Online Competitive Auctions

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Joint work with George Pierrakos

Digital goods auctions

Digital good auction

- We want to sell a digital good (with no replication cost)
- n bidders who have a **private valuation** for the good
- Objective: Maximize the profit

Types of auctions

Offline All bidders are present

Online Bidders appear online

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How to model uncertainty?

Models

Adversarial: The input is designed by a powerful adversary who knows the algorithm and tailors the set of bids to defeat it

Stochastic: There is a known or unknown probability distribution.

- Independent bids: Each bid is selected independently from the others
- Correlated bids: The probability distribution is for all bids and not for each one separately

Random-order (online): The adversary selects the set of bids and they are presented in a random order, as in the **secretary problem**

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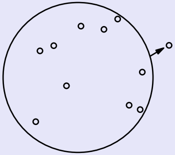
Some truthful offline auctions

Truthfulness

An auction is **truthful** if and only if the price offered to a bidder is independent of his bid

Some auctions

DOP (offline) To every bidder offer the optimal **single price** of the other bidders

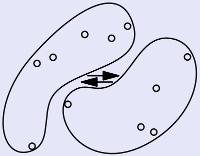


Some truthful offline auctions

Some auctions

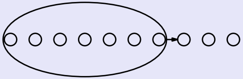
RSOP (offline)

- Partition the bidders randomly into two sets
- Find the optimal **single price** for each set and offer it to the bidders of the other set



SCS (offline) Similar to RSOP but try to extract the profit of each set instead of offering its optimal price

BPSF (online) To every bidder offer the optimal single price for the revealed bids (the online version of DOP)



How to evaluate an auction?

Notation: Let $b_1 > b_2 > \cdots > b_n$ be the bids

Compare a mechanism against ?

- Sum of all bids: $\sum_i$ b$_i$ (unrealistic)
- Optimal single-price profit: $\max_i i \cdot b_i$ (problem: highest bid impossible to get)
- A reasonable benchmark:

$$F^{(2)} = \max_{i>=2} i \cdot b_i$$

The **optimal** profit of

- a single-price auction
- which sells the good to at least 2 bidders

- This is the benchmark we adopt
- We call an algorithm ρ-**competitive** if its profit is at least $F^{(2)}/\rho$

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Optimal competitive ratio for the **adversarial offline** case?

- Symmetric deterministic: unbounded
- Randomized: $\in [2.42, 3.24]$
- RSOP is 4.64-competitive
- Conjecture: RSOP is 4-competitive

(Goldberg-Hartline-Karlin-Saks-Wright, Hartline-McGrew)

Optimal competitive ratio for the stochastic case?

- Again $\in [2.42, 3.24]$
- Why the same? Because of Yao's lemma
- Theorem: For bid-independent distributions the answer is 2.42

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Optimal online competitive ratio for the random-order case?

- Theorem: There is a generic transformation of offline auctions to online auctions, with only a loss of a factor of 2 in the competitive ratio.
- Competitive ratio $\in [4, 6.48]$
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Stochastic case: worst-case distribution

- Suppose that the bids are drawn from a known probability distribution
- We can then design the auction with the best competitive ratio
- How high can it be?
- For which distribution?

Yao's lemma / minmax property

The competitive ratio of the worst-case distribution provides a (usually tight) lower bound for randomized algorithms in the worst-case input.

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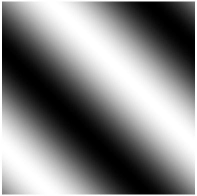
Independent vs correlated distributions



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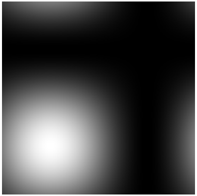


i.i.d.

We will only consider i.i.d.'s or simply i.d's

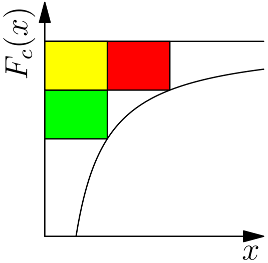
The equal-reveneue distribution

The equal-revenue distributions

- The equal-revenue cumulative distributions are of the form

$$F_c(x) = \begin{cases} 0 & x < c \\ 1 - \frac{c}{x} & x \geq c \end{cases}$$

- It has profit $x(1 - F_c(x)) = c$ independent of the price offered



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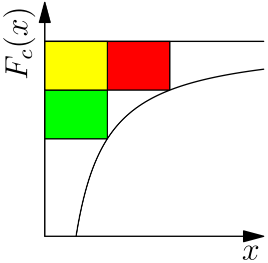

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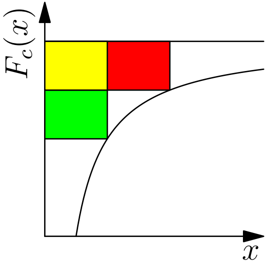

The worst-case independent distribution

Theorem

Among the independent distributions, the equal-revenue distributions have maximum competitive ratio.

