 1 - \sum_{i=2}^{n} \left( \frac{-1}{n} \right)^{i-1} \cdot \frac{i}{i-1} \cdot \binom{n-1}{i-1} \right)$$

- The competitive ratio ranges from 2 (when $n = 2$) to 2.42 (when $n \to \infty$)

Conjecture

The optimal offline competitive ratio is 2.42
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The online problem

Assumptions

- Unknown bids $b_1 > b_2 > \cdots > b_n$
- They arrive in order $b_{\pi_1}, \ldots, b_{\pi_n}$, where $\pi$ is a random permutation
- For each bid we offer a take-it-or-leave price
- We assume that we learn the actual bid
- The bidders cannot control their arrival time

Question

- What is the best price $p(b_{\pi_1}, \ldots, b_{\pi_{t-1}})$ to offer to $b_{\pi_t}$?
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Natural (?) pricing algorithms

### Pricing algorithms
- MIN, MEAN, MEDIAN: unbounded competitive ratio
- Why? Consider bids $1, 1, 0, 0, \ldots, 0$

### Theorem
*The algorithm (MAX) which offers the maximum revealed bid has competitive ratio $k/(H_k - 1)$, where $F^{(2)} = kb_k$."

### Proof.
The exact (!) profit of MAX is

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\frac{1}{2}b_2 + \cdots + \frac{1}{n}b_n
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The ratio $k/(H_k - 1)$ is not bad for small values of $k$ (it is less than 4 for $k \leq 5$).
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Transforming an offline mechanism to online

How to transform an offline algorithm to online

- Simply run the offline algorithm for the set of revealed bids and the current (unrevealed bid)
- For example, the online version of DOP is the BPSF auction
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Theorem

The competitive ratio of the online algorithm is at most $k/(k - 1) \leq 2$ times greater than the offline competitive ratio, where $F^{(2)} = kb_k$. 
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*The competitive ratio of the online algorithm is at most $k / (k - 1) \leq 2$ times greater than the offline competitive ratio, where $F^{(2)} = kb_k$.***
Proof

- Let \( \rho \) be the offline competitive ratio
- Let \( F^{(2)}(b_1, \ldots, b_n) = k \cdot b_k \)
- Expected online profit at step \( t = \frac{1}{t} \cdot \) expected offline profit of the first \( t \) bids
- With probability \( \binom{t}{m} \binom{n-t}{k-m} / \binom{n}{k} \) the first \( t \) bids have \( m \) of the high \( k \) bids
- offline profit \( \geq \frac{1}{\rho} \cdot m \cdot b_k \), when \( m \geq 2 \)
- Putting everything together

\[
\text{online profit} \geq \sum_{t=2}^{n} \sum_{m=2}^{\min\{t,k\}} \frac{\binom{t}{m} \binom{n-t}{k-m}}{\binom{n}{k}} \cdot \frac{1}{t\rho} \cdot mb_k
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Theorem

The online competitive ratio is between 4 and 6.48

Why?

- The lower bound comes from specific cases: 2 distinct bids or $b = (2 + \epsilon, 2 - \epsilon, 1)$
- For the upper bound, take the offline auction of Hartline-McGrew with competitive ratio 3.24 and transform it into an online auction

Conjecture

The online competitive ratio is 4. Stronger: BPSF has competitive ratio 4.
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"Almost" 4-competitive

- Let $F^{(2)} = k \cdot b_k$
- MAX has competitive ratio $\frac{k}{H_{k-1}} \leq 4$ for $k \leq 5$
- Online-SCS has competitive ratio $\frac{k}{k-1} \left(\frac{1}{2} - \left(\left\lfloor \frac{k-1}{k-1}\right\rfloor \cdot 2^{-k}\right)\right)^{-1}$, which is less than 4 for $k \geq 5$.
- If we know $k$, we can achieve 4-competitiveness.
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Deterministic vs randomized

**Offline auctions**

No offline symmetric deterministic auction has bounded competitive ratio [GHKSW06]

**Online auctions**

- Order seems to matter!
- BPSF has bounded competitive ratio (open!)
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Let $S = \{b_{j_1} > b_{j_2} > \cdots > b_{j_r}\}$, a subset of bids

Define $y(S) = \max\{1 \cdot b_{j_1}, 2 \cdot b_{j_2}, \ldots, r \cdot b_{j_r}\}$ the optimal single price profit of $S$

Define $z(S)$ the profit from offering the optimal single price of $S$ to the other side

$z(S) = (j_i - i) b_{j_i}$, where $i = \arg\max y(S)$

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\text{RSOP} = \sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot 2^{-(n-1)}
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RSOP and BPSF

Let $S = \{ b_{j_1} > b_{j_2} > \cdots > b_{j_r} \}$, a subset of bids.

Define $y(S) = \max \{ 1 \cdot b_{j_1}, 2 \cdot b_{j_2}, \ldots, r \cdot b_{j_r} \}$ the optimal single price profit of $S$.

Define $z(S)$ the profit from offering the optimal single price of $S$ to the other side.

$z(S) = (j_i - i) b_{j_i}$, where $i = \arg\max y(S)$.

RSOP = \[
\sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot 2^{-(n-1)}
\]

BPSF = \[
\sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot \left(\frac{n-1}{|S|}\right)^{-1} \cdot n^{-1}
\]
Conjectures

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RSOP is 4-competitive. Equivalently, for every set of bids $b$:

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\sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot 2^{-(n-1)} \geq y(b_2, b_2, b_3, \ldots, b_n)
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A coupling argument

\[ \sum_{S \in \{b_2, \ldots, b_n\}} y(S) \geq \]

\[ \sum_{S \in \{b_2, \ldots, b_n\}} y(S) = \]

\[ 2^{n-i} \sum_{j=0}^{i-2} \binom{i-2}{j} \cdot (j + 1) \cdot b_i = \]

\[ 2^{n-3} \cdot i \cdot b_i \]

Lemma

\[ \sum_{S \in \{b_2, \ldots, b_n\}} y(S) \geq 2^{n-3} \cdot F(2) \]
Relations between $z$ and $y$

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This will show that RSOP is 4-competitive

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$$\sum_{S \in \{b_3, \ldots, b_n\}} z(S) \geq \sum_{S \in \{b_3, \ldots, b_n\}} y(S)$$

The second conjecture implies the first because

$$z(b_{j_1}, \ldots, b_{j_r}) \geq y(b_{j_1}, \ldots, b_{j_r}) - y(b_{j_2}, \ldots, b_{j_r})$$
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Open problems

- Prove or disprove that the worst-case distribution is bid-independent
- Prove that BPSF is 4-competitive
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Thank you!