The Newton polytope of implicit curves

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Oberwolfach Nov'07

Outline

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Toric elimination theory

Newton polytopes

The support A_i of a polynomial $f_i \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, s.t.

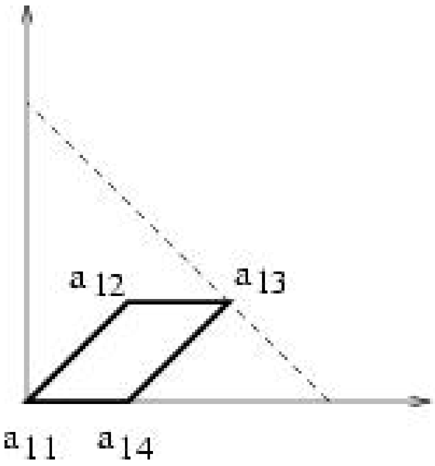
$$f_i = \sum_j c_{ij} x^{a_{ij}}, \qquad c_{ij} \neq 0,$$

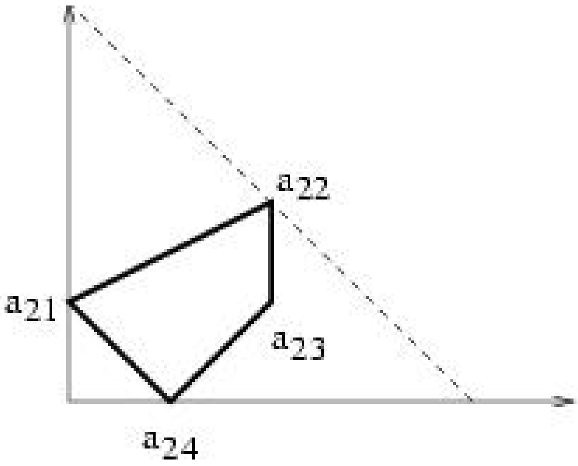
is defined as the set $A_i := \{a_{ij} \in \mathbb{Z}^n : c_{ij} \neq 0\}$.

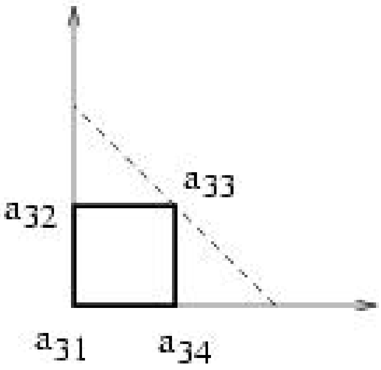
The Newton polytope $Q_i \subset \mathbb{R}^n$ of f_i is the Convex Hull of all $a_{ij} \in A_i$.

Example:

$$\begin{aligned} f_1 &= c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x \\ f_2 &= c_{21}y + c_{22}x^2y^2 + c_{23}x^2y + c_{24}x + c_{25}xy \\ f_3 &= c_{31} + c_{32}y + c_{33}xy + c_{34}x \end{aligned}$$







Mixed volume

1. The mixed volume $\mathrm{MV}(P_1, \ldots, P_n) \in \mathbb{R}$ of convex polytopes $P_i \subset \mathbb{R}^n$

- is multilinear wrt Minkowski addition and scalar multiplication:

$$\mathrm{MV}(P_1, \ldots, \lambda P_i + \mu P_i', \ldots, P_n) =$$
$$= \lambda \mathrm{MV}(P_1, \ldots, P_i, \ldots, P_n) + \mu \mathrm{MV}(P_1, \ldots, P_i', \ldots, P_n), \qquad \lambda, \mu \in \mathbb{R},$$

- st. $\mathrm{MV}(P_1, \ldots, P_1) = n!\ \mathrm{vol}(P_1)$.

2. Equivalently, $\mathrm{vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ is a polynomial in scalar variables $\lambda_1, \ldots, \lambda_n$, with multilinear term $\mathrm{MV}(P_1, \ldots, P_n)\ \lambda_1 \cdots \lambda_n$.

