Compact formulae in sparse elimination

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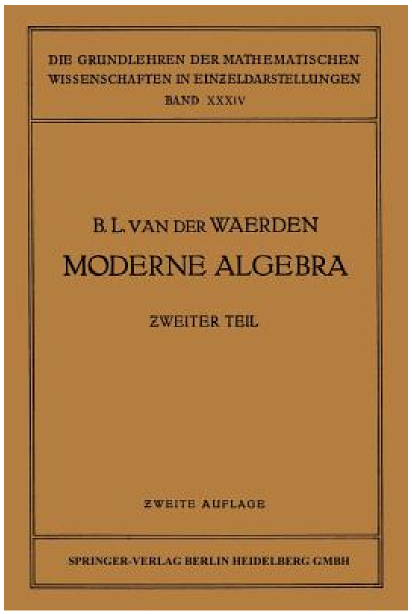
Context: Effective algebraic elimination



Polynomials and Power series,
May they forever rule the world!

Shreeram S. Abhyankar [Shankar]

Polynomials and Power series,
May they forever rule the world!
Eliminate, eliminate, eliminate!
Eliminate the eliminators of Elimination theory!

As you must resist the superbourbaki coup,
so must you fight the little bourbakis too!
Kronecker, Kronecker, Kronecker above all!
Kronecker, Mertens, Macaulay, and Sylvester!
Not the theology of Hilbert,
But the Constructions of Gordon!
Not the surface of Riemann,
But the Algorithm of Jacobi!
Ah! the beauty of the identity of Rogers and Ramanujan!
Can it be surpassed by Dirichlet and his principle?

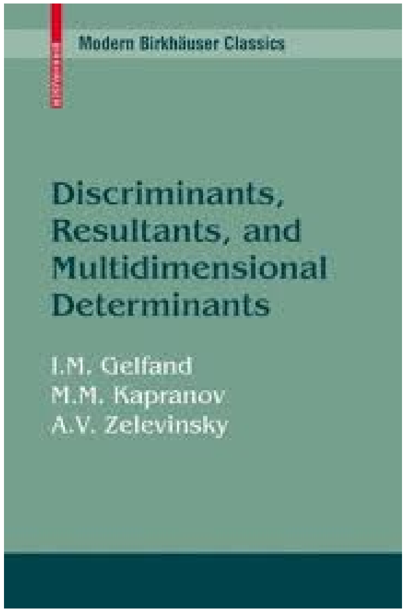
Germs, viruses, fungi, and functors,
Stacks and sheaves of the lot
Fear them not . . .
We shall be victors!
Come ye forward who dare present a functor,
We shall eliminate you!
By Resultants, Discriminants, Circulants, and Alternants!
Given to us by Kronecker, Mertens, Macaulay, and Sylvester!
Let not here enter the omologists, homologists,
And their cohorts the cohomologists crystalline
For this ground is sacred!
Onward Soldiers! defend your fortress!
Fight the Tor with a Determinant long and tall,
But shun the Ext above all!
Morphic injectives, toxic projectives,
Etal, eclat, devious devisage,
Arrows poisonous large and small!
May the armour of Tschirnhausen
Protect us from the scourge of them all!
You cannot conquer us with the rings of Chow
And shrieks of Chern!
For we too are armed, with Polygons of Newton
And Algorithms of Perron!
To arms, to arms, Fractions, continued or not,
Fear not the scheming ghost of Grothendieck!
For the power of Power series is with you!
May they converge or not
(May they be Polynomials or not)
(May they terminate or not)
Can the followers of G by mere ‘smooth’ talk
Ever make the singularity simple?
Long live Karl Weierstrass!
What need have we for rings Japanese, excellent or bad,
When, in person, Nagata himself is on our side?
What need to tensorise,
When you can Uniformise!
What need to homologise,
When you can Desingularise!
(Is Hironaka on our side?)
Alas! Princeton and fair Harvard you too,
Reduced to satellites in the Bur-Paris zoo!

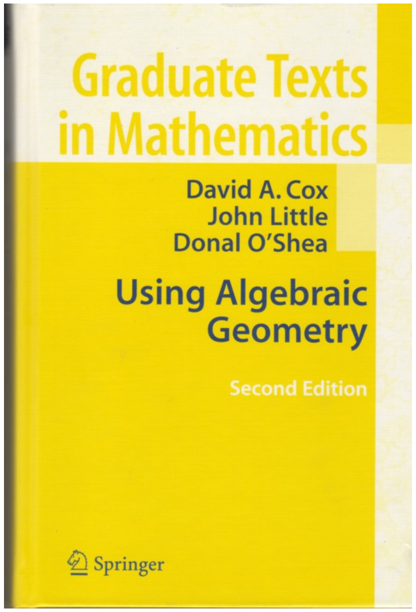
Topics

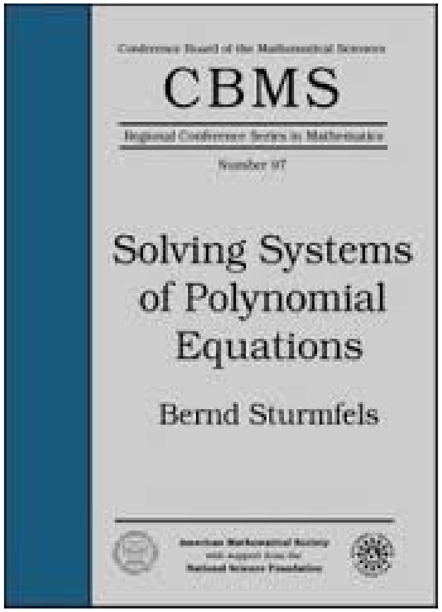
Sparse elimination theory

Sparse (toric) theory of algebraic elimination

- Revisit algebraic elimination with a combinatorial lens
- Fundamental results, opening up of new avenues
- A major contribution of computational algebra


Modern Birkhäuser Classics
Discriminants, Resultants, and Multidimensional Determinants
I.M. Gelfand
M.M. Kapranov
A.V. Zelevinsky


Graduate Texts in Mathematics
David A. Cox
John Little
Donal O'Shea
Using Algebraic Geometry
Second Edition
Springer


CBMS
Solving Systems of Polynomial Equations
Bernd Sturmfels

Newton polytopes

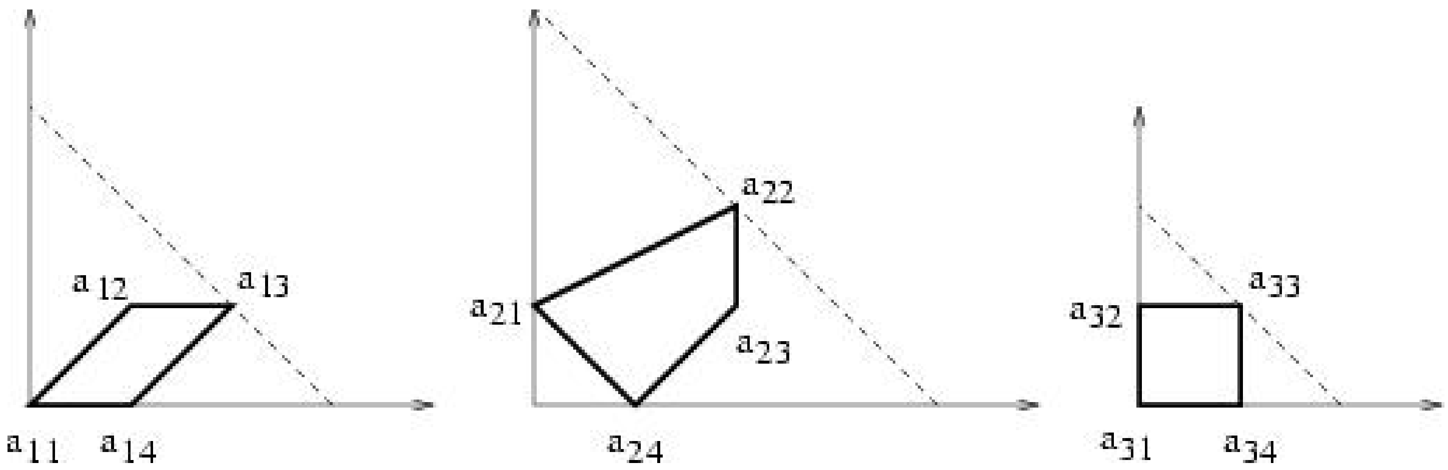
The support A_i of a polynomial $f_i \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, s.t.

$$f_i = \sum_j c_{ij} x^{a_{ij}}, \qquad c_{ij} \neq 0,$$

is defined as the set $A_i := \{a_{ij} \in \mathbb{Z}^n : c_{ij} \neq 0\}$.

