# The Newton polytope of implicit curves

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## Outline

# Toric elimination theory

# Newton polytopes

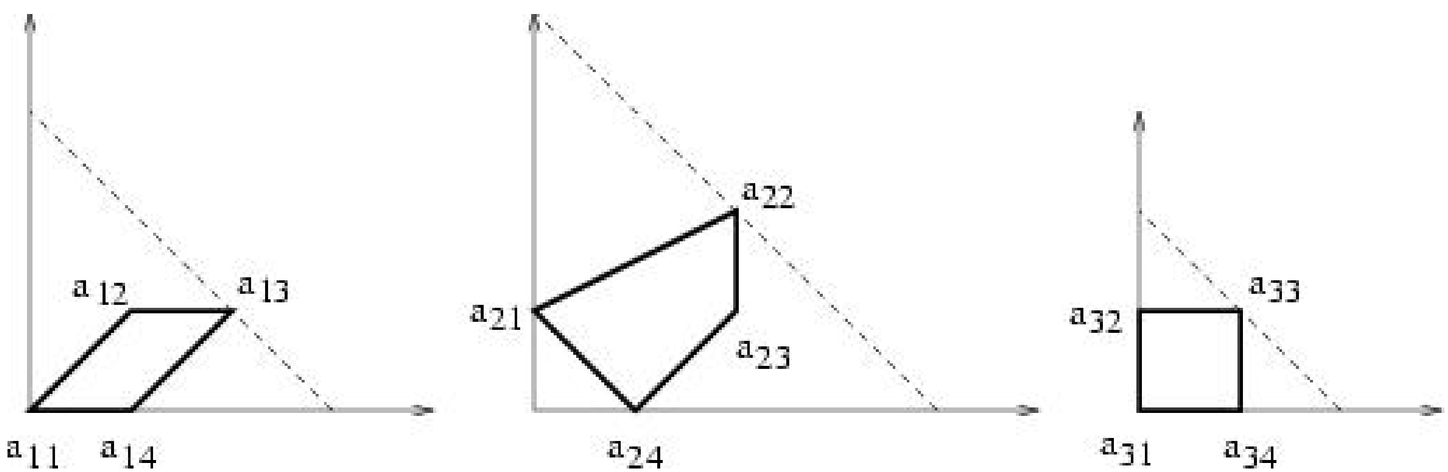
The support $A_i$ of a polynomial $f_i \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, s.t.

$$f_i = \sum_j c_{ij} x^{a_{ij}}, \qquad c_{ij} \neq 0,$$

is defined as the set $A_i := \{a_{ij} \in \mathbb{Z}^n : c_{ij} \neq 0\}$.
The Newton polytope $Q_i \subset \mathbb{R}^n$ of $f_i$ is the Convex Hull of all $a_{ij} \in A_i$.

Example:

$$\begin{aligned} f_1 &= c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x \\ f_2 &= c_{21}y + c_{22}x^2y^2 + c_{23}x^2y + c_{24}x + c_{25}xy \\ f_3 &= c_{31} + c_{32}y + c_{33}xy + c_{34}x \end{aligned}$$



## Mixed volume

1. The mixed volume $\mathrm{MV}(P_1,\ldots,P_n) \in \mathbb{R}$ of convex polytopes $P_i \subset \mathbb{R}^n$

- is multilinear wrt Minkowski addition and scalar multiplication:

$$\mathrm{MV}(P_1,\ldots,\lambda P_i + \mu P_i',\ldots,P_n) =$$
$$= \lambda \mathrm{MV}(P_1,\ldots,P_i,\ldots,P_n) + \mu \mathrm{MV}(P_1,\ldots,P_i',\ldots,P_n), \qquad \lambda,\mu \in \mathbb{R},$$

- st. $\mathrm{MV}(P_1,\ldots,P_1) = n!\ \mathrm{vol}(P_1)$.

2. Equivalently, $\mathrm{vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ is a polynomial in scalar variables $\lambda_1,\ldots,\lambda_n$, with multilinear term $\mathrm{MV}(P_1,\ldots,P_n)\ \lambda_1 \cdots \lambda_n$.

# Mixed subdivisions

## Regular subdivisions

For $Q_i \subset \mathbb{R}^n$, $$(Q_i)_{i\in I} \rightarrow Q = \sum_{i\in I} Q_i \; : \; (q_i)_{i\in I} \mapsto \sum_{i\in I} q_i.$$

Consider lifting functions $l_i : \mathbb{R}^n \rightarrow \mathbb{R}$, which define

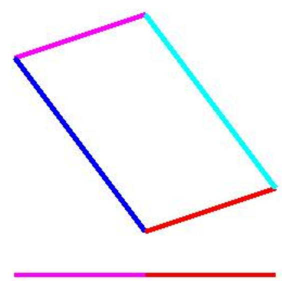
$$\widehat{Q}_i := \mathrm{CH}\{(p_i, l_i(p_i)) : q_i \in Q_i\} \subset \mathbb{R}^{n+1}.$$

Let $\widehat{Q}$ be the Minkowski sum $\sum_i \widehat{Q}_i$.
$\forall$ face in the lower-hull of $\widehat{Q}$ is written uniquely as $\sum_i \widehat{F}_i$, for faces $\widehat{F}_i \subset \widehat{Q}_i$.

$\widehat{Q}$ projects onto $Q$, so the lower-hull faces induce a regular subdivision of $Q$, with faces (cells) $\sum_i F_i$.
In particular, facets on the lower-hull project to maximal cells (dim$= n$).

## Coherent subdivisions

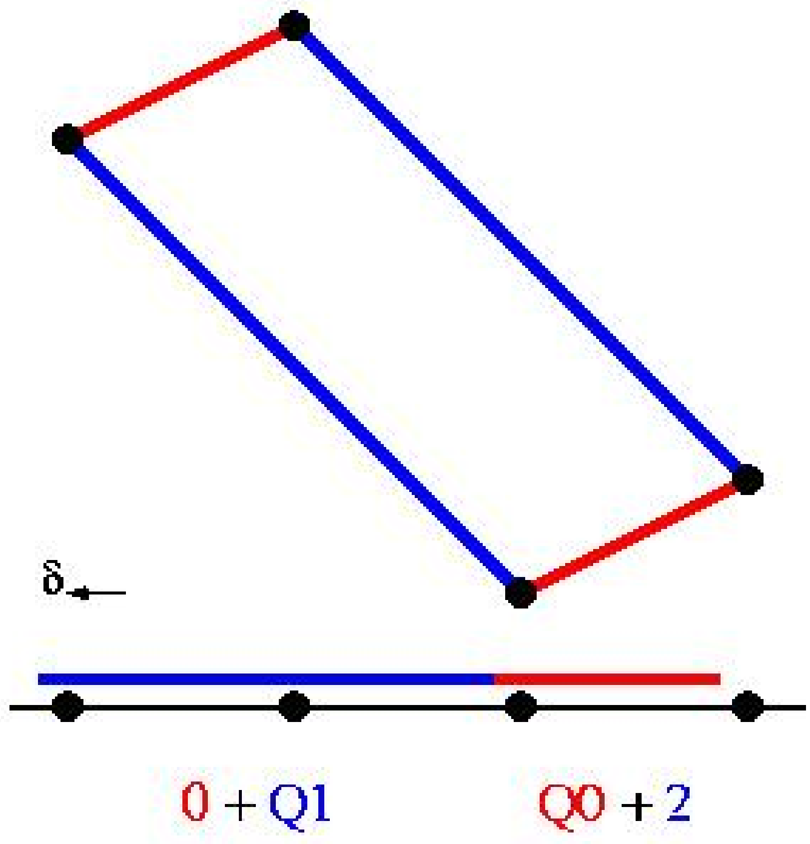
Induced subdivisions are coherent,
i.e. there is a continuous change of the unique expression of every cell as we move to its subcells and adjacent cells.
We also say that the cells intersect properly as Minkowski sums.