Mixed subdivisions

Regular subdivisions

For $Q_i \subset \mathbb{R}^n$, $\qquad (Q_i)_{i\in I} \to Q = \sum_{i\in I} Q_i \;:\; (q_i)_{i\in I} \mapsto \sum_{i\in I} q_i.$

Consider lifting functions $l_i : \mathbb{R}^n \to \mathbb{R}$, which define

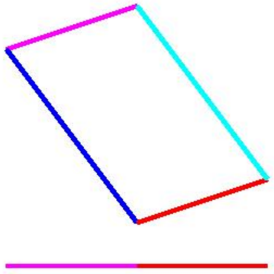
$$\widehat{Q}_i := \mathrm{CH}\{(p_i, l_i(p_i)) : q_i \in Q_i\} \subset \mathbb{R}^{n+1}.$$

Let $\widehat{Q}$ be the Minkowski sum $\sum_i \widehat{Q}_i$.
$\forall$ face in the lower-hull of $\widehat{Q}$ is written uniquely as $\sum_i \widehat{F}_i$, for faces $\widehat{F}_i \subset \widehat{Q}_i$.

$\widehat{Q}$ projects onto Q, so the lower-hull faces induce a regular subdivision of Q, with faces (cells) $\sum_i F_i$.
In particular, facets on the lower-hull project to maximal cells (dim$= n$).

Coherent subdivisions

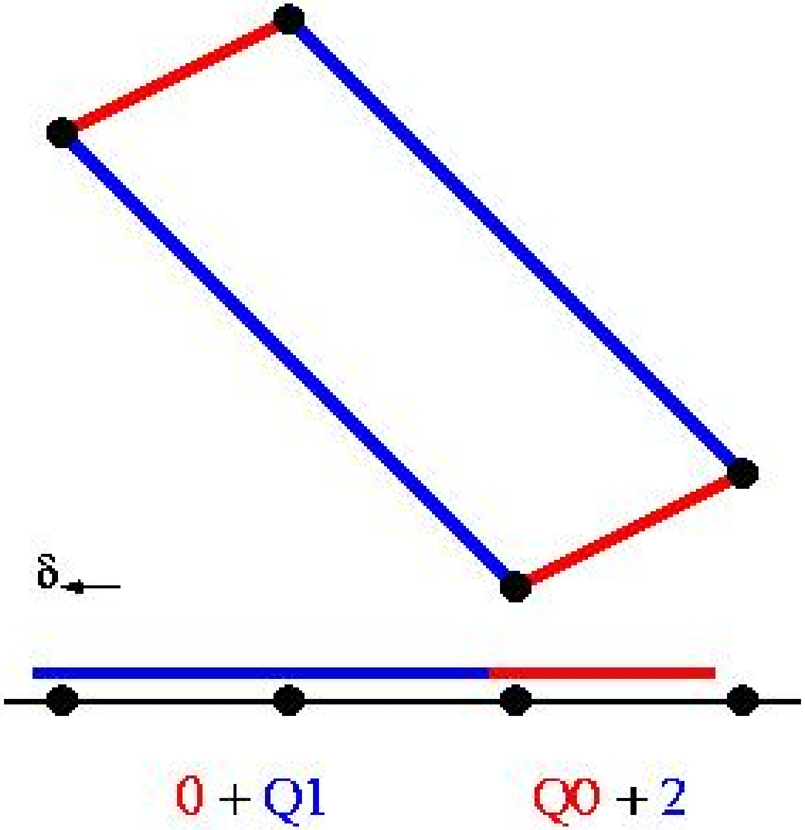
Induced subdivisions are coherent,
i.e. there is a continuous change of the unique expression of every cell as we move to its subcells and adjacent cells.
We also say that the cells intersect properly as Minkowski sums.



Eg: Incoherent subdivision of $Q_0 + Q_1,\ Q_i = [0,1]$.
Leftmost cell = $\mathrm{proj}(\widehat{0} + \widehat{Q}_1)$, so $\widehat{0} + \widehat{1} \mapsto 1 \in \mathbb{R}$.
Rightmost cell = $\mathrm{proj}(\widehat{1} + \widehat{Q}_1)$, so $\widehat{1} + \widehat{0} \mapsto 1$: different expression.

Lifting in the Sylvester case

$$f_0 = c_{00} + c_{01}x,\ f_1 = c_{10} + c_{11}x + c_{12}x^2$$



Point $2 = 0 + 2$ from both maximal cells.

Tight coherent mixed subdivisions

In general: $\dim\left(\sum_i F_i\right) \le \sum_i \dim F_i$.
A generic lifting implies equality, i.e. a tight subdivision.
In particular, for a maximal-dimension cell, $n = \sum_i \dim F_i$.
Also, the lower-hull of $\widehat{Q}$ corresponds bijectively to Q.