The Newton polytope $Q_i \subset \mathbb{R}^n$ of f_i is the Convex Hull of all $a_{ij} \in A_i$.

Example:
$$\begin{aligned} f_1 &= c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x \\ f_2 &= c_{21}y + c_{22}x^2y^2 + c_{23}x^2y + c_{24}x + c_{25}xy \\ f_3 &= c_{31} + c_{32}y + c_{33}xy + c_{34}x \end{aligned}$$



Root bounds

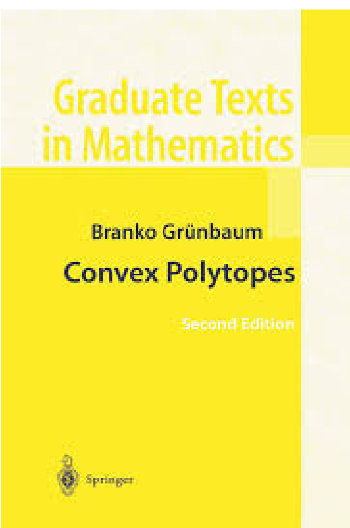
Mixed Volume definition

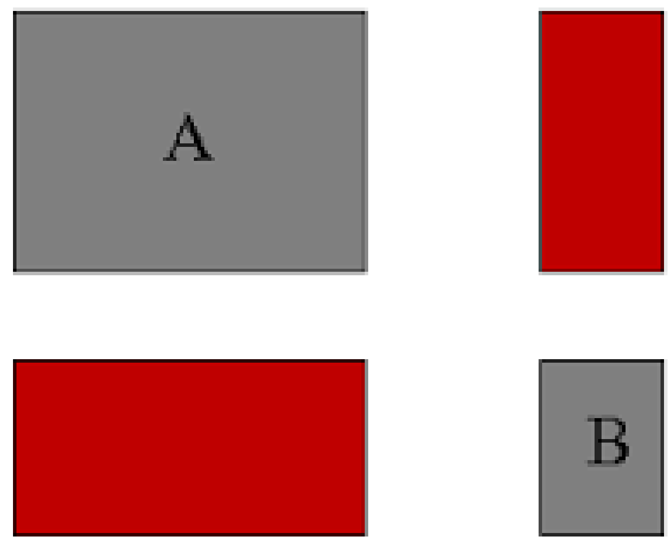
The mixed volume $\text{MV}(P_1, \ldots, P_n) \in \mathbb{R}$ of convex polytopes $P_i \subset \mathbb{R}^n$ is

- multilinear wrt Minkowski addition and scalar multiplication:

$$\text{MV}(P_1, \ldots, \lambda P_i + \mu P_i', \ldots, P_n) =$$
$$= \lambda \text{MV}(P_1, \ldots, P_i, \ldots, P_n) + \mu \text{MV}(P_1, \ldots, P_i', \ldots, P_n), \qquad \lambda, \mu \in \mathbb{R},$$

- st. $\text{MV}(P_1, \ldots, P_1) = n! \text{ vol}(P_1)$,
- or, equivalently, the coefficient of multilinear term $\text{MV}(P_1, \ldots, P_n)\, \lambda_1 \cdots \lambda_n$ in the polynomial $\text{vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ with scalar variables $\lambda_1, \ldots, \lambda_n$.





Mixed Volume characterization

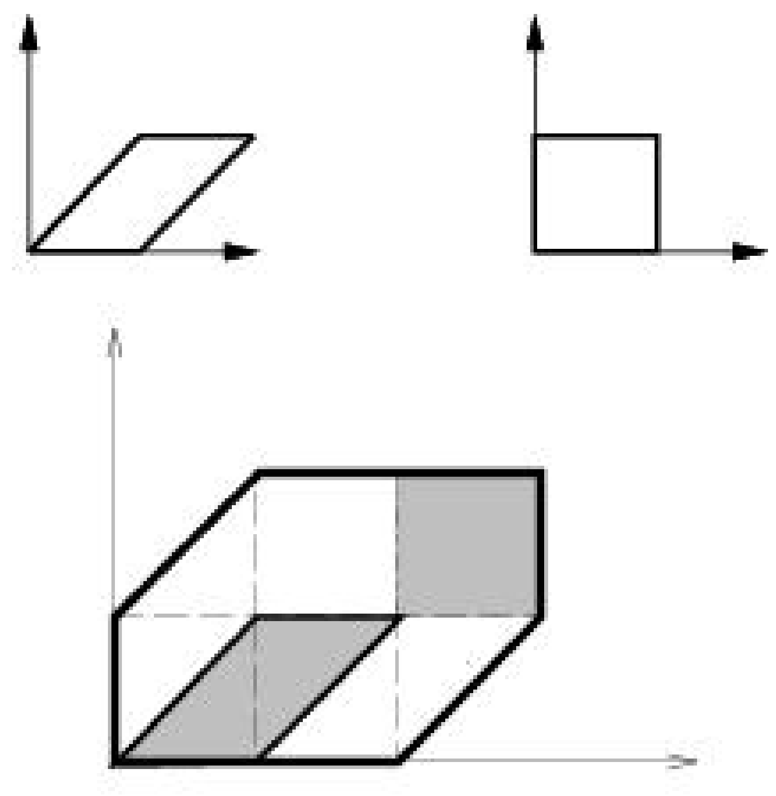
Property	MV: vtx$(Q_i) \subset \mathbb{Z}^n$	Generic number of isolated solutions
$\in \mathbb{Z}_{\geq 0}$	$\mathrm{MV}(\ldots, Q_i, \ldots)$	$\#\{x \in (\overline{K}^*)^n \mid \cdots = f_i(x) = \cdots = 0\}$
Invariance by permutation	$\mathrm{MV}(\ldots, Q_j, \ldots, Q_i, \ldots) =$ $= \mathrm{MV}(\ldots, Q_i, \ldots, Q_j, \ldots)$	$\#\{x \mid \cdots = f_j(x) = \cdots = f_i(x) = \cdots = 0\} =$ $= \#\{x \mid \cdots = f_i(x) = \cdots = f_j(x) = \cdots = 0\}$
Linearity wrt Minkowski addition	$\mathrm{MV}(\ldots, Q_i + Q_i', \ldots) =$ $= \mathrm{MV}(\ldots, Q_i, \ldots) +$ $+ \mathrm{MV}(\ldots, Q_i', \ldots)$	$\#\{x \mid \cdots = (f_i f_i')(x) = \cdots = 0\} =$ $= \#\{x \mid \cdots = f_i(x) = \cdots = 0\} +$ $+ \#\{x \mid \cdots = f_i'(x) = \cdots = 0\}$
Linearity wrt scalar product	$\mathrm{MV}(\ldots, \lambda Q_i, \ldots) =$ $= \lambda\ \mathrm{MV}(\ldots, Q_i, \ldots)$	$\#\{x \mid \cdots = (f_i(x))^\lambda = \cdots = 0\} =$ $= \lambda\ \#\{x \mid \cdots = f_i(x) = \cdots = 0\}$
Monotone wrt volume	$\mathrm{MV}(\ldots, Q_i \cup \{a\}, \ldots) \geq$ $\geq \mathrm{MV}(\ldots, Q_i, \ldots)$	$\#\{x \mid \cdots = f_i(x) + cx^a = \cdots = 0\} \geq$ $\geq \#\{x \mid \cdots = f_i(x) = \cdots = 0\}$
[Kushnirenko]	$\mathrm{MV}(Q_1, \ldots, Q_1) = n! V(Q_1)$	$\#\{x \mid f_1(x) = \cdots = f_n(x) = 0\} = n! V(Q_1)$

Bernstein (BKK) bound

Theorem [Bernstein'75,Kushnirenko'75,Khovanskii'78] [Danilov'78]:
Given polynomials $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, with Newton polytopes $Q_1, \ldots, Q_n$, the number of common isolated zeros, multiplicities counted, in the corresponding toric variety, which contains $(\mathbb{C}^*)^n$ as a dense subset, is bounded by the mixed volume $\mathrm{MV}(Q_1, \ldots, Q_n)$, irrespective of the toric variety's dimension.