Eg: Incoherent subdivision of $Q_0 + Q_1,\ Q_i = [0,1]$.
Leftmost cell = proj$(\widehat{0} + \widehat{Q}_1)$, so $\widehat{0} + \widehat{1} \mapsto 1 \in \mathbb{R}$.
Rightmost cell = proj$(\widehat{1} + \widehat{Q}_1)$, so $\widehat{1} + \widehat{0} \mapsto 1$: different expression.

## Lifting in the Sylvester case

$f_0 = c_{00} + c_{01}x$, $f_1 = c_{10} + c_{11}x + c_{12}x^2$




Point $2 = 0 + 2$ from both maximal cells.

## Tight coherent mixed subdivisions

In general: $\dim(\sum_i F_i) \leq \sum_i \dim F_i$.
A generic lifting implies equality, i.e. a tight subdivision.
In particular, for a maximal-dimension cell, $n = \sum_i \dim F_i$.
Also, the lower-hull of $\widehat{Q}$ corresponds bijectively to $Q$.

E.g: NOT tight subdivision: 2 segments lifted in parallel:
$\dim(F_0 + F_1) = 1 < \dim F_0 + \dim F_1 = 1 + 1$.

One computes a (tight coherent) mixed subdivision, which partitions $Q$.

Example lifting for the over-constrained problem

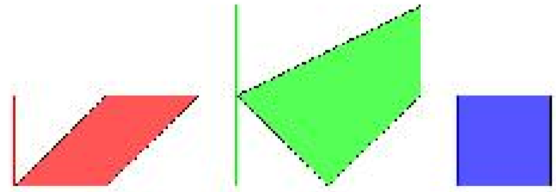


Figure 1: The given polytopes.

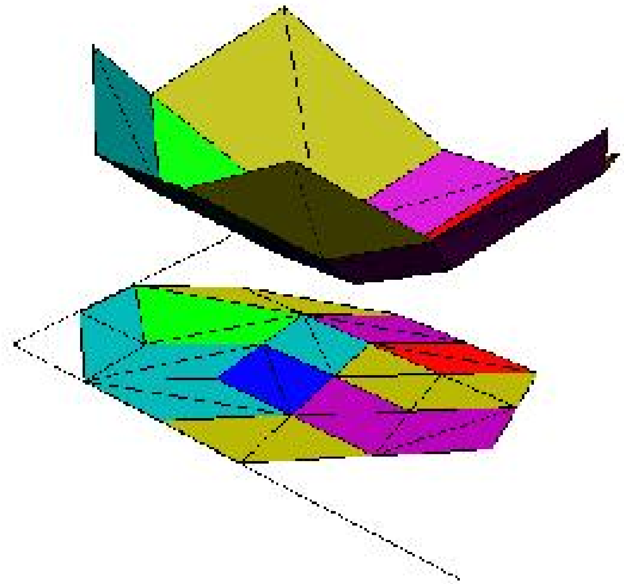


Figure 2: The lower hull of the lifted Minkowski Sum and its planar projection.

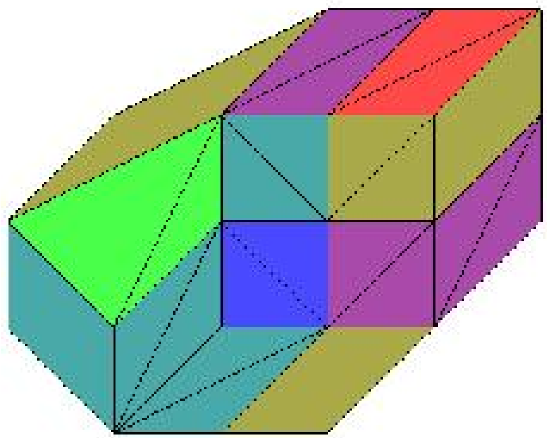


Figure 3: The mixed subdivision.

## Mixed cells

A maximal cell $\sigma$, in a mixed decomposition $\Delta$, is mixed iff it has $n$ linear summands, ie. $n$ edges $F_i$ : $\dim F_i = 1$.

- $n$ polytopes: $Q = Q_1 + \cdots + Q_n$, mixed cells are sums of edges.

Thm: $\mathrm{MV}\,(Q_1, \ldots, Q_n) = \sum_{\sigma} \mathrm{vol}(\sigma)$, over all mixed cells $\sigma \in \Delta$.

- $n+1$ polytopes: $Q = Q_0 + Q_1 + \cdots + Q_n$, $i$-mixed cells are sums of edges plus vertex $a_i \in Q_i$.

Thm: $$\mathrm{MV}(Q_0, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n) = \sum_{\sigma} \mathrm{vol}(\sigma),$$

over all $i$-mixed cells $\sigma \in \Delta$.

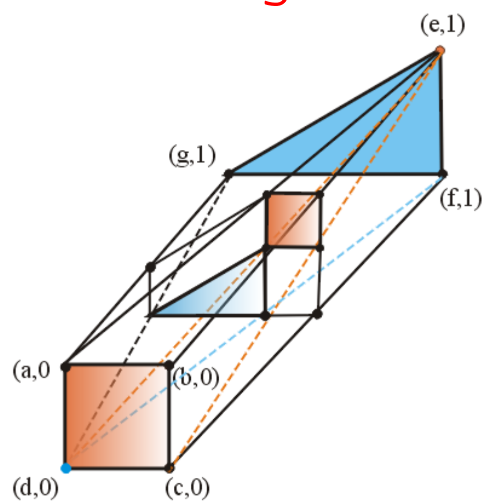
## Enumerating mixed subdivisions

The Cayley trick introduces point set $C \subset \mathbb{Z}^{2n}$:

$$C := \{(e_i, a_{ij}) \;:\; i = 0, \dots, n,\; a_{ij} \in A_i\},$$

where the $e_i$ constitute an affine basis of $\mathbb{N}^n$. So $|C| = |A_0| + \cdots + |A_n|$.

Theorem. The set of all mixed subdivisions of $A_0, \dots, A_n \subset \mathbb{Z}^n$ corresponds bijectively to the set of all regular triangulations of $C$.

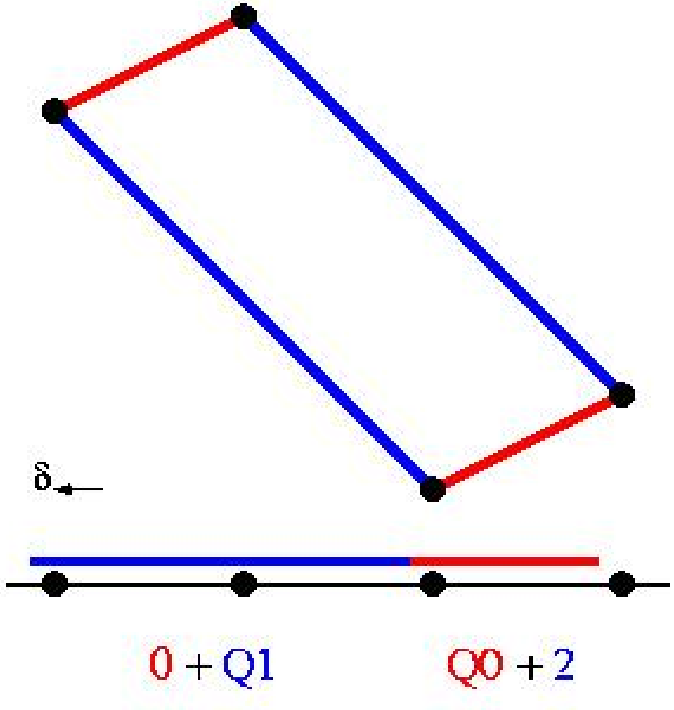


# Caley trick in sparse Sylvester case

$f_0 = c_{00} + c_{01}x,$
$f_1 = c_{10} + c_{12}x^2.$

Cayley point set

$$C := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$




Triangulations: $\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ shown,

and also $\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$

# Resultants

## Resultant definition

Given $n+1$ Laurent polynomials $f_0, \ldots, f_n \in K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ with indeterminate coefficients $\vec{c}$, their projective, resp. toric / sparse, *resultant* is the unique (up to sign) irreducible polynomial $R(\vec{c}) \in \mathbb{Z}[\vec{c}]$ such that

$$R(\vec{c}) = 0 \Leftrightarrow \exists\, \xi = (\xi_1, \ldots, \xi_n) \in X \,:\, f_0(\xi) = \cdots = f_n(\xi) = 0$$

where the variety $X$ equals:

- the projective space $\mathbb{P}^n$ over the algebraic closure $\overline{K}$,
- resp. the toric variety $X$, $(\overline{K}^*)^n \subset X \subset \mathbb{P}^N$.