E.g: NOT tight subdivision: 2 segments lifted in parallel:
$\dim(F_0 + F_1) = 1 < \dim F_0 + \dim F_1 = 1 + 1$.

One computes a (tight coherent) mixed subdivision, which partitions Q.

Example lifting for the over-constrained problem

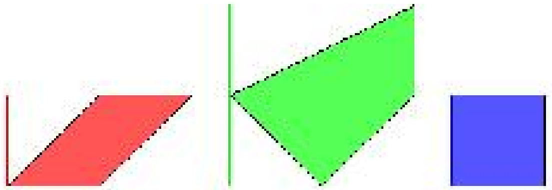


Figure 1: The given polytopes.

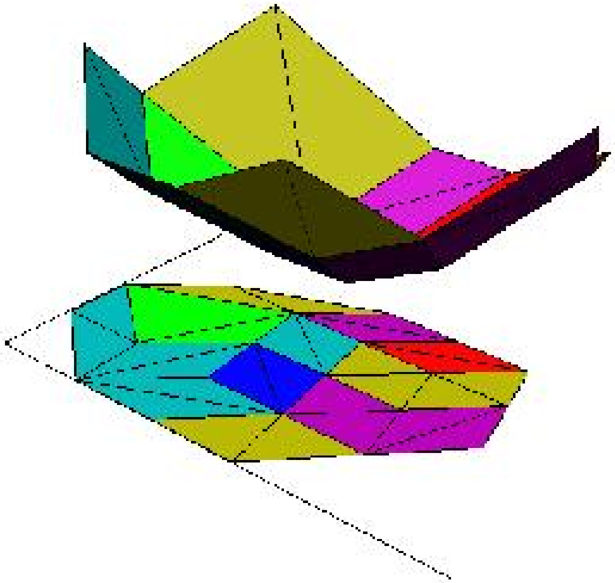


Figure 2: The lower hull of the lifted Minkowski Sum and its planar projection.

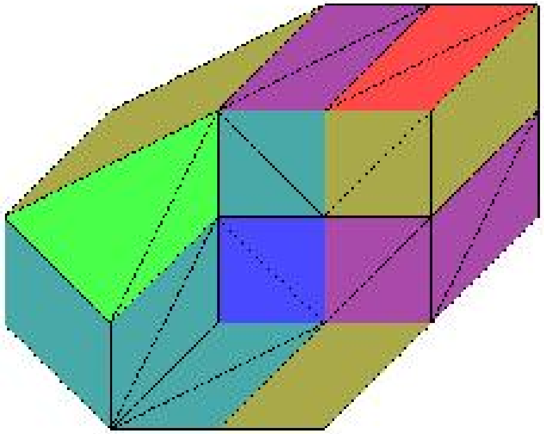


Figure 3: The mixed subdivision.

Mixed cells

A maximal cell σ, in a mixed decomposition Δ, is mixed iff it has n linear summands, ie. n edges $F_i\ :\ \dim F_i = 1$.

- n polytopes: $Q = Q_1 + \cdots + Q_n$, mixed cells are sums of edges.

Thm: $\mathrm{MV}\ (Q_1, \ldots, Q_n) = \sum_\sigma \mathrm{vol}(\sigma)$, over all mixed cells $\sigma \in \Delta$.

- $n+1$ polytopes: $Q = Q_0 + Q_1 + \cdots + Q_n$, i-mixed cells are sums of edges plus vertex $a_i \in Q_i$.

Thm: $$\mathrm{MV}(Q_0, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n) = \sum_\sigma \mathrm{vol}(\sigma),$$

over all i-mixed cells $\sigma \in \Delta$.

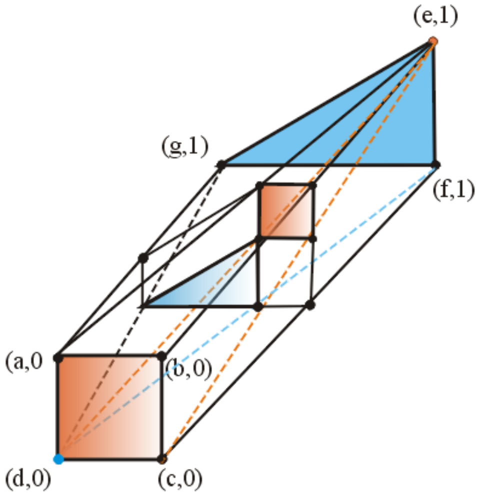
Enumerating mixed subdivisions

The Cayley trick introduces point set $C \subset \mathbb{Z}^{2n}$:

$$C := \{(e_i, a_{ij}) \ : \ i = 0, \ldots, n, \ a_{ij} \in A_i\},$$

where the e_i constitute an affine basis of $\mathbb{N}^n$. So $|C| = |A_0| + \cdots + |A_n|$.