Dense homogeneous: $\mathrm{MV}(Q_1, \ldots, Q_n) = \prod_{i=1}^n d_i$ = Bézout's bound, where $d_i = \deg(f_i)$ and $Q_i = \mathrm{simplex}\{0, (d_i, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_i)\}$.

Example: Mixed volume computation



Given $f_1 = c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x$, $f_3 = c_{31} + c_{32}y + c_{33}xy + c_{34}x$,

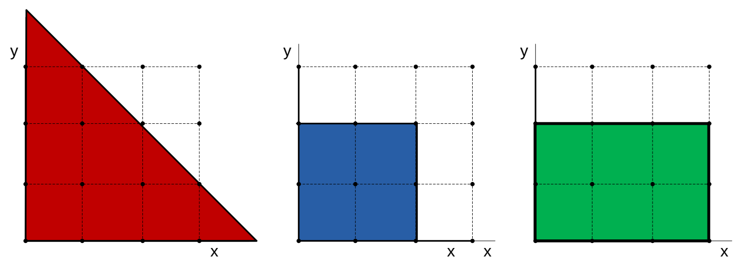
- construct their Newton polytopes in $\mathbb{R}^2$,
- compute a mixed subdivision of the Minkowski Sum (3 mixed cells),
- compute the Mixed Volume using the formula MV$= \sum_\sigma V(\sigma)$, over all mixed cells σ of the mixed subdivision (here MV=3).

Multihomogeneous polynomials

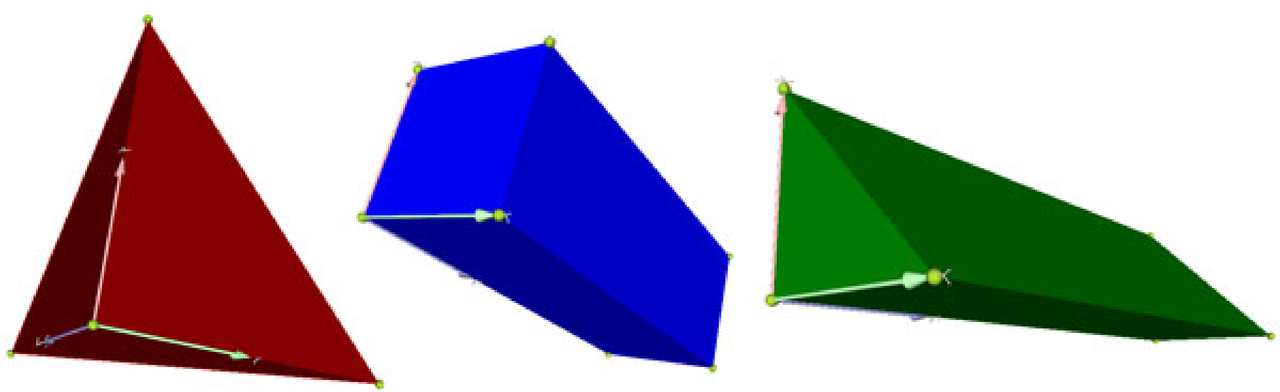
Partition variables into r subsets: the polynomial is homogeneous in each subset. The i-th subset has $n_i + 1$ homogeneous variables, of total degree d_i: type $(n_1, \ldots, n_r; d_1, \ldots, d_r)$.

Type $(2, 1; 2, 1)$, $(x_0 : x_1 : x_2,\ y_0 : y_1) \in \mathbb{P}^2 \times \mathbb{P}^1 : c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2 + c_4 x_1^2 + c_5 x_2^2 + c_6 y_1 + c_7 x_1 y_1 + c_8 x_2 y_1 + c_9 x_1 x_2 y_1 + c_{10} x_1^2 y_1 + c_{11} x_2^2 y_1$.

$n = 2$:



$n = 3$



Application: Game theory

A game is decribed by each player's (pure) strategies and their payoff for each combination of strategies.

Example: Prisoner's dilemma. Payoff matrix of first player:

	2nd tells	2nd not
1st tells	-5	-10
1st not	0	-2

Each player has two pure strategies.

Expected payoff

Assume r players, where the ℓ-th player has m_ℓ pure strategies.

The expexted payoff of player j choosing strategy k_j is

$$\sum_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_r} c^{(j)}_{k_1,\ldots,k_j,\ldots,k_r} \cdot p^{(1)}_{k_1} \cdots p^{(j-1)}_{k_{j-1}} p^{(j+1)}_{k_{j+1}} \cdots p^{(r)}_{k_r},$$

where $c^{(j)}_{k_1,\ldots,k_j,\ldots,k_r}$ is the specific payoff when player i opts for pure strategy $k_i \in \{1,\ldots,m_i\}$ with probability $p^{(i)}_{k_i}$.

These are multilinear polynomials in the unknown probabilities partitioned in r subsets, and miss $p^{(j)}_{k_j}$ for all k_j.

(Multihomogeneous) m-Bézout bound

Consider a system of n equations in n affine variables, partitioned into r subsets so that the j-th subset includes n_j variables: $n = n_1 + \cdots + n_r$. Let d_{ij} be the degree of the i-th equation in the j-th variable subset.

Theorem. The coefficient of $y_1^{n_1} \cdots y_r^{n_r}$ in

$$\prod_{i=1}^{n} (d_{i1} y_1 + \cdots + d_{ir} y_r)$$

bounds the number of isolated complex roots in

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}.$$

For generic coefficients this bound is tight.

For dense systems, it equals the mixed volume.

Permanent

The permanent of square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{bmatrix}$$

is defined similarly to the determinant but without the sign multiplier:

$$\sum_{\pi \in S_r} \prod_{i=1}^{r} a_{i\pi(i)}.$$

The permanent equals the coefficient of $x_1 x_2 \cdots x_r$ in

$$\prod_{i=1}^{r} (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ir}x_r).$$

Products of simplices

For a system of $n = n_1 + \cdots + n_r$ equations on

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r},$$

assume the i-th equation has degree a_{ij} in the j-th variable block:

$$Q_i = a_{i1}\Delta_1 \times \cdots \times a_{ir}\Delta_r.$$

Theorem [McLennan'99] $A = [a_{ij}]$ is an $n \times n$ matrix with column $[a_{1j}, \ldots, a_{nj}]^T$ repeated n_j times, $j \in \{1, \ldots, r\}$. The mixed volume equals

$$\frac{1}{n_1! \cdots n_r!} \text{ perm } A.$$

Remark. MV is #P-complete, harder to approximate than *perm*.

Nash equilibria

Definition. A Nash equilibrium is a combination of players' strategies where no player improves her payoff by unilaterally changing her strategy

A Totally mixed Nash equilibrium (TMNE) occurs when every player plays all strategies with positive probability.
Then, for each player, all payoffs are equal.