Proof.

- Let F be a cumulative distribution with competitive ratio ρ
- The optimal pricing mechanism selects price p which maximizes $p(1 - F(p))$
- Let c be its profit
- Then for every x: $x(1 - F(x)) \leq c$, or equivalently, $F(x) \geq 1 - c/x$.
- Thus, $F(x)$ dominates the equal-revenue distribution $F_c(x)$.

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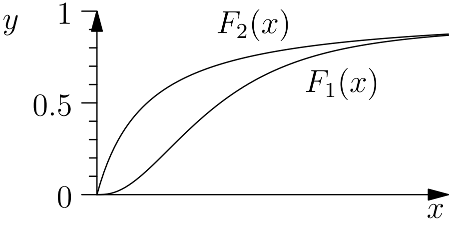
A crucial lemma

Lemma

Let F_1, F_2 be two cumulative distributions with $F_1(x) \leq F_2(x)$ for every x. Let also $G : \mathbb{R}^n \to \mathbb{R}$ be a function which is ***non-decreasing*** *in all its variables. Then*

$$E_{b \in F_1^n}[G(b)] \geq E_{b \in F_2^n}[G(b)]$$



Proof (cont.)

The proof of the lemma

- For a single variable the proof depends on the following property of integrals

$$\int_0^\infty F'(x)G(x)\,dx = \int_0^\infty (1 - F(x))G'(x)\,dx + G(0)$$

- For many variables, we can apply this inductively
- The independence of variables is crucial for the induction

- The benchmark $F^{(2)}(b)$ is non-decreasing in each bid
- Therefore the equal-revenue distributions have maximum competitive ratio

Proof (cont.)

The proof of the lemma

- For a single variable the proof depends on the following property of integrals

$$\int_0^\infty F'(x)G(x)\,dx = \int_0^\infty (1 - F(x))G'(x)\,dx + G(0)$$

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- [GHKSW06] has shown that if $b_1, \ldots, b_n$ are drawn from the equal-revenue distribution F_1, the expected value of $F^{(2)}$ is

$$n \cdot \left(1 - \sum_{i=2}^{n} \left(\frac{-1}{n}\right)^{i-1} \cdot \frac{i}{i-1} \cdot \binom{n-1}{i-1}\right)$$

- The competitive ratio ranges from 2 (when $n = 2$) to 2.42 (when $n \to \infty$)

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The online problem

Assumptions

- Unknown bids $b_1 > b_2 > \cdots > b_n$
- They arrive in order $b_{\pi_1}, ..., b_{\pi_n}$, where π is a random permutation
- For each bid we offer a take-it-or-leave price
- **We assume that we learn the actual bid**
- **The bidders cannot control their arrival time**

Question

- What is the best price $p(b_{\pi_1}, \ldots, b_{\pi_{t-1}})$ to offer to b_{π_t}?

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Natural (?) pricing algorithms

Pricing algorithms

- MIN, MEAN, MEDIAN: unbounded competitive ratio
- Why? Consider bids $1, 1, 0, 0, \ldots, 0$

Theorem

The algorithm (MAX) which offers the maximum revealed bid has competitive ratio $k/(H_k - 1)$, where $F^{(2)} = kb_k$.

Proof.

The exact (!) profit of MAX is

$$\frac{1}{2}b_2 + \cdots + \frac{1}{n}b_n$$

□

The ratio $k/(H_k - 1)$ is not bad for small values of k (it is less than 4 for $k \leq 5$).

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How to transform an offline algorithm to online

- Simply run the offline algorithm for the set of revealed bids and the current (unrevealed bid)
- For example, the online version of DOP is the BPSF auction
- Is it good? We compare with $F^{(2)}$ of **all** bids

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The competitive ratio of the online algorithm is at most $k/(k-1) \leq 2$ *times greater than the offline competitive ratio, where* $F^{(2)} = kb_k$.

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- Let $F^{(2)}(b_1, \ldots, b_n) = k \cdot b_k$
- Expected online profit at step $t =$ $\frac{1}{t} \cdot$ expected offline profit of the first t bids
- With probability $\binom{t}{m}\binom{n-t}{k-m}/\binom{n}{k}$ the first t bids have m of the high k bids
- offline profit $\geq \frac{1}{\rho} \cdot m \cdot b_k$, **when** $m \geq 2$
- Putting everything together

$$\text{online profit} \geq \sum_{t=2}^{n} \sum_{m=2}^{\min\{t,k\}} \frac{\binom{t}{m}\binom{n-t}{k-m}}{\binom{n}{k}} \cdot \frac{1}{t\rho} \cdot m b_k$$

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Consequences

Theorem

The online competitive ratio is between 4 and 6.48

Why?