[van der Waerden, Gelfand-Kapranov-Zelevinsky, Cox-Little-O'Shea]

## Newton polytope of the toric resultant

Given are supports $A_0, \ldots, A_n$ s.t. $k := \sum_i |A_i|$ and $\dim(\sum_i A_i) = n$. Consider the toric resultant $R \in \mathbb{Z}[c]$ and its Newton polytope in $\mathbb{R}^k$.

Let lifting $l \in \mathbb{R}^k$ define a (tight coherent) mixed subdvision of $Q_0 + \cdots + Q_n$. Consider the trailing monomial of $R$ with respect to $l$, which corresponds to the vertex of $\mathrm{supp}(R) \subset \mathbb{Z}^k$ with inner normal $l$.

Theorem. [Sturmfels'94] This trailing monomial is

$$\prod_{i=0}^{n} \prod_{i-\text{mixed } \sigma} \mathrm{coef}(f_i, a_i)^{\mathrm{vol}(\sigma)},$$

where $\mathrm{vol}(\cdot)$ denotes Euclidean volume and the $i$-mixed cells are $\sigma = F_0 + \cdots + a_i + \cdots + F_n : \dim a_i = 0$.

## Newton polytope of the toric resultant (cont'd)

Corollary. A surjection exists from the set of mixed-cell configurations onto the extreme monomials of $R$ (vertices of its Newton polytope).

Corollary. The coefficient of the trailing term is $\pm 1$.

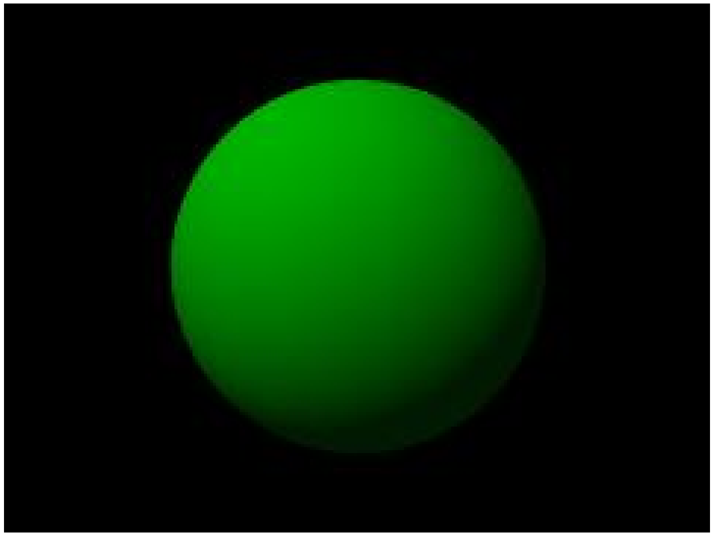
# Implicitization

## Example: sphere

The sphere in $\mathbb{R}^3$ is the set of values $(x, y, z)$:

$$x = \frac{t_1^2 - t_2^2 - 1}{t_1^2 + t_2^2 + 1}, y = \frac{2t_1}{t_1^2 + t_2^2 + 1}, z = \frac{2t_1 t_2}{t_1^2 + t_2^2 + 1}, \; t_1, t_2 \in [0, 1],$$

as well as the set of roots of $H(x, y, z) := x^2 + y^2 + z^2 - 1 = 0$.



Modeling/CAD use parametric *and* implicit/algebraic representations
$\Rightarrow$ must implicitize a (hyper)surface given a (rational) parameterization

## Implicitization by linear algebra

$S =$ monomials forming (a superset of) the implicit support.
$C =$ unknown coefficients of implicit equation wrt $S$, $|C| = |S|$.

- $MC = \vec{0}$, where matrix $M$ is $|S| \times |S|$, and contains values of $S$ at points $(s_i, t_i), i = 1, \ldots, |S|$. Try roots of unity.

- $(SS^T)C = \vec{0}$, substitute $x, y, z$ by parametric expressions in $K[s,t]$, integrate over $s, t$; solve for $C$ [Corless-Galligo-Kotsireas-Watt'00].
Example: supp$(H) \subset \{x^3y, x^3, x^3y^2, y^2z^3\}$, then

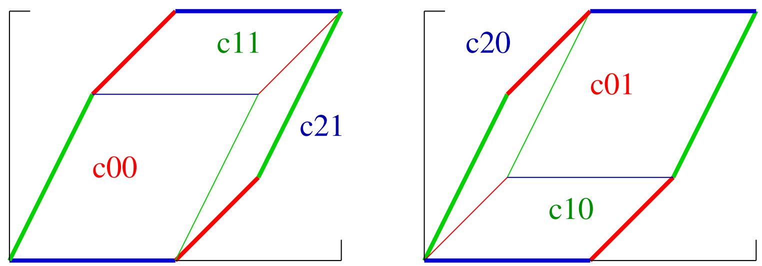
$$SS^T = \begin{bmatrix} x^6y^2 & x^6y & x^6y^3 & x^3y^3z^3 \\ x^6y & x^6 & x^6y^2 & x^3y^2z^3 \\ x^6y^3 & x^6y^2 & x^6y^4 & x^3y^4z^3 \\ x^3y^3z^3 & x^3y^2z^3 & x^3y^4z^3 & y^4z^6 \end{bmatrix} \Rightarrow C = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

- Approximate implicitization [Dokken].

## Enumerate mixed subdivisions

A sparse example [Buchberger'88]

$$f_0 = c_{00} - c_{01}st, \; f_1 = c_{10} - c_{11}st^2, \; f_2 = c_{20} - c_{21}s^2.$$

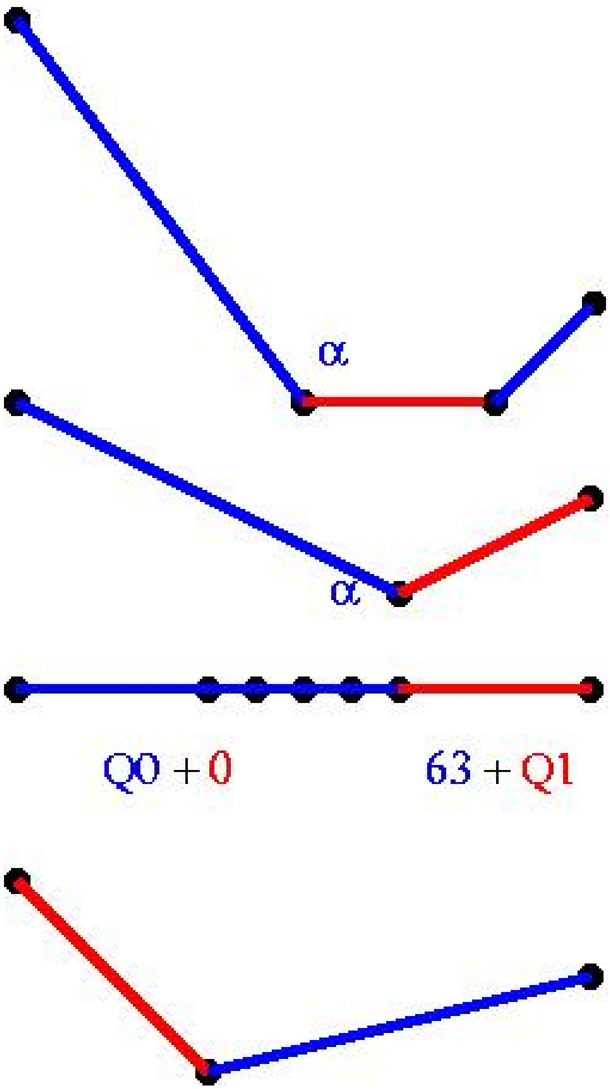



The mixed subdivisions yield extreme monomials $c_{00}^4 c_{11}^2 c_{21}$, $c_{01}^4 c_{10}^2 c_{20}$.