Theorem. The set of all mixed subdivisions of $A_0, \ldots, A_n \subset \mathbb{Z}^n$ corresponds bijectively to the set of all regular triangulations of C.

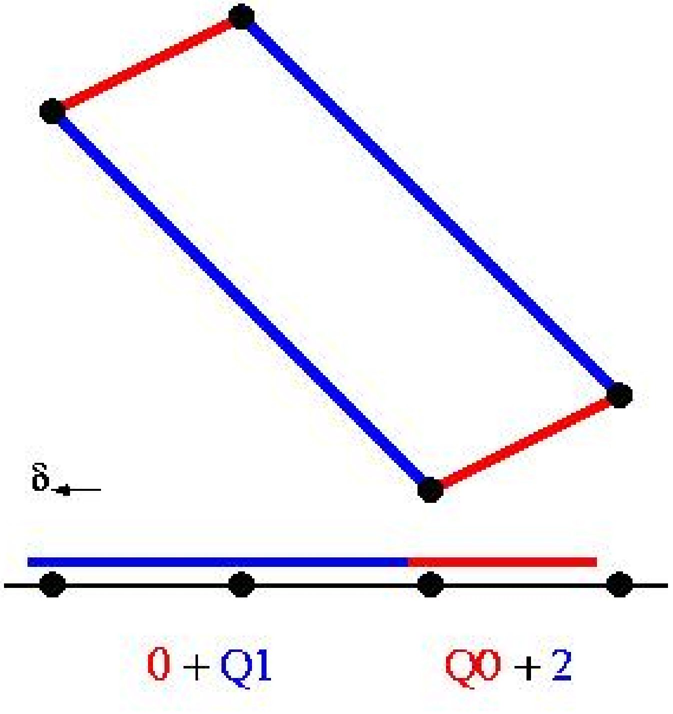


Caley trick in sparse Sylvester case

$f_0 = c_{00} + c_{01}x$,
$f_1 = c_{10} + c_{12}x^2$.

Cayley point set

$$C := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$



Triangulations: $$\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$ shown,

and also $$\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

Resultants

Resultant definition

Given $n+1$ Laurent polynomials $f_0, \dots, f_n \in K[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ with indeterminate coefficients $\vec{c}$, their projective, resp. toric / sparse, *resultant* is the unique (up to sign) irreducible polynomial $R(\vec{c}) \in \mathbb{Z}[\vec{c}]$ such that

$$R(\vec{c}) = 0 \Leftrightarrow \exists\, \xi = (\xi_1, \dots, \xi_n) \in X \,:\, f_0(\xi) = \cdots = f_n(\xi) = 0$$

where the variety X equals:

- the projective space $\mathbb{P}^n$ over the algebraic closure $\overline{K}$,
- resp. the toric variety X, $(\overline{K}^*)^n \subset X \subset \mathbb{P}^N$.

[van der Waerden, Gelfand-Kapranov-Zelevinsky, Cox-Little-O'Shea]

Newton polytope of the toric resultant

Given are supports $A_0, \ldots, A_n$ s.t. $k := \sum_i |A_i|$ and $\dim(\sum_i A_i) = n$. Consider the toric resultant $R \in \mathbb{Z}[c]$ and its Newton polytope in $\mathbb{R}^k$.

Let lifting $l \in \mathbb{R}^k$ define a (tight coherent) mixed subdvision of $Q_0 + \cdots + Q_n$. Consider the trailing monomial of R with respect to l, which corresponds to the vertex of $\text{supp}(R) \subset \mathbb{Z}^k$ with inner normal l.

Theorem. [Sturmfels'94] This trailing monomial is

$$\prod_{i=0}^{n} \prod_{i-\text{mixed } \sigma} \text{coef}(f_i, a_i)^{\text{vol}(\sigma)},$$

where $\text{vol}(\cdot)$ denotes Euclidean volume and the i-mixed cells are $\sigma = F_0 + \cdots + a_i + \cdots + F_n : \dim a_i = 0$.