- The lower bound comes from specific cases: 2 distinct bids or $b = (2+\epsilon, 2-\epsilon, 1)$
- For the upper bound, take the offline auction of Hartline-McGrew with competitive ratio 3.24 and transform it into an online auction

Conjecture

The online competitive ratio is 4. Stronger: BPSF has competitive ratio 4.

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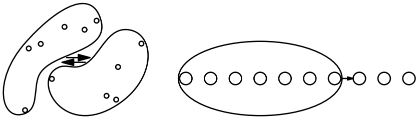
RSOP and BPSF



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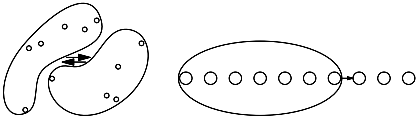
- Define $y(S) = \max\{1 \cdot b_{j_1}, 2 \cdot b_{j_2}, \ldots, r \cdot b_{j_r}\}$ the optimal single price profit of S
- Define $z(S)$ the profit from offering the optimal single price of S to the other side
- $z(S) = (j_i - i)b_{j_i}$, where $i = \operatorname{argmax} y(S)$

$$\text{RSOP} = \sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot 2^{-(n-1)}$$

$$\text{BPSF} = \sum_{S \subseteq \{b_2, \ldots, b_n\}} z(S) \cdot \binom{n-1}{|S|}^{-1} \cdot n^{-1}$$

RSOP and BPSF



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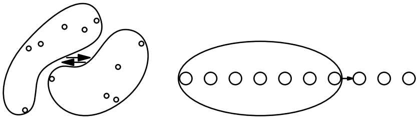
RSOP and BPSF



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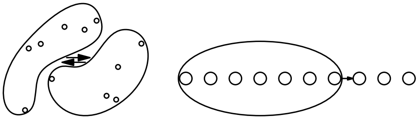

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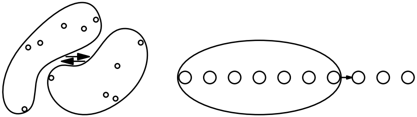

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Conjectures

Conjecture

RSOP is 4-competitive. Equivalently, for every set of bids b:

$$\sum_{S \subseteq \{b_2,\ldots,b_n\}} z(S) \cdot 2^{-(n-1)} \geq y(b_2, b_2, b_3, \ldots, b_n)$$

Conjecture

BPSF is 4-competitive. Equivalently, for every set of bids b:

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BPSF is 4-competitive. Equivalently, for every set of bids b:

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A coupling argument

S	#of b_2's	#of b_3's	#of b_4's
$b_2b_3b_4$	b_2	$2b_3$	$3b_4$
b_2b_3	b_2	$2b_3$	-
b_2b_4	b_2	-	$2b_4$
b_2	b_2	-	-
b_3b_4	-	b_3	$2b_4$
b_3	-	b_3	-
b_4	-	-	b_4
/	-	-	-
Σ # ib_i's	$2*2b_2$	$2*3b_3$	$2*4b_4$

$$\sum_{\substack{S \in \{b_2,\dots,b_n\} \\ b_2 \in S}} y(S) \geq$$

$$\sum_{\substack{S \in \{b_2,\dots,b_n\} \\ b_i \in S}} y(S) =$$

$$2^{n-i} \sum_{j=0}^{i-2} \binom{i-2}{j} \cdot (j+1) \cdot b_i =$$

$$2^{n-3} \cdot i \cdot b_i$$

Lemma

$$\sum_{\substack{S \in \{b_2,\dots,b_n\} \\ b_2 \in S}} y(S) \geq 2^{n-3} \cdot F^{(2)}$$

Relations between z and y

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$$\sum_{S \in \{b_2, \dots, b_n\}} z(S) \geq \sum_{\substack{S \in \{b_2, \dots, b_n\} \\ b_2 \in S}} y(S)$$

This will show that RSOP is 4-competitive

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$$\sum_{S \in \{b_3, \dots, b_n\}} z(S) \geq \sum_{S \in \{b_3, \dots, b_n\}} y(S)$$

The second conjecture implies the first because

$$z(b_{j_1}, \dots, b_{j_r}) \geq y(b_{j_1}, \dots, b_{j_r}) - y(b_{j_2}, \dots, b_{j_r})$$

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Open problems

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- Prove or disprove that the worst-case distribution is bid-independent
- Prove that BPSF is 4-competitive
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Thank you!