The toric resultant turns out to be $R = c_{00}^4 c_{11}^2 c_{21} - c_{01}^4 c_{10}^2 c_{20}$.

# The Fröberg-Dickenstein example

$$x = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, \quad y = t^{32}.$$




$Q_0' + 0, a + Q_1, Q_0'' + 32$ $\quad$ $\pm y^a c_{0a}^{32} c_{1,32}^{63-a}$

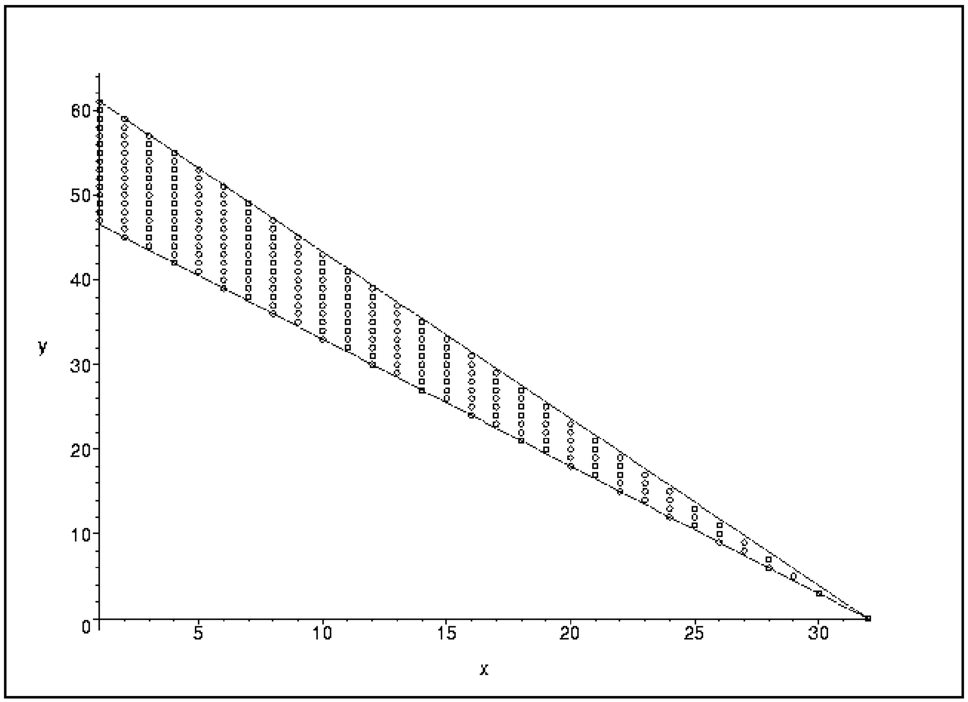
$a = 48, 56, 60, 62, 63$ $\quad$ yields $\pm y^a$

$a = 63$ $\quad$ $\pm y^{63} c_{0,63}^{32}$

$0 + Q_1, Q_0 + 32, (a = 0)$ $\quad$ $\pm x^{32} c_{1,32}^{63}$

## The Fröberg-Dickenstein example (cont'd)

The projected support is defined by points $(0, 48), (0, 63), (32, 0)$:
This triangle includes 257 lattice points, optimally:



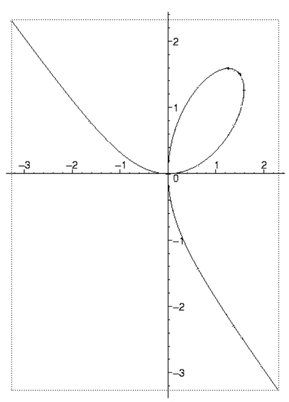

Degree bounds yield quad with additional vertices $(0, 0), (32, 31)$.

## Implicitization examples [E-Kotsireas'03]

[Descartes' folium]

[1596-1650]

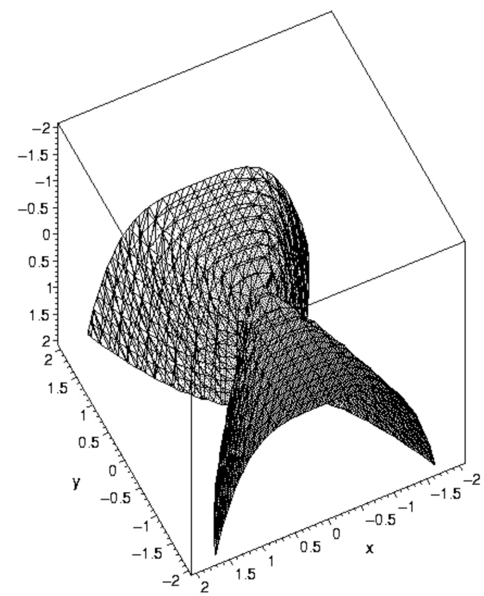
$$(x, y) = \left( \frac{3\ t^2}{t^3 + 1}, \frac{3\ t}{t^3 + 1} \right)$$



$$H = x^3 + y^3 - 3\ x\ y$$

[Buchberger'88]

$$(x, y, z) = (st, st^2, s^2)$$




$$H = x^4 - y^2 z$$

[Busé'01]

$$x = \frac{s^2}{s^3 + t^3},$$

$$y = \frac{s^3}{s^3 + t^3},$$

$$z = \frac{t^2}{s^3 + t^3}$$

$$H = x^3 - 2x^3 y + x^3 y^2 - y^2 z^3$$

## Implicit polytope

Consider parameterizations with fixed supports and generic coefficients.

- ○ Compute the resultant's Newton polytope, then specialize.

– [E-Kotsireas'03] developed Maple code based on Topcom [Rambau].

– [EKP'07] algorithm for projecting polytope to $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$.

- Tropical geometry for the polytope of Laurent-polynomial hypersurfaces and varieties of codim$> 1$. For curves, determines implicit vertices [Sturmfels-Tevelev-Yu'06].

- Implicit edges for all curves, using linear factors:

– [D'Andrea-Sombra'07] use mixed fiber polytopes [Esterov-Khovanskii'07], characterize polygons which realize as implicit.

– [Dickenstein-Sturmfels'07] use tropical discriminants.