Newton polytope of the toric resultant (cont'd)

Corollary. A surjection exists from the set of mixed-cell configurations onto the extreme monomials of R (vertices of its Newton polytope).

Corollary. The coefficient of the trailing term is ± 1.

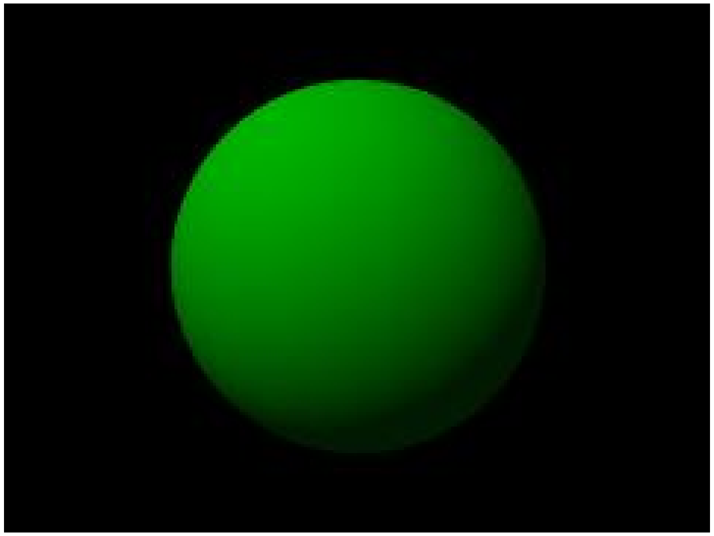
Implicitization

Example: sphere

The sphere in $\mathbb{R}^3$ is the set of values (x, y, z):

$$x = \frac{t_1^2 - t_2^2 - 1}{t_1^2 + t_2^2 + 1}, y = \frac{2t_1}{t_1^2 + t_2^2 + 1}, z = \frac{2t_1 t_2}{t_1^2 + t_2^2 + 1}, \quad t_1, t_2 \in [0, 1],$$

as well as the set of roots of $H(x, y, z) := x^2 + y^2 + z^2 - 1 = 0.$



Modeling/CAD use parametric *and* implicit/algebraic representations

$\Rightarrow$ must implicitize a (hyper)surface given a (rational) parameterization

Implicitization by linear algebra

$S =$ monomials forming (a superset of) the implicit support.
$C =$ unknown coefficients of implicit equation wrt S, $|C| = |S|$.

- $MC = \vec{0}$, where matrix M is $|S| \times |S|$, and contains values of S at points $(s_i, t_i), i = 1, \ldots, |S|$. Try roots of unity.

- $(SS^T)C = \vec{0}$, substitute x, y, z by parametric expressions in $K[s, t]$, integrate over s, t; solve for C [Corless-Galligo-Kotsireas-Watt'00].
Example: supp$(H) \subset \{x^3y, x^3, x^3y^2, y^2z^3\}$, then

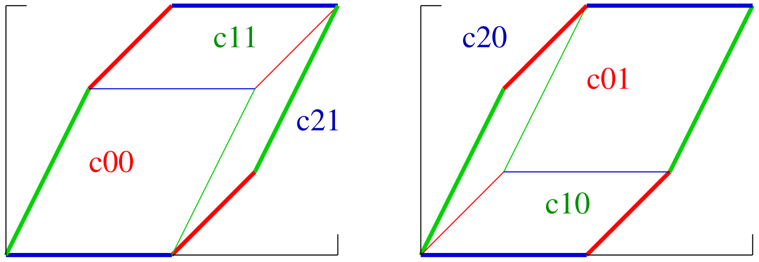
$$SS^T = \begin{bmatrix} x^6y^2 & x^6y & x^6y^3 & x^3y^3z^3 \\ x^6y & x^6 & x^6y^2 & x^3y^2z^3 \\ x^6y^3 & x^6y^2 & x^6y^4 & x^3y^4z^3 \\ x^3y^3z^3 & x^3y^2z^3 & x^3y^4z^3 & y^4z^6 \end{bmatrix} \Rightarrow C = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

- Approximate implicitization [Dokken].