A Nash equilibrium always exist, but their computation is hard.
Enumerative question: How many Nash equilibria exist?
Known, for 2 players with $\lesssim 6$ strategies [von Steghel] [Vidunas'14].
Little is known for more than two players.

Totally Mixed Nash Equilibria

Let $E(n_1, \ldots, n_r)$ denote the number of TMNE's where the i-th player has $n_i + 1$ pure strategies. In the formula above, A contains 0's and 1's

- $E(n, n) = 1, \quad E(n_1, n_2) = 0$, if $n_1 \neq n_2$;
- $E(n_1, n_2, \ldots, n_r) = 0$, for $n_1 > n_2 + \cdots + n_r$;
- $E(n_1, n_2, \ldots, n_r) = \dfrac{n_1!}{n_2! \cdots n_r!}$, for $n_1 = n_2 + \cdots + n_r$;
- $E(1, 1, \ldots, 1) = r! \displaystyle\sum_{i=0}^{r} \frac{(-1)^i}{i!}$;
- $E(n, n, n) = \displaystyle\sum_{i=0}^{n} \binom{n}{i}^3$: the Franel numbers.

Arbitratry direct products

Consider a system of n equations, where the i-th Newton polytope is the direct product

$$Q_i = a_{i1}\Gamma_1 \times \cdots \times a_{ir}\Gamma_r, \quad 1 \le i \le n,$$

where $a_{ij} \in \mathbb{R}$, and $\Gamma_j \subset \mathbb{R}^{n_j}$ is a n_j-dimensional polytope in a separate complementary subspace of $\mathbb{R}^n$, $n = n_1 + \cdots + n_r$.

Theorem. [E-Vidunas'14] Let matrix $A = [a_{ij}]$ have column $[a_{1j}, \ldots, a_{nj}]^T$ repeated n_j times. Then,

$$\mathrm{MV}(Q_1, \ldots, Q_n) = \operatorname{perm} A \prod_{j=1}^{r} \mathrm{vol}(\Gamma_j).$$

Generating functions

Let $x_1, \ldots, x_r$ be formal variables, and let V be the diagonal matrix $\text{diag}(x_1, \ldots, x_r)$.

Theorem. [MacMahon's Master Theorem] Given a complex $r \times r$ matrix $A = (a_{ij})$, the coefficient of $x_1^{n_1} \cdots x_r^{n_r}$ in

$$\prod_{j=1}^{r} (a_{j1}x_1 + \cdots + a_{jr}x_r)^{n_j}$$

is the coefficient of $x_1^{n_1} \cdots x_r^{n_r}$ in the multivariate Taylor expansion of

$$1/\det(I - VA)$$

around $(x_1, x_2, \ldots, x_r) = (0, 0, \ldots, 0)$.

Semi-mixed homogeneous systems

Consider a system of n equations on

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r},$$

partitioned into r subsets of $n_1, \ldots, n_r$ equations. Assume the equations in the i-th subset have degree a_{ij} in the j-th variable block.

Theorem. [E-Vidunas'14] The system's m-Bézout bound equals the coefficient of $x_1^{n_1} \cdots x_r^{n_r}$ in the multivariate Taylor expansion of

$$1/\det(I - VA)$$

around $(x_1, \ldots, x_r) = (0, \ldots, 0)$, where A is the $r \times r$ matrix of the a_{ij}'s.

Corollary. There exists a polynomial-time algorithm to compute $E(n_1, \ldots, n_r)$ by using Laguerre polynomials [Gillis].

Resultants

Resultant definition

Given $n+1$ Laurent polynomials $f_0, \ldots, f_n \in K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ with indeterminate coefficients $\vec{c}$, their projective, resp. toric / sparse, *resultant* is the unique (up to sign) irreducible polynomial $R(\vec{c}) \in \mathbb{Z}[\vec{c}]$ such that

$$R(\vec{c}) = 0 \Leftrightarrow \exists\, \xi = (\xi_1, \ldots, \xi_n) \in X \,:\, f_0(\xi) = \cdots = f_n(\xi) = 0$$

where the variety X equals:

- the projective space $\mathbb{P}^n$ over the algebraic closure $\overline{K}$,
- resp. the toric variety X, $(\overline{K}^*)^n \subset X \subset \mathbb{P}^N$.

[van der Waerden, Gelfand-Kapranov-Zelevinsky, Cox-Little-O'Shea]

Resultant degree

The projective, resp. toric, resultant polynomial $R \in \mathbb{Z}[\vec{c}]$ is separately homogeneous in the coefficients of each f_i, with *degree* equal to $\prod_{j \neq i} \deg f_j$ (Bézout's number), resp. the n-fold mixed volume:

$$\mathrm{MV}_{-i} := \mathrm{MV}(f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n),$$

provided the supports of the f_i generate $\mathbb{Z}^n$.

Generalizations

The toric resultant reduces to:

- the determinant of the coefficient matrix of a *linear* system,
- the Sylvester or Bézout determinant of 2 *univariate* polynomials,
- the projective resultant for $n+1$ *dense* polynomials, where the toric variety equals $\mathbb{P}^n$ and $\mathrm{MV}_{-i} = \prod_{j \neq i} \deg f_j$.

Matrix formulae

- Resultant matrix: The resultant divides the determinant.
- Rational, Macaulay-type formula: The resultant equals the ratio of two determinants.
- Determinantal (optimal) formula: the resultant equals a determinant
- Polynomial formula: A power of the resultant equals the determinant, Pfaffian when $R = \sqrt{\det M}$.

- Poisson formula.
- Determinantal from rational formula [Kaltofen-Koiran'08]

- Matrix formulae allow system solving by: an eigenproblem, u-resultant, primitive/separating element (RUR).

Resultant matrices

- $n = 1$: Bézout 1779, Sylvester 1840.

- Bézout: [Chtcherba-Kapur'00], [Kapur et.al], [Cardinal-Mourrain], [Busé et al.].

- Homogeneous: Macaulay, [GKZ'94], [Jouanolou'97], [D'Andrea-Dickenstein'01], [CoxMatera08], complexes [Eisenbud-Schreyer'03].

- Toric: [Canny-E'93], [E-Canny'93]*, generalized [Sturmfels'94], Jacobian [Cattani-Dickenstein-Sturmfels], [D'Andrea-E'01], complexes [Khetan'02], rational [D'Andrea'02], [E-Konaxis'09].

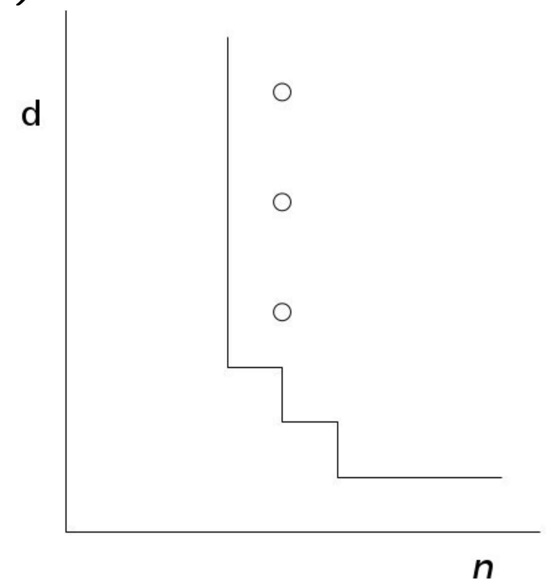
- m-homogeneous: Dixon, [GKZ], [Chionh-Goldman-Zhang98,ZG00], [Dickenstein-E'03, E-Mantzaflaris'09], [Awane-Chkiriba-Goze'05].