- Specify implicit vertices for all curves [E-Konaxis-Palios'07]

## Polynomially parameterized curves

$$x_i = P_i \in K[t],\ i = 0, 1,$$

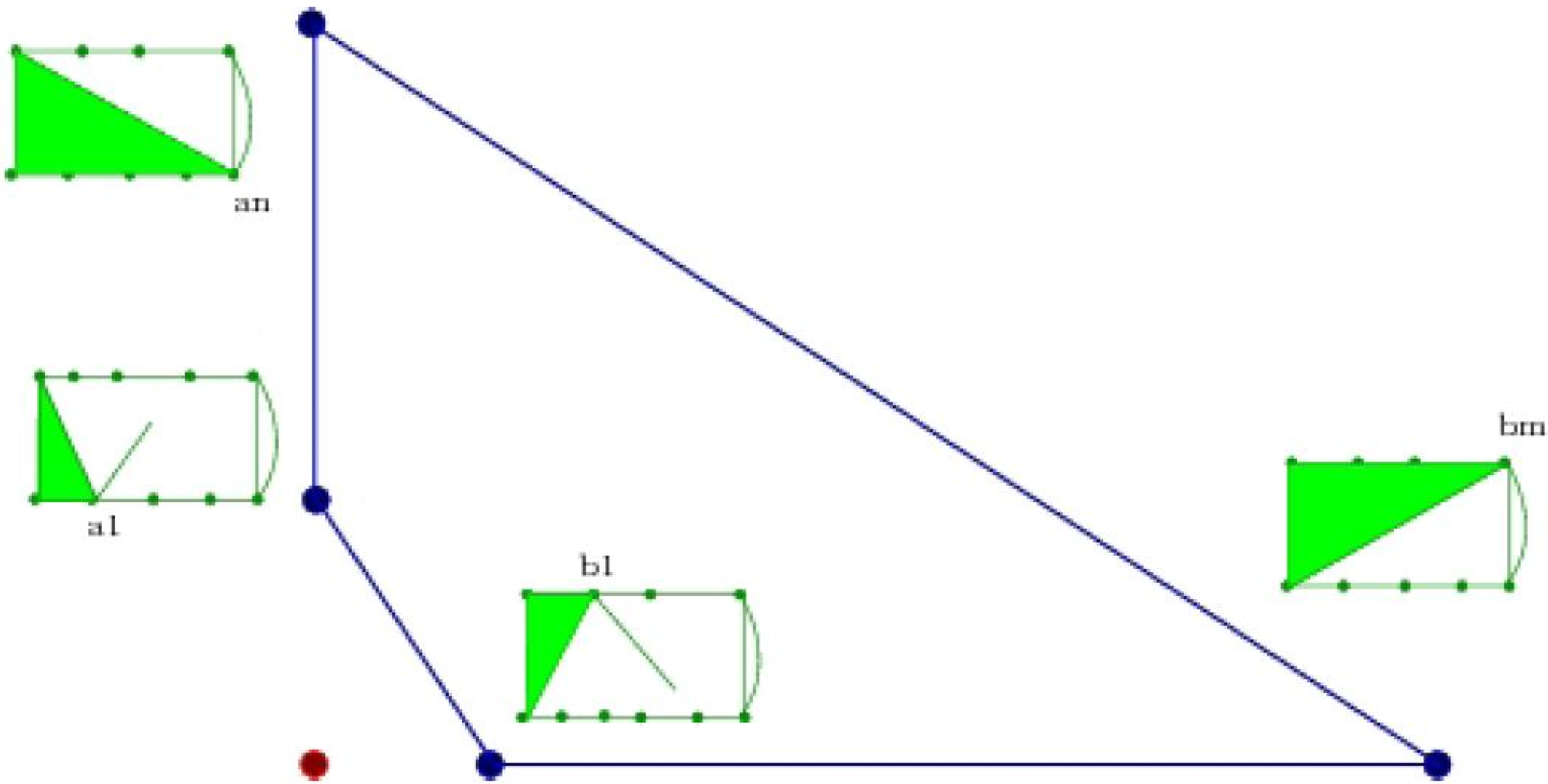
with supports $\{a_0, \dots, a_n\}, \{b_0, \dots, b_m\} \subset \mathbb{N}$.

Cayley's trick: $\{(a_0, 0), \dots, (a_n, 0), (b_0, 1) \dots, (b_m, 1)\} \subset \mathbb{N}^2$.
Now, consider triangulations of this point set.

Triangle $(a_0, b_i, b_j) \to x^{b_j - b_i}$, $(a_i, a_j, b_0) \to y^{a_j - a_i}$.
Triangles $(a_i, *, b_j), i, j \neq 0$, do not contribute to $x, y$.
We shall say $a_0, b_0$ are selected.

# Implicit polygon of Polynomial curves

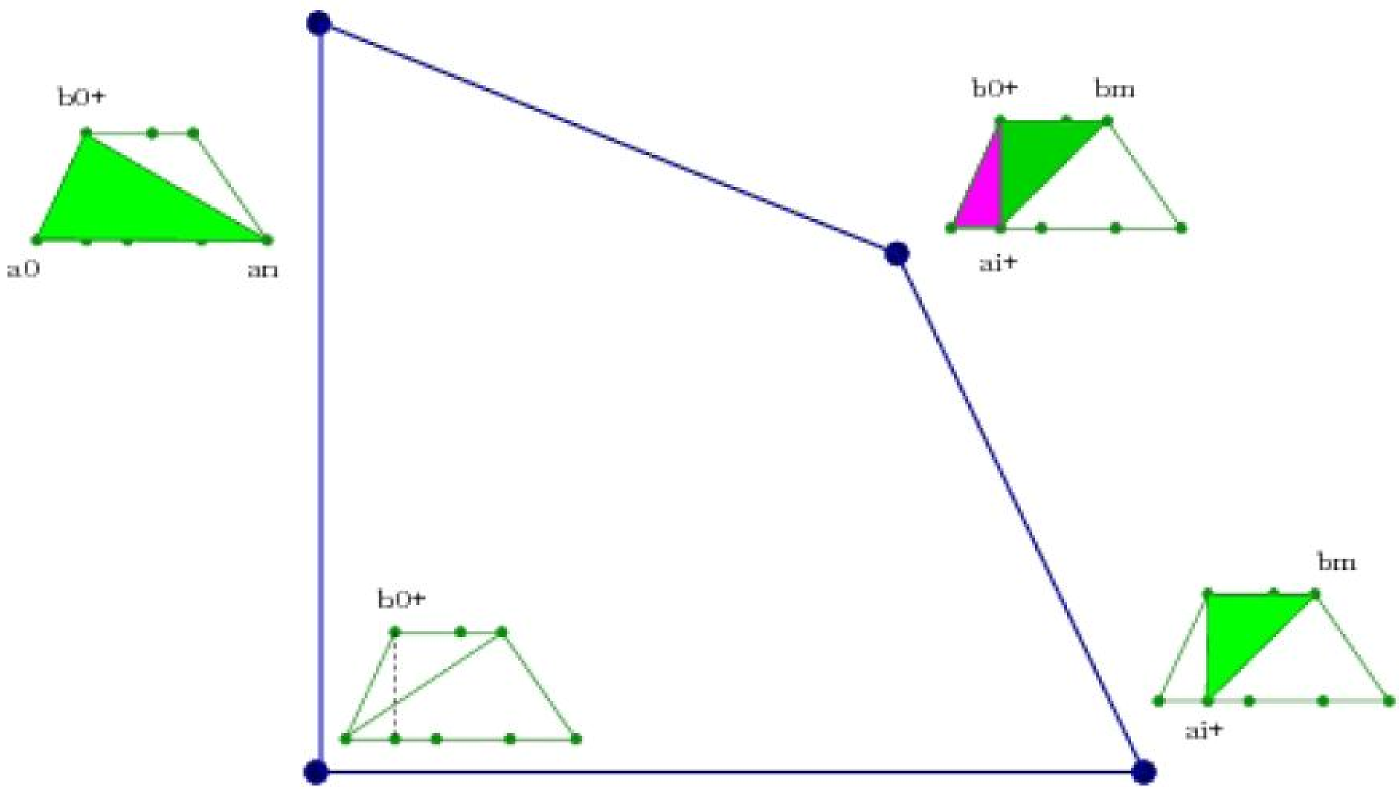
$\exists$ constant term in some $P_i(t) \Rightarrow \exists$ implicit vertex $(0,0)$:




Cor: coef$(x^{b_m}) = \pm(-c_{1m})^{a_n}$, coef$(y^{a_n}) = \pm(-c_{0n})^{b_m}$.