Enumerate mixed subdivisions

A sparse example [Buchberger'88]

$$f_0 = c_{00} - c_{01}st,\ f_1 = c_{10} - c_{11}st^2,\ f_2 = c_{20} - c_{21}s^2.$$

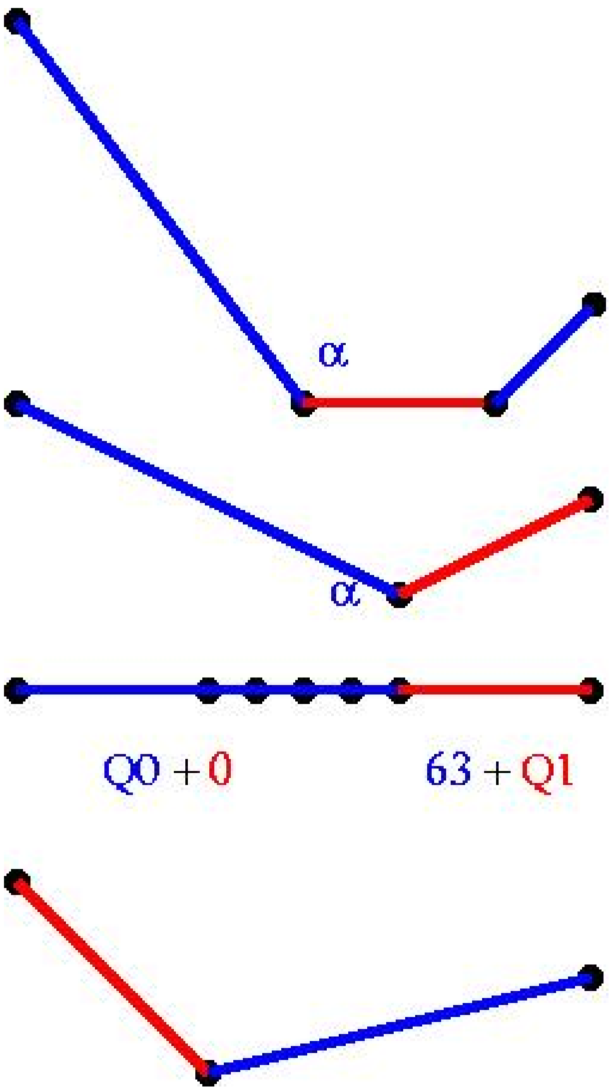


The mixed subdivisions yield extreme monomials $c_{00}^4 c_{11}^2 c_{21},\ c_{01}^4 c_{10}^2 c_{20}$.

The toric resultant turns out to be $R = c_{00}^4 c_{11}^2 c_{21} - c_{01}^4 c_{10}^2 c_{20}$.

The Fröberg-Dickenstein example

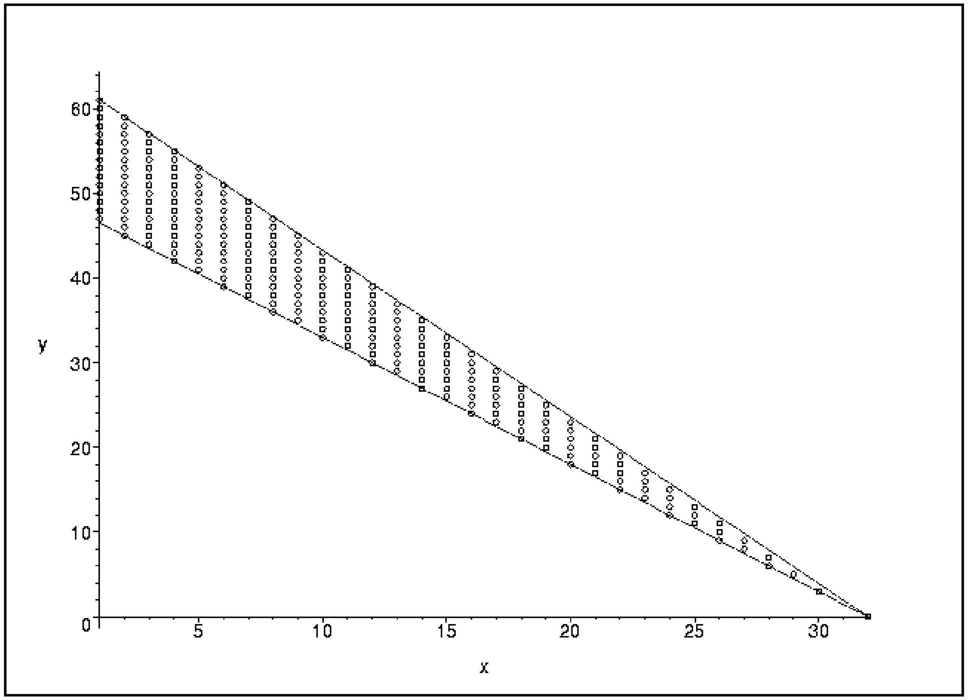
$$x = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, \quad y = t^{32}.$$