* My first ISSAC paper in Kiev 

Homogeneous unmixed systems

State of the art. Given $n+1$ dense polynomials in n variables of total degree d, an optimal (Sylvester, Bézout or polynomial) formula is known for

- $d = 1$
- $n \leq 3$
- $n = 4$, $d \leq 3$, $d = 4, 6, 8$
- $n = 5$, $d = 2$



- [D'Andrea-Dickenstein'01] [Khetan'02]
- Koszul-Weyman complexes [GKZ'94]
- Chow complex [Eisenbud-Schreyer'03]

Polynomials of arbitrary support

Matrices of Sylvester-type

Algorithms: subdivision-based [Canny-E'93,'00], incremental [E-Canny'95] yield a square matrix M of the sparse/toric resultant, such that:

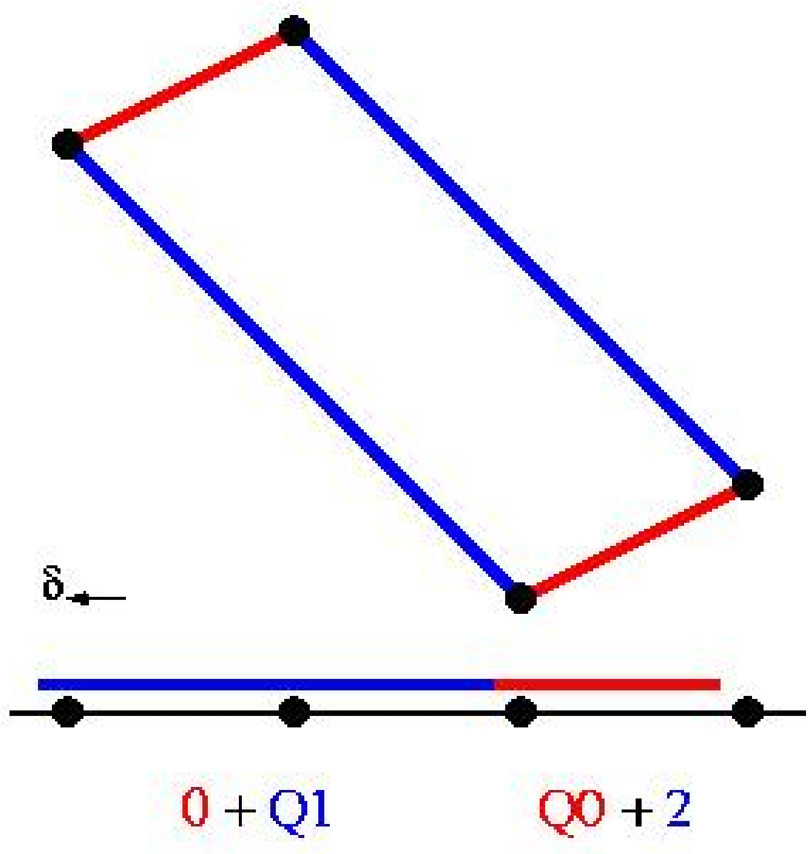
$$\begin{aligned} \det(M) &\not\equiv 0, \\ R &\mid \det(M), \\ \deg_{f_0} \det(M) &= \deg_{f_0} R, \end{aligned}$$

where R is the toric resultant.

Rational form [D'Andrea'02]: $R = \det(M)/\det(M')$,
where M' is a submatrix of M, generalizing Macaulay's construction.

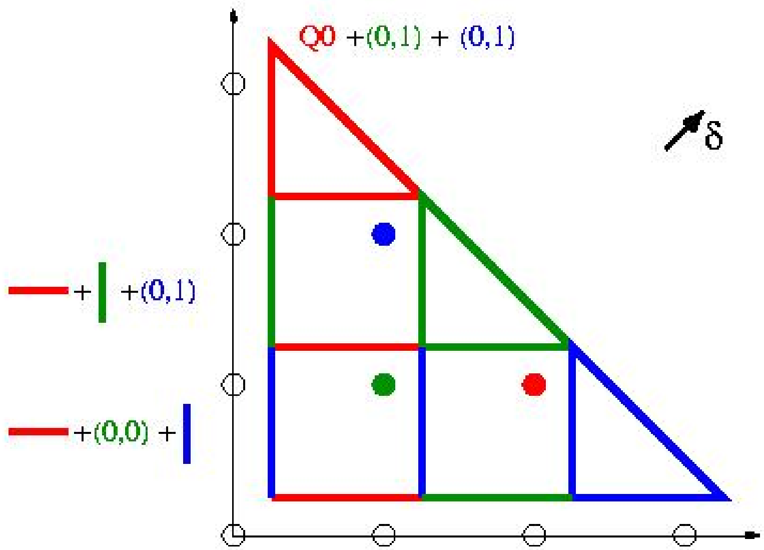
Lifting in the Sylvester case

$$f_0 = c_{00} + c_{01}x, \; f_1 = c_{10} + c_{11}x + c_{12}x^2$$



RC(2) = (1; 2) ie. $x^2 \mapsto x^{2-2} f_1$.

Mixed subdivision of a linear system



RC(1, 2) = [2, (0, 1)] ie. $x_1 x_2^2 \mapsto x^{(1,2)-(0,1)} f_2 = x^{(1,1)} f_2$
RC(1, 1) = [1, (0, 0)] ie. $x_1 x_2 \mapsto x^{(1,1)-(0,0)} f_1 = x^{(1,1)} f_1$
RC(2, 1) = [0, (1, 0)] ie. $x_1^2 x_2 \mapsto x^{(2,1)-(1,0)} f_0 = x^{(1,1)} f_0$

$$
\begin{array}{cc}
 & \begin{array}{ccc} x_1^2 x_2 & x_1 x_2^2 & x_1 x_2 \end{array} \\
M = & \begin{bmatrix} c_{01} & c_{02} & c_{03} \\ c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}
\end{array}
\qquad
\begin{array}{c} x_1 x_2 f_0 \\ x_1 x_2 f_1 \\ x_1 x_2 f_2 \end{array}
$$

Example: subdivision-based matrix

$$
\begin{aligned}
f_1 &= c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x, \\
f_2 &= c_{21}y + c_{22}x^2y^2 + c_{23}x^2y + c_{24}x, \\
f_3 &= c_{31} + c_{32}y + c_{33}xy + c_{34}x.
\end{aligned}
$$

	1, 0	2, 0	0, 1	1, 1	2, 1	3, 1	0, 2	1, 2	2, 2	3, 2	4, 2	1, 3	2, 3	3, 3	4
1, 0)x	c_{11}	c_{14}	0	0	c_{12}	c_{13}	0	0	0	0	0	0	0	0	0
2, 0)x	c_{31}	c_{34}	0	c_{32}	c_{33}	0	0	0	0	0	0	0	0	0	0
0, 1)y	0	0	c_{11}	c_{14}	0	0	0	c_{12}	c_{13}	0	0	0	0	0	0
1, 1)xy	0	0	0	c_{11}	c_{14}	0	0	0	c_{12}	c_{13}	0	0	0	0	0
2, 1)	c_{24}	0	c_{21}	0	c_{23}	0	0	0	c_{22}	0	0	0	0	0	0
3, 1)x	0	c_{24}	0	c_{21}	0	c_{23}	0	0	0	c_{22}	0	0	0	0	0
0, 2)y	0	0	c_{31}	c_{34}	0	0	c_{32}	c_{33}	0	0	0	0	0	0	0
1, 2)xy	0	0	0	c_{31}	c_{34}	0	0	c_{32}	c_{33}	0	0	0	0	0	0
2, 2)x^2y^2	0	0	0	0	0	0	0	0	c_{11}	c_{14}	0	0	0	c_{12}	c
3, 2)x^2y	0	0	0	0	c_{31}	c_{34}	0	0	c_{32}	c_{33}	0	0	0	0	0
4, 2)x^2y	0	0	0	0	0	c_{24}	0	0	c_{21}	0	c_{23}	0	0	0	c
1, 3)xy^2	0	0	0	0	0	0	0	c_{31}	c_{34}	0	0	c_{32}	c_{33}	0	0
2, 3)y	0	0	0	c_{24}	0	0	c_{21}	0	c_{23}	0	0	0	c_{22}	0	0
3, 3)x^2y^2	0	0	0	0	0	0	0	0	c_{31}	c_{34}	0	0	c_{32}	c_{33}	0
4, 3)x^3y^2	0	0	0	0	0	0	0	0	0	c_{31}	c_{34}	0	0	c_{32}	c

dim $M = 15$, greedy [Canny-Pedersen]: 14, incremental [E-Canny]: 12.