## Laurent-polynomial parameterization

$\{a_0, \ldots, a_n\}, \{b_0, \ldots, b_m\} \subset \mathbb{Z}$, unique selected $a_i^+, b_0^+$.




Up-right vertex $= (b_m, |a_0|)$ iff $\det \begin{bmatrix} |a_0| & a_n \\ |b_0| & b_m \end{bmatrix} > 0$, $(|b_0|, a_n)$ iff $\det < 0$.

## Rational curves, different denominators

$$x_i = \frac{P_i(t)}{Q_i(t)}, \text{gcd}(P_i, Q_i) = 1 \rightarrow f_i = x_i Q_i(t) - P_i(t) \in K[t],\ i = 0, 1,$$

$$\text{where supp}(f_0) = \{a_0, \ldots, a_n\},\ \text{supp}(f_1) = \{b_0, \ldots, b_m\} \subset \mathbb{N}.$$

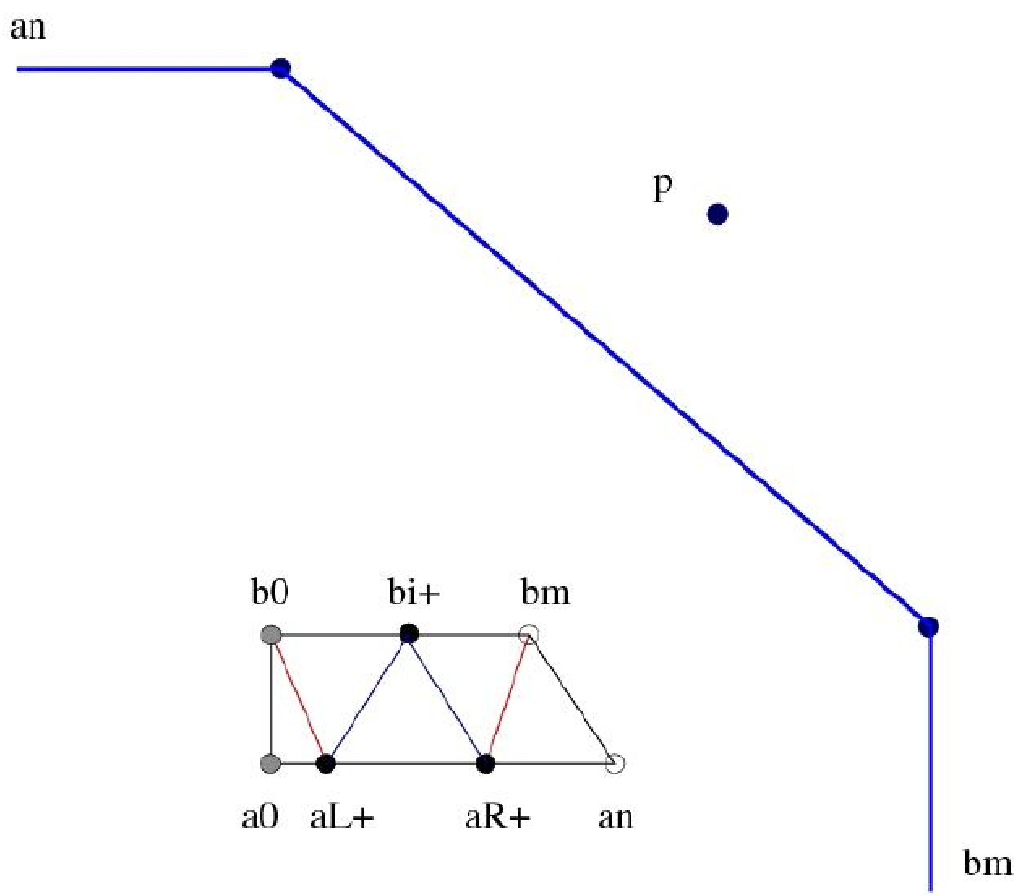
**Lemma**. Consider direction $(1, 1)$. The upper [resp. lower] hull of the Newton polygon has vertices of the form:

$$\left( \sum_{k,l,r} \text{vol}(a_k^+, b_l, b_r), \sum_{l,r,j} \text{vol}(a_l, a_r, b_j^+) \right) \in \mathbb{N}^2,$$

where $a_k^+ \in \text{supp}(Q_0)$ [resp. $\text{supp}(Q_0) - \text{supp}(P_0)$],
and $b_j^+ \in \text{supp}(Q_1)$ [resp. $\text{supp}(Q_1) - \text{supp}(P_1)$]

Proof. The significant resultant coefficients are the following: the monomials $c_i x_i$ and the binomials $c_i x_i + c_i' \in K[x_i]$ [resp. the monomials $c_i x_i \in K[x_i]$ ].

# Upper-right corner




$$x_{\max} = b_m,$$

$$y_{\max} = (a_R^+ - a_L^+) + \mathcal{X}(b_m^+) \cdot (a_n - a_R^+)$$

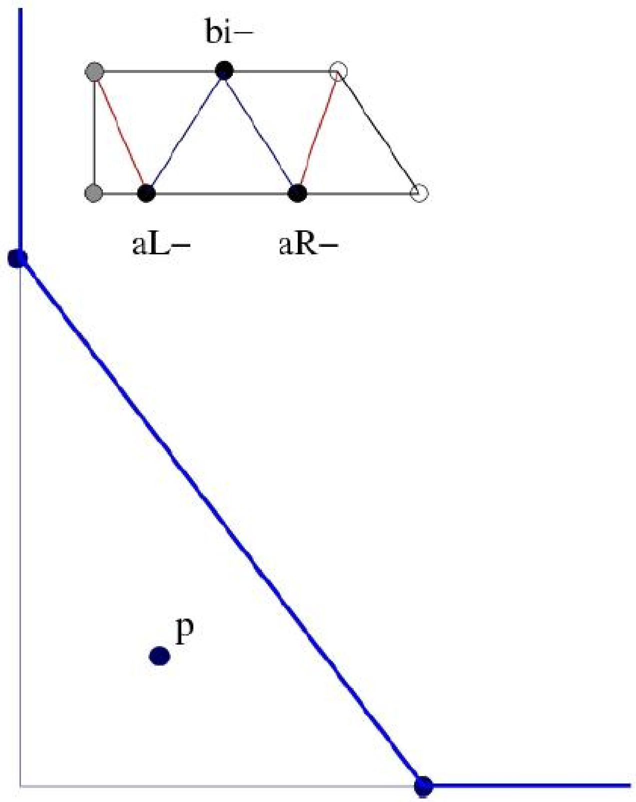
if $a_0^-, b_0^-, a_n^-, b_m^-$ then:

$$\exists\ p = (b_R^+, a_n - a_L^+) \Leftrightarrow \det \begin{bmatrix} a_n - a_R^+ & a_L^+ \\ b_L^+ & b_m - b_R^+ \end{bmatrix} > 0,$$

$$p = (b_m - b_L^+, a_R^+) \Leftrightarrow \det < 0$$

Selected $a_k^+ \in \text{supp}(Q_0), b_j^+ \in \text{supp}(Q_1)$, not selected $a_k^-, b_j^-$.

# Lower-left corner




$$x_{\text{min}} = 0,$$

$$y_{\text{min}} = \mathcal{X}(b_0^+)a_L^+ + \\ +\mathcal{X}(b_m^+)(a_n - a_R^-)$$

if $a_0^+, b_0^+, a_n^+, b_m^+$ then:

$$\exists\ p = (b_L^-, a_n - a_R^-) \Leftrightarrow$$

$$\det \begin{bmatrix} a_n - a_R^- & a_L^- \\ b_m - b_R^- & b_L^- \end{bmatrix} < 0,$$

$$p = (b_m - b_R^-, a_L^-) \Leftrightarrow \det > 0$$

Selected $a_k^+ \in \text{supp}(Q_0) - \text{supp}(P_0)$, $b_j^+$, not selected $a_k^-, b_j^-$.