$Q_0' + 0, a + Q_1, Q_0'' + 32$	$\pm y^a c^{32}_{0a} c^{63-a}_{1,32}$
$a = 48, 56, 60, 62, 63$	yields $\pm y^a$
$a = 63$	$\pm y^{63} c^{32}_{0,63}$
$0 + Q_1, Q_0 + 32, (a = 0)$	$\pm x^{32} c^{63}_{1,32}$

The Fröberg-Dickenstein example (cont'd)

The projected support is defined by points $(0, 48), (0, 63), (32, 0)$:
This triangle includes 257 lattice points, optimally:



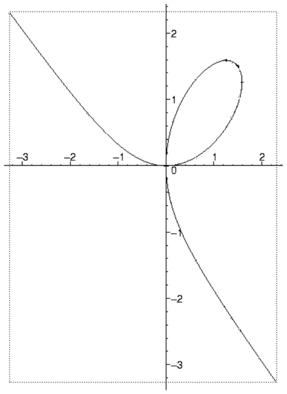
Degree bounds yield quad with additional vertices $(0, 0), (32, 31)$.

Implicitization examples [E-Kotsireas'03]

[Descartes' folium]

[1596-1650]

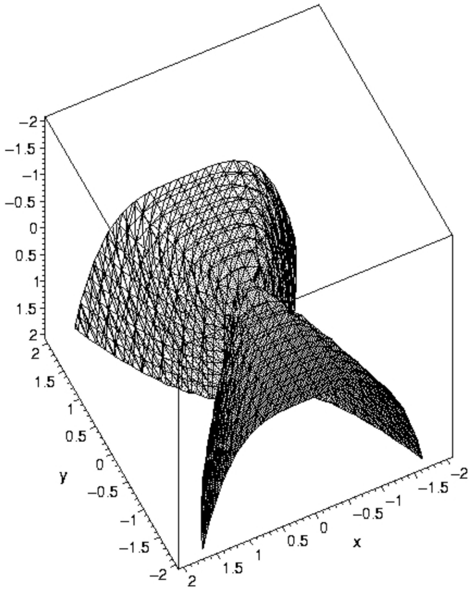
$$(x, y) = \left(\frac{3\ t^2}{t^3 + 1}, \frac{3\ t}{t^3 + 1} \right)$$



$$H = x^3 + y^3 - 3\ x\ y$$

[Buchberger'88]

$$(x, y, z) = (st, st^2, s^2)$$



$$H = x^4 - y^2 z$$

[Busé'01]

$$x = \frac{s^2}{s^3 + t^3},$$

$$y = \frac{s^3}{s^3 + t^3},$$

$$z = \frac{t^2}{s^3 + t^3}$$

$$H = x^3 - 2x^3 y + x^3 y^2 - y^2 z^3$$

Implicit polytope

Consider parameterizations with fixed supports and generic coefficients.

○ Compute the resultant's Newton polytope, then specialize.

– [E-Kotsireas'03] developed Maple code based on Topcom [Rambau].

– [EKP'07] algorithm for projecting polytope to $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$.

● Tropical geometry for the polytope of Laurent-polynomial hypersurfaces and varieties of codim> 1. For curves, determines implicit vertices [Sturmfels-Tevelev-Yu'06].

● Implicit edges for all curves, using linear factors:

– [D'Andrea-Sombra'07] use mixed fiber polytopes [Esterov-Khovanskii'07], characterize polygons which realize as implicit.

– [Dickenstein-Sturmfels'07] use tropical discriminants.

● Specify implicit vertices for all curves [E-Konaxis-Palios'07]

Polynomially parameterized curves

$$x_i = P_i \in K[t],\ i = 0, 1,$$

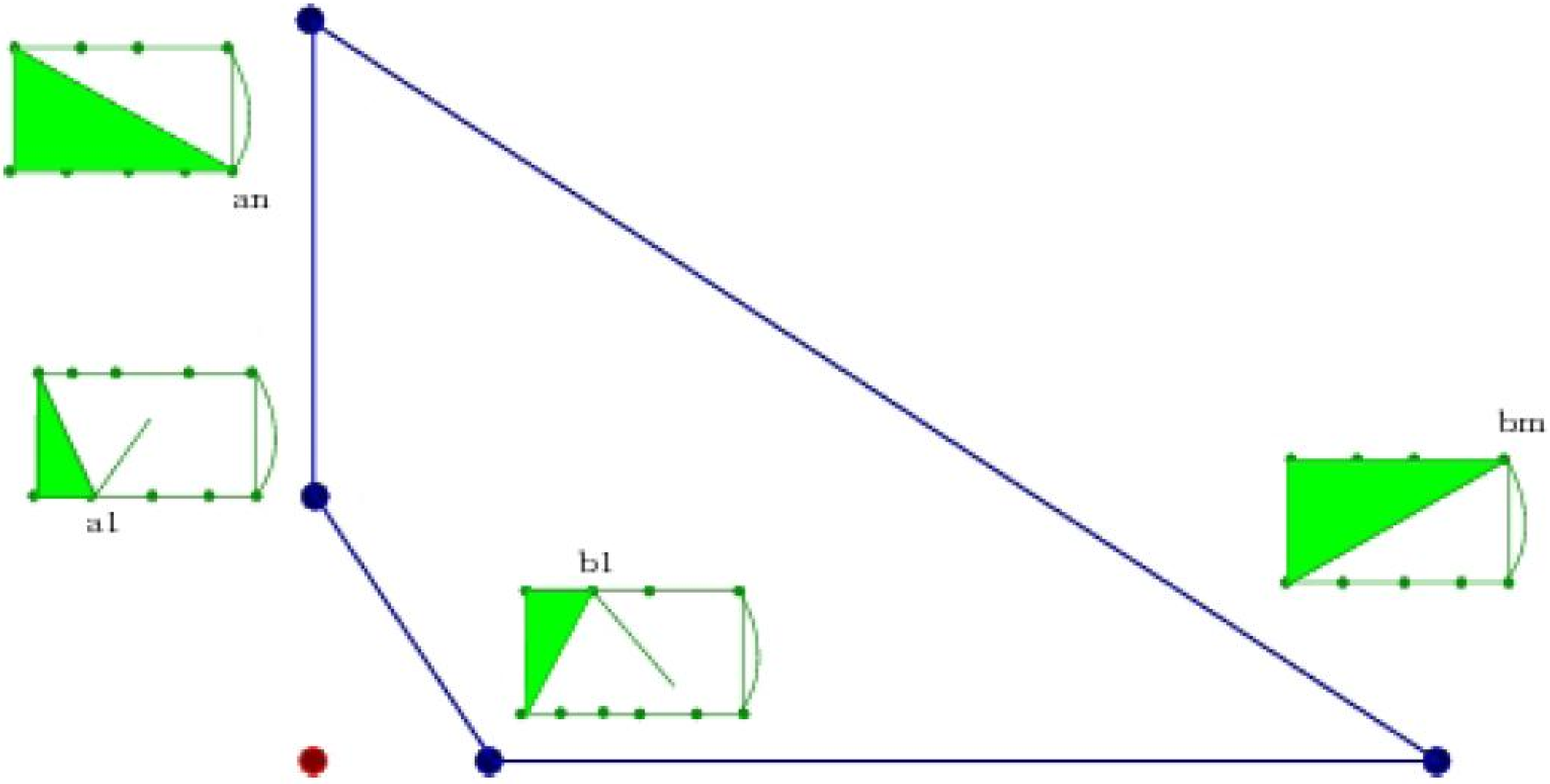
with supports $\{a_0, \ldots, a_n\}, \{b_0, \ldots, b_m\} \subset \mathbb{N}$.

Cayley's trick: $\{(a_0, 0), \ldots, (a_n, 0), (b_0, 1) \ldots, (b_m, 1)\} \subset \mathbb{N}^2$.
Now, consider triangulations of this point set.

Triangle $(a_0, b_i, b_j) \to x^{b_j - b_i}$, $(a_i, a_j, b_0) \to y^{a_j - a_i}$.
Triangles $(a_i, *, b_j), i, j \neq 0$, do not contribute to x, y.
We shall say a_0, b_0 are selected.

Implicit polygon of Polynomial curves