Mixed volumes $= 4, 3, 4 \Rightarrow$ deg $R_{tor} = 11$ while deg(classical resultant) $= 26$.

Rational form

Recursive lifting on n, using the subdivision algorithm [D'Andrea'01].

Bilinear: $f_i = a_i + b_i x_1 + c_i x_2 + d_i x_1 x_2$, $i = 0, 1, 2$.
Linear lift $(-\infty, \ldots), (0, 1, 1, 2), (0, 0, 7, 7)$, $\delta = (\frac{2}{3}, \frac{1}{2}) \Rightarrow \dim M = 16$ (numerator):

$$M = \begin{pmatrix}
a_1 & b_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_0 & b_0 & 0 & c_0 & d_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_1 & b_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_0 & 0 & 0 & c_0 & d_0 & b_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_2 & b_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & b_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & c_1 & d_1 & b_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & c_2 & d_2 & b_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_2 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 & d_2 & b_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & c_2 & d_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & c_2 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & c_2 & d_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & c_1 & d_1
\end{pmatrix}$$

Rational form: denominator

$$M' = \begin{pmatrix} a_1 & 0 & c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & c_2 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & c_1 & d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_2 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & 0 & c_2 & d_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & c_2 \end{pmatrix}$$

$\det(M) = \pm R \cdot \det(M')$: M' is a submatrix of M,

$|M'| = -c_2^3(-c_1a_2 + a_1c_2)b_2(c_1d_2 - d_1c_2)(-b_2c_1 + b_1c_2)$

Main step: lifting of some $b \in Q_0$ is very negative.
The mixed subdivision provides all info.

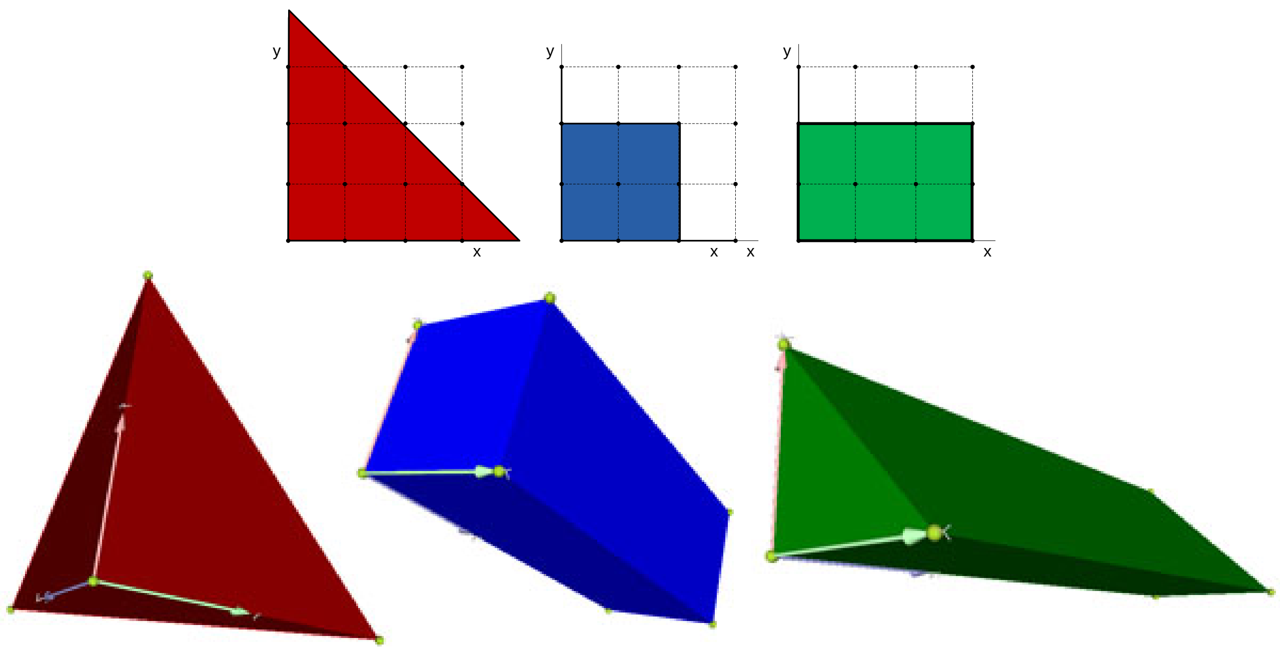
Open: $\exists$ single lifting yielding both numerator and denominator?
YES if $n = 2$, unmixed system, or sufficiently different Newton polytopes
[E-Konaxis'11]

Multi-homogeneous resultant

Unmixed (multi)homogeneous systems

Partition variables into r subsets: every polynomial is homogeneous in each subset. The i-th subset has $l_i + 1$ homogeneous variables, of total degree d_i: type $(l_1, \ldots, l_r; d_1, \ldots, d_r)$.

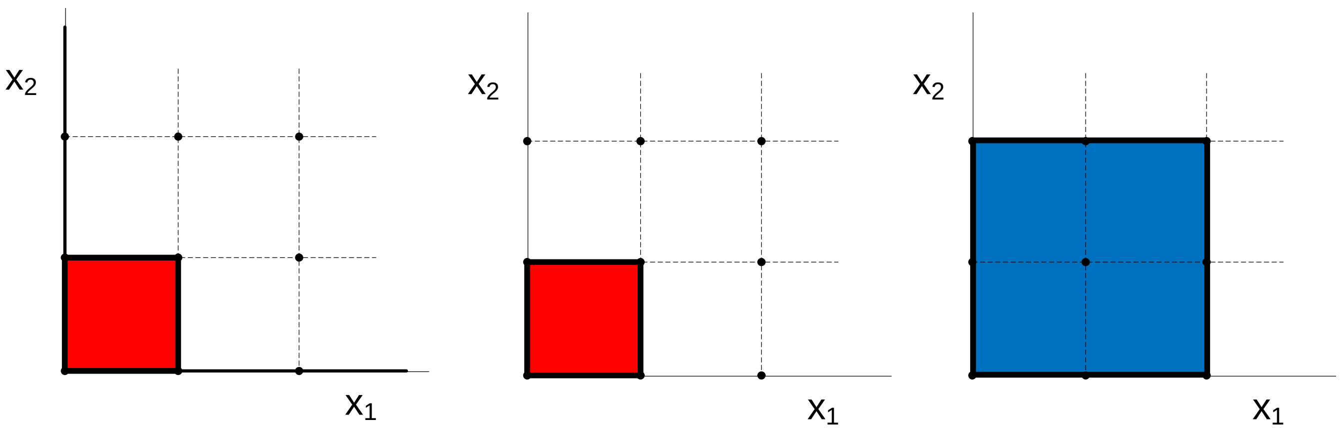
Type $(2, 1; 2, 1)$, $(x_0 : x_1 : x_2, y_0 : y_1) \in \mathbb{P}^2 \times \mathbb{P}^1 : c_0 + c_1x_1 + c_2x_2 + c_3x_1x_2 + c_4x_1^2 + c_5x_2^2 + c_6y_1 + c_7x_1y_1 + c_8x_2y_1 + c_9x_1x_2y_1 + c_{10}x_1^2y_1 + c_{11}x_2^2y_1$.