## Rational curves, common denominator

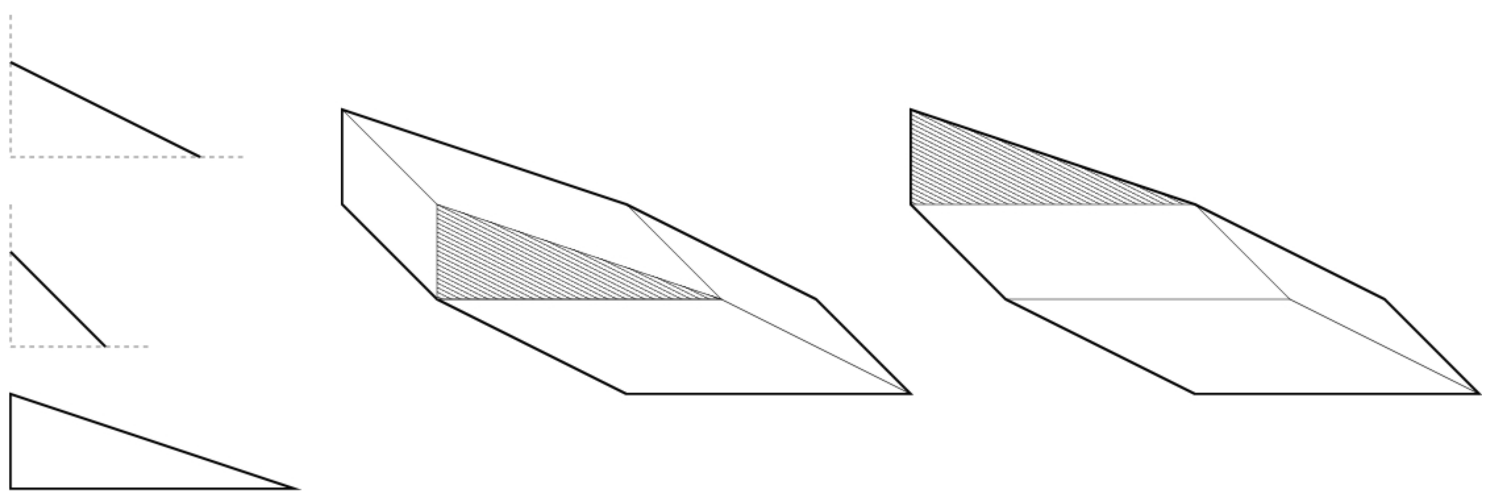
$$x_0 = \frac{P_0(t)}{Q(t)},\ x_1 = \frac{P_1(t)}{Q(t)},\ \text{gcd}(P_i, Q) = 1,$$

$$B_i = \text{supp}(P_i) = \{b_{iL}, \ldots, b_{iR}\}, i = 0, 1,\ B_2 = \text{supp}(Q) = \{b_{2L}, \ldots, b_{2R}\} \subset \mathbb{N}$$

$$\text{Let } f_0 = x_0 r - P_0(t),\ f_1 = x_1 r - P_1(t),\ f_2 = r - Q(t) \in K[t, r],$$

$$A_i = \text{supp}(f_i) = \{a_{i0} = (0, 1), a_{iL} = (b_{iL}, 0), \ldots, a_{iR} = (b_{iR}, 0)\}, i = 0, 1, 2$$

**Example:** Folium of Descartes: $x = \frac{3t^2}{1+t^3}, y = \frac{3t}{1+t^3}$.

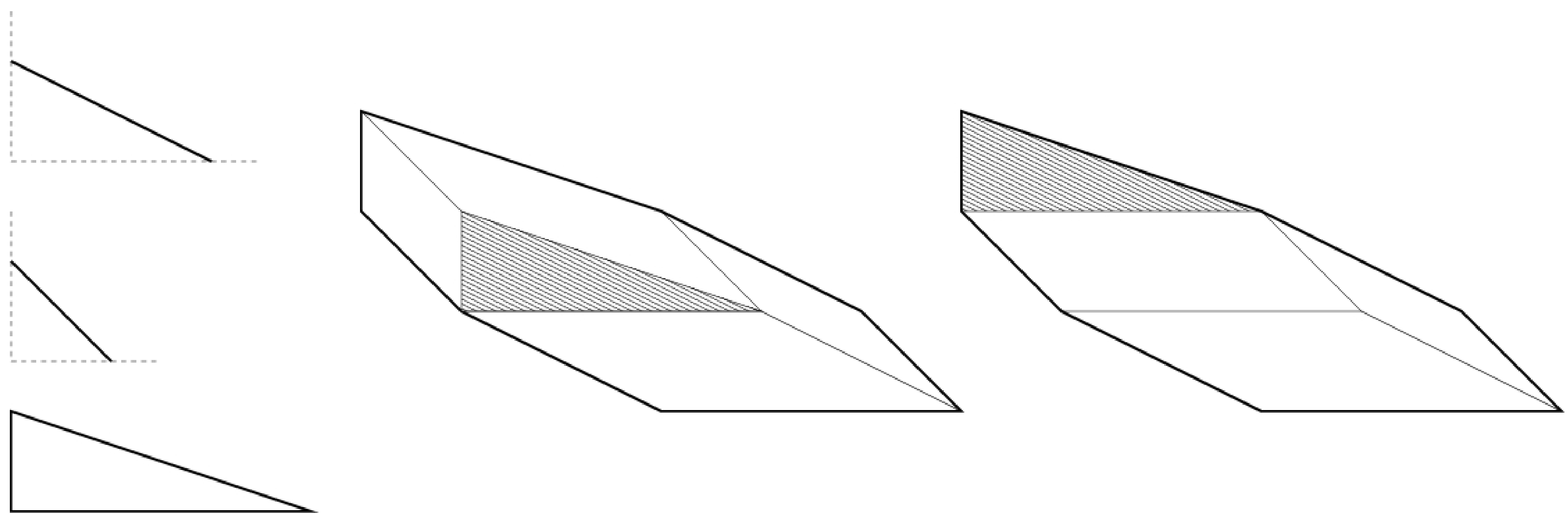


## Mixed subdivisions

Lemma. In Minkowski sum $\mathrm{CH}(A_0) + \mathrm{CH}(A_1) + \mathrm{CH}(A_2)$, all $a_{i0}$-mixed cells are as follows, for $\{i, j, k\} = \{0, 1, 2\}$:

$$a_{i0} + E_j + E_k, \text{ where edge } E_j = (a_{j0}, a_{jt}) \subset A_j,$$

and $E_k \subset A_k$ is either non-horizontal $(a_{k0}, a_{km})$, or horizontal $(a_{kl}, a_{km})$.



Cor. It suffices to consider subdivisions of segment $((0, 2), (u, 2))$.