Scaled (multi)homogeneous systems

Scaled case: $\boxed{\deg f_i = s_i d \in \mathbb{N}^r}$

- Cardinalities $\ell = (\ell_1, \ldots, \ell_r) \in \mathbb{N}^r$
- Base degrees $d = (d_1, \ldots, d_r) \in \mathbb{N}^r$
- Scalars $s = (s_0, \ldots, s_n) \in \mathbb{N}^{n+1}$



Running example: $\ell = (1,1),\ d = (1,1),\ s = (1,1,2)$:

$$f_0 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2,$$
$$f_1 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2,$$
$$f_2 = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2 + c_4 {x_1}^2 + c_5 {x_1}^2 x_2 + c_6 {x_2}^2 + \\ + c_7 x_1 x_2^2 + c_8 x_1^2 x_2^2$$

Bilinear system: Sylvester-type matrix

$$\begin{aligned} f_0 &= a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2, \\ f_1 &= b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2, \\ f_2 &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2, \end{aligned}$$

of type $(1,1;1,1)$, $\deg R = 3 \cdot \deg_{f_i} R = 3\binom{2}{1,1} = 6$.

A determinantal pure Sylvester formula:

$$R = \det \begin{array}{c} \begin{matrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_1^2 x_2 \end{matrix} \\ \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ c_0 & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & a_0 & 0 & a_2 & a_1 & a_3 \\ 0 & b_0 & 0 & b_2 & b_1 & b_3 \\ 0 & c_0 & 0 & c_2 & c_1 & c_3 \end{bmatrix} \end{array} \quad \begin{matrix} f_0 \\ f_1 \\ f_2 \\ x_1 f_0 \\ x_1 f_1 \\ x_1 f_2 \end{matrix}$$

Bilinear system: Bézout-type and hybrid matrices

The Bezoutian polynomial

$$B = \det \begin{bmatrix} f_0(x_1,x_2) & f_0(y_1,x_2) & f_0(y_1,y_2) \\ f_1(x_1,x_2) & f_1(y_1,x_2) & f_1(y_1,y_2) \\ f_2(x_1,x_2) & f_2(y_1,x_2) & f_2(y_1,y_2) \end{bmatrix} \,/\, (x_1-y_1)(x_2-y_2),$$

supported by $\{1,x_2\},\{1,y_1\}$, yields a determinantal pure Bézout formula:

$$R = \det \begin{bmatrix} [123] & [023] \\ -[103] & [012] \end{bmatrix} \quad : \qquad [ijk] = \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix}$$

A hybrid determinantal formula [Cattani,Dickenstein,Sturmfels]:

$$\begin{matrix} 1 & x_1 & x_2 & x_1x_2 \end{matrix}$$

$$R = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ [012] & [013] & [032] & -[123] \end{bmatrix}.$$

Weyman complex

Definition. Given degree vector $m \in \mathbb{Z}^r$, the Weyman complex of modules $K_\nu(m)$, $\nu = n+1, \ldots, -n$, over systems of type (l, d) generalizes the Cayley-Koszul complex and depends only on l, d, m:

$$0 \to K_{n+1}(m) \to \cdots \to K_1(m) \to K_0(m) \to K_{-1}(m) \to \cdots \to K_{-n} \to 0$$

- Maps $\lambda_i : K_i \to K_{i-1}$ s.t. $\lambda_i \circ \lambda_{i-1} = 0$, i.e. $\mathrm{im}(\lambda_i) \subset \ker(\lambda_{i-1})$.

Exact complex iff equality.

- The determinant of the complex, for some maximal minors of λ_i, is:

$$\det(K_\nu) = \frac{\cdots \det(\lambda_3) \det(\lambda_1) \det(\lambda_{-1}) \cdots}{\cdots \det(\lambda_4) \det(\lambda_2) \det(\lambda_0) \cdots}$$

Determinantal formula

Theorem [Weyman-Zelevinsky'94, Sturmfels-Zelevinsky'94] A (hybrid) determinantal formula exists, for unmixed systems (scaling $s = 1$), iff all defects $\delta_k := l_k - \lceil \frac{l_k}{d_k} \rceil \leq 2,\ k = 1, \ldots, r$.

Theorem [Dickenstein-E] A determinantal formula of pure Sylvester type exists iff all defects vanish iff a determinantal formula of pure Bézout type exists.

Characterize all determinantal formulae by the following data: For any permutation $\pi : [1, r] \rightarrow [1, r]$,

$$m_k^\pi = \left(1 - \delta_k + \sum_{\pi(j) \geq \pi(k)} l_j\right) d_k - l_k \in \mathbb{Z}, \quad k = 1, \ldots, r.$$

Further results for non-determinantal formulae.

Example I: Hybrid determinantal formula

$$l = (3,2), d = (2,3)$$

then $\deg R = 6 \cdot \deg_{f_i} R = 6\binom{5}{3,2}2^3 3^2 = 4320.$

Found 30 determinantal m, $\min\{\dim M\} = 1320$, for $m = (6,3), (2,12)$.
For $m = (6,3)$:

$$K_2 = 0 \to K_1 \to K_0 \to K_{-1} = 0 \qquad \text{where}$$

$$K_1 = H^0(4,0)^{\binom{6}{1}} \oplus H^2(0,-6)^{\binom{6}{3}} \oplus H^5(-6,-15)^{\binom{6}{6}} \to$$
$$\to H^0(6,3)^{\binom{6}{0}} \oplus H^2(2,-3)^{\binom{6}{2}} \oplus H^5(-4,-12)^{\binom{6}{5}} = K_0.$$

Let $\phi_{\alpha,\beta} : H^\alpha \to H^\beta$.
Then $\phi_{02} = \phi_{05} = \phi_{25} = 0$, 3 pure Sylvester, 3 pure Bézout maps

Example I (cont'd): the hybrid matrix

$$M: \quad \begin{array}{c} \\ 210 \\ 220 \\ 910 \end{array} \begin{array}{c} \begin{array}{ccc} 840 & 150 & 330 \end{array} \\ \left[\begin{array}{ccc} \phi_{00} & 0 & 0 \\ \phi_{20} & \phi_{22} & 0 \\ \phi_{50} & \phi_{52} & \phi_{55} \end{array}\right] \end{array} = \left[\begin{array}{ccc} S_{00} & 0 & 0 \\ B_{20}^{x_2} & \phi_{22} & 0 \\ B_{50} & B_{52}^{x_1} & S_{55}^{T} \end{array}\right]$$

The partial Bézout-type formula for $B_{52}^{x_1}$ is composed of $\binom{6}{2}$ "smaller" Bézout-type formulae.

Each is defined by omitting $f_j,\ j \in J$, for all $J \subset \{0, \ldots, 5\}, |J| = 2$.

Only the x_{1j} are substituted by y_{1j}, yielding 4×4 matrices, which define the Bezoutian polynomials.

Determinantal formula for scaled systems

Theorem A determinantal formula exists in the scaled case ($s \neq 1$):

• [D'Andrea-Dickenstein'01] [Cox-Matera'08] for scaled homogeneous systems ($r = 1$ blocks), iff $s_2 + \cdots + s_n - n < s_0 + s_1$,

• [E-Mantzaflaris] for multi-homogeneous systems a determinantal pure-Sylvester formula exists iff $n = 1$ or $\ell = (1, 1)$.
No pure-Bézout formula exists.

Characterize all formulae:

$$n = 1 \Rightarrow m = d \sum_{0}^{n} s_i - 1, \text{ or } m = -1 \text{ (both classic Sylvester).}$$

$$\ell = (1, 1) \Rightarrow m = \left(-1,\ d_2 \sum_{i=0}^{2} s_i - 1\right), \text{ or } m = \left(d_1 \sum_{i=0}^{2} s_i - 1, -1\right).$$

Example II: Scaled determinantal Sylvester

$\ell = d = (1,1),$
$s = (1,1,2)$

$$f_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_1x_2$$
$$f_1 = b_0 + b_1x_1 + b_2x_2 + b_3x_1x_2$$
$$f_2 = c_0 + c_1x_1 + c_2x_2 + c_3x_1x_2 + c_4x_1^2 + \\ + c_5x_1^2x_2 + c_6x_2^2 + c_7x_1x_2^2 + c_8x_1^2x_2^2$$

$$m = (3,-1) \Rightarrow 0 \to H^1(1,-3) \oplus H^1(0,-4)^2 \to H^1(2,-2)^2 \oplus H^1(1,-3) \to 0.$$

$$[H^0(1) \otimes H^0(1)^*] \oplus [H^0(0) \otimes H^0(2)^*]^2 \to [H^0(2) \otimes H^0(0)^*]^2 \oplus [H^0(1) \otimes H^0(1)^*].$$

$$[g_0, g_1, g_2] \quad \begin{matrix} H^1(1,-3) \\ H^1(0,-4) \\ H^1(0,-4) \end{matrix} \overset{\begin{matrix} H^1(2,-2) & H^1(2,-2) & H^1(1,-3) \end{matrix}}{\begin{bmatrix} -M(f_1) & M(f_0) & 0 \\ -M(f_2) & 0 & M(f_0) \\ 0 & -M(f_2) & M(f_1) \end{bmatrix}}$$

$$= [-g_0f_1 - g_1f_2,\ g_0f_0 - g_2f_2,\ g_1f_0 + g_2f_1]$$

$$M(f_i) \ : \quad \mathrm{POLS}(u - \deg f_i) \ni g \quad \longmapsto \quad gf_i \in \mathrm{POLS}(u)$$

Example II (cont'd): determinantal Sylvester

$\ell = (1,1),$
$d = (1,1),$
$s = (1,1,2)$

$$f_0 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2$$
$$f_1 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2$$
$$f_2 = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2 + c_4 {x_1}^2 + c_5 {x_1}^2 x_2 + \\ + c_6 x_2^2 + c_7 x_1 x_2^2 + c_8 x_1^2 x_2^2$$

$\deg R = 4 + 4 + 2 = 10 \Rightarrow$ optimal matrix:

$$R(f_0, f_1, f_2) = \det \begin{bmatrix} -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_8 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\ -c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ -c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\ 0 & 0 & 0 & -c_4 & -c_5 & -c_8 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2 \end{bmatrix}$$

Discriminant formula

Polynomial Discriminant

Given a support of m points $A \subset \mathbb{Z}^n$, let

$$f_A = \sum_{a \in A} c_a t^a, \quad a \in A,\ c_a \in \mathbb{R}^*,\ t := (t_1, \ldots, t_n).$$

Definition. [GKZ] The *A-discriminant (or sparse discriminant)* is the irreducible integer polynomial $\Delta_A = \Delta_A(c)$ in the coefficients

$$c = \{\, c_a : a \in A \,\},$$

defined up to sign, which vanishes for each choice of c for which f_A and all partial derivatives $\partial f_A / \partial t_i, i = 1, \ldots, n$, have a common root in $(\mathbb{C}^*)^n$.

Δ_A is homogenous.
Example: $A = \{0, 1, 2\} \subset \mathbb{Z}$, $\Delta_A = c_1^2 - 4c_0c_2$.

System discriminant

Definition. Given a well-constrained system of n polynomials F_i in n variables with real coefficients and fixed supports

$$A_1, \ldots, A_n \subset \mathbb{Z}^n,$$

their *mixed (sparse) discriminant* $\Delta(F_1, \ldots, F_n)$ is the irreducible integer polynomial (defined up to sign) in the coefficients of the F_i which vanishes whenever the system

$$F_1 = \cdots = F_n = 0$$

has a multiple root in $(\mathbb{C}^*)^n$, provided this discriminantal variety is a hypersurface; otherwise, $\Delta(F_1, \ldots, F_n) = 1$.

Discriminant formulae

- No matrix formulae known although closely related to the resultant.
- $E \cdot \Delta(f_1, \ldots, f_n) = R(f_1, \ldots, f_n, J)$.
- $R(f_0, \ldots, f_n) = \Delta(f_0 + f_1 y_1 + \cdots + f_n y_n)$: Cayley trick

Sparse multilinear systems

Bilinear systems over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$F_1 = a_0 y_1 z_1 + a_1 y_1 z_0 + a_2 y_0 z_1 + a_3 y_0 z_0,$$

$$F_2 = b_0 x_1 z_1 + b_1 x_1 z_0 + b_2 x_0 z_1 + b_3 x_0 z_0,$$

$$F_3 = c_0 x_1 y_1 + c_1 x_1 y_0 + c_2 x_0 y_1 + c_3 x_0 y_0.$$

The system's roots correspond to Nash equilibria of games with 3 players, 2 pure strategies each, and probabilities x_i, y_i, z_i.

Discriminant formula

Theorem. [E-Vidunas'15]

$$\Delta(F_1, F_2, F_3) = \det \begin{bmatrix} 0 & 0 & c_0 & c_1 & b_0 & b_1 \\ 0 & 0 & c_2 & c_3 & b_2 & b_3 \\ c_0 & c_2 & 0 & 0 & a_0 & a_1 \\ c_1 & c_3 & 0 & 0 & a_2 & a_3 \\ b_0 & b_2 & a_0 & a_2 & 0 & 0 \\ b_1 & b_3 & a_1 & a_3 & 0 & 0 \end{bmatrix}.$$

Properties

Generalization to sparse multihomogeneous systems over $\mathbb{P}^k \times \mathbb{P}^\ell \times \mathbb{P}^m$ when equation blocks of equal size.

Open. Full multilinear systems.

Intuition. The rows of the discriminant matrix above contain the coefficients in the partial derivatives wrt x_1, x_0, y_1, y_0 etc. In fact, its block structure roughly follows the pattern of matrix

$$\begin{bmatrix} 0 & \partial F_3/\partial x & \partial F_2/\partial x \\ \partial F_3/\partial y & 0 & \partial F_1/\partial y \\ \partial F_2/\partial z & \partial F_1/\partial z & 0 \end{bmatrix},$$

whose determinant equals the Jacobian!

Sketch of Proof

Lemma. Let $(x_1 : x_0), (y_1 : y_0), (z_1 : z_0)$ be a multiple root, and consider the transposed Jacobian matrix:

$$\begin{bmatrix} 0 & \partial F_2/\partial x_1 & \partial F_3/\partial x_1 \\ \partial F_1/\partial y_1 & 0 & \partial F_3/\partial y_1 \\ \partial F_1/\partial z_1 & \partial F_2/\partial z_1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 0.$$

It follows that the kernel of the discriminant matrix is

$$\left[\frac{x_1}{\lambda_1} : \frac{x_0}{\lambda_1} : \frac{y_1}{\lambda_2} : \frac{y_0}{\lambda_2} : \frac{z_1}{\lambda_3} : \frac{z_0}{\lambda_3}\right].$$

Then, the Euler identities $F_i = \sum_k v_k \partial F_i/\partial v_k$ vanish, for variables v_k.

Conclusions

Outlook

- Examine the open questions.

- Develop good software

(now just prototypes in C, Maple, Singular, Macaulay)

- Address the relevant applications.

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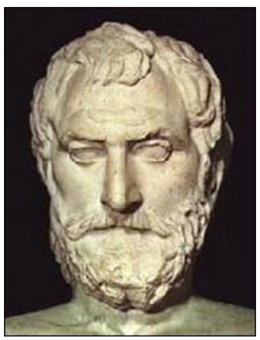
Algebraic Represenations for Computer-Aided Design of complEx Shapes (M.S.Curie ITN, H2020) *[open PhD positions]*

Cloudflow

CloudFlow (I4MS, ICT, FP7)



Advanced Geometric Computing (Thales Net, GR)

ESPRESSO: Exploiting Structure in Polynomial Equation & System Solving for geOmetric mOdeling & game theOry (Excellence grant, GR)

Thank you!