## Implicit equation

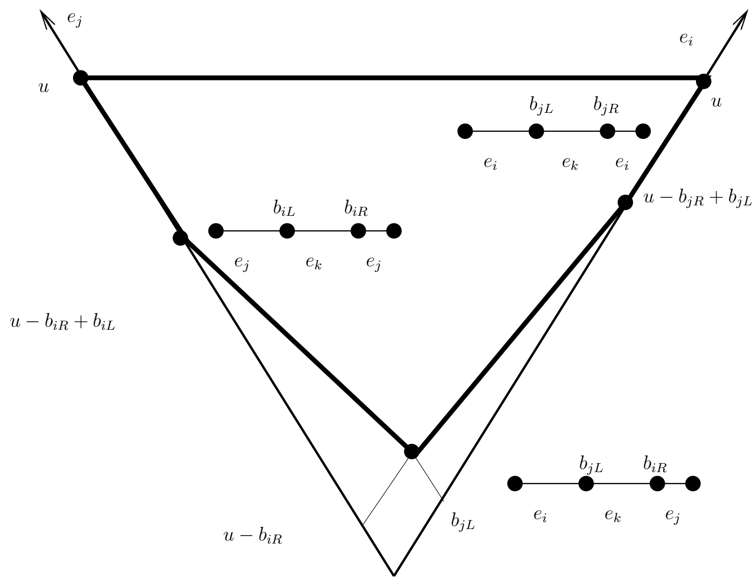
Lemma. Implicit equation $\phi \in K[x_0, x_1]$ has

$$\deg \phi = \text{vol}[\text{CH}(\cup_i \text{supp}(f_i))] = \max_i\{b_{iR}\} - \min_i\{b_{iL}\} = u - 0.$$

Cor. $\text{Res}_{t,r}(f_0, f_1, f_2)$ is homogeneous, of degree $u$, in the 3 coefficients corresponding to the $a_{00}, a_{10}, a_{20}$.
$\Phi$ := specialization of $\text{Res}_{t,r}(f_0, f_1, f_2)$ to polynomial in the 3 coeffs: $N(\Phi) \subset \mathbb{R}^3$ projects bijectively to $N(\phi) \subset \mathbb{R}^2$.

Thm. For computing the vertices of $N(\phi)$ (or $N(\Phi)$), it suffices to consider subdivisions defined by the vertices of the $\text{CH}(A_i), i = 0, 1, 2$, i.e. the relevant liftings of each $A_i$ are linear in $\mathbb{R}^2$.

# Implicit vertices (A)




Thm. If all CH$(B_i \cup B_j) = [0, u]$, then $N(\phi) =$CH$((0,0),(0,u),(u,0))$. Otherwise, if $\exists!\, B_k = [0, u]$, then the $(e_i, e_j)$-vertices of $N(\phi)$ lie in

$$\{(u,0),(0,u),(0,u-b_{iR}+b_{iL}),(b_{jL},u-b_{iR}),(u-b_{jR}+b_{jL},0)\},$$

where $\{i,j,k\} = \{0,1,2\}$, and $b_{iL}(u - b_{jR}) \geq b_{jL}(u - b_{iR})$.

## Implicit vertices (B)

Thm. If $\forall\, B_t \neq [0, u]$, then choose $\{i, j, k\} = \{0, 1, 2\}$ s.t.:

$$0 < b_{iL} \leq b_{iR} = u, 0 = b_{jL} \leq b_{jR} < u, 0 \leq b_{kL} \leq b_{kR} < u.$$

If $b_{kL} > 0$, the $(e_i, e_j)$-vertices lie in

$$\{(b_{jR}, 0), (b_{kR}, u - b_{kR}), (b_{kL}, u - b_{kL}), (0, u - b_{0L}), (0, 0)\}.$$

If $b_{kL} = 0$, the 3rd and 4th vertices are replaced by $(0, u)$.

## Implicit polygon cuts

Corollary. Start with a triangle or quadrilateral that has a vertex at $(0, 0)$ and incident edges which lie on the axes.

- Polynomial parameterizations:

Take a right triangle, apply at most one corner cut excluding the origin.

- Rational parameterizations with equal denominators:

Take a right triangle, apply at most two cuts (same or different corners).

- Rational parameterizations with different denominators:

Take a quadrilateral, apply at most two cuts (same or different corners).

## Conclusions

Inverse. If $\phi(x, y)$ admits a polynomial parameterization, then $N(\phi)$ has one edge on its upper hull wrt $(1, 1)$.
Then, if $N(\phi) =$ segment, it contains no interior lattice points.

Future:

- Specify genericity conditions, determine extremal coefficients.
- Polytope of implicit Surfaces.
- When are linear liftings sufficient? Pyramids? Separated variables?
- Project resultant polytope to low dimension.
- Numerical implicitization.

# Secondary polytopes

## Secondary polytope

Consider the graph of regular triangulations of point-set $C \subset \mathbb{Z}^d$, where edges correspond to (bistellar) flips.

Theorem [Gelfand-Kapranov-Zelevinsky, Billera-Sturmfels]
If $C$ affinely spans $\mathbb{R}^d$, then the graph can be embedded in $\mathbb{R}^{|C|-d-1}$ as the secondary polytope $\Sigma(C)$. For triangulation $T$,

$$(v_T)_i = \sum_{i \in \text{vtx}(\sigma) : \sigma \in T} \text{vol}(\sigma), \qquad i = 1, \dots, |C|,$$

where vtx($\sigma$) are the vertices of simplex $\sigma$.

E.g. $C \subset \mathbb{Z}^2$, $|C| = 4$:

## Circuits

A circuit $Z = \{c_1, \ldots, c_t\}$ is a minimal affinely-dependent subset of $C$, satisfying $\lambda_1 c_1 + \cdots + \lambda_t c_t = 0$, where $\lambda_i \neq 0$, $\sum_i \lambda_i = 0$.

$Z$ admits triangulations $Z^+ = \{Z \setminus \{c_i\} \mid \lambda_i > 0\}$, $Z^- = \{Z \setminus \{c_i\} \mid \lambda_i < 0\}$. Each flip $T \leftrightarrow T'$ corresponds to precisely one circuit $Z$ s.t.

$$T' \simeq T \setminus Z^+ \cup Z^-$$

E.g. $Z = C$, $$-\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$Z^+ = T =$  $= T' = Z^-$

## Projecting $\Sigma(C)$ to $\mathbb{R}$

Project to 1st coordinate, corresponding to $c_1 \in C \subset \mathbb{Z}^d$ (point set).

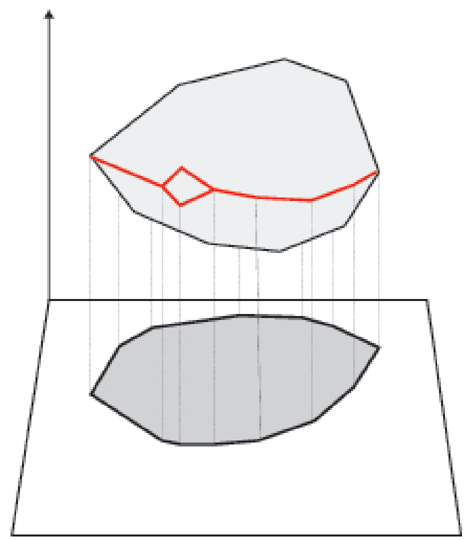
- Let $T$ be a regular triangulation, and consider flip $T \leftrightarrow T'$. Then $c_1$ is a vertex of every new simplex iff $(v_T)_1 < (v_{T'})_1$.

- Let $Z_j$ be the flips that make $(v)_1$ increase, and $\sigma_j$ the unique simplex vanishing with $Z_j$ not containing $c_1$. Then, the triangulation $T$ maximizing $(v_T)_1$ is s.t. the volume of simplices containing $\sigma_j$ is max.

- Hence $(v)_1 \uparrow$; if strictly$\uparrow$ then min-path.

[E-Konaxis-Palios'07]

# Projecting $\Sigma(C)$ to $\mathbb{R}^k, k \geq 2$



- Complexity: Time $= O^*(s^2 m)$LP$(\dim \Sigma, s)$, Space $= O(ns)$,

$s =$ max #any-dim simplices$= O(k^n)$, $m =$ #mixed-cell config's.

- Gift-wrapping, CCW$(u, v, w) = \det \begin{bmatrix} 1 & u_1 & u_2 \\ 1 & v_1 - u_1 & v_2 - u_2 \\ 1 & w_1 - v_1 & w_2 - v_2 \end{bmatrix}$.

Goal: complexity proportional to #silhouette-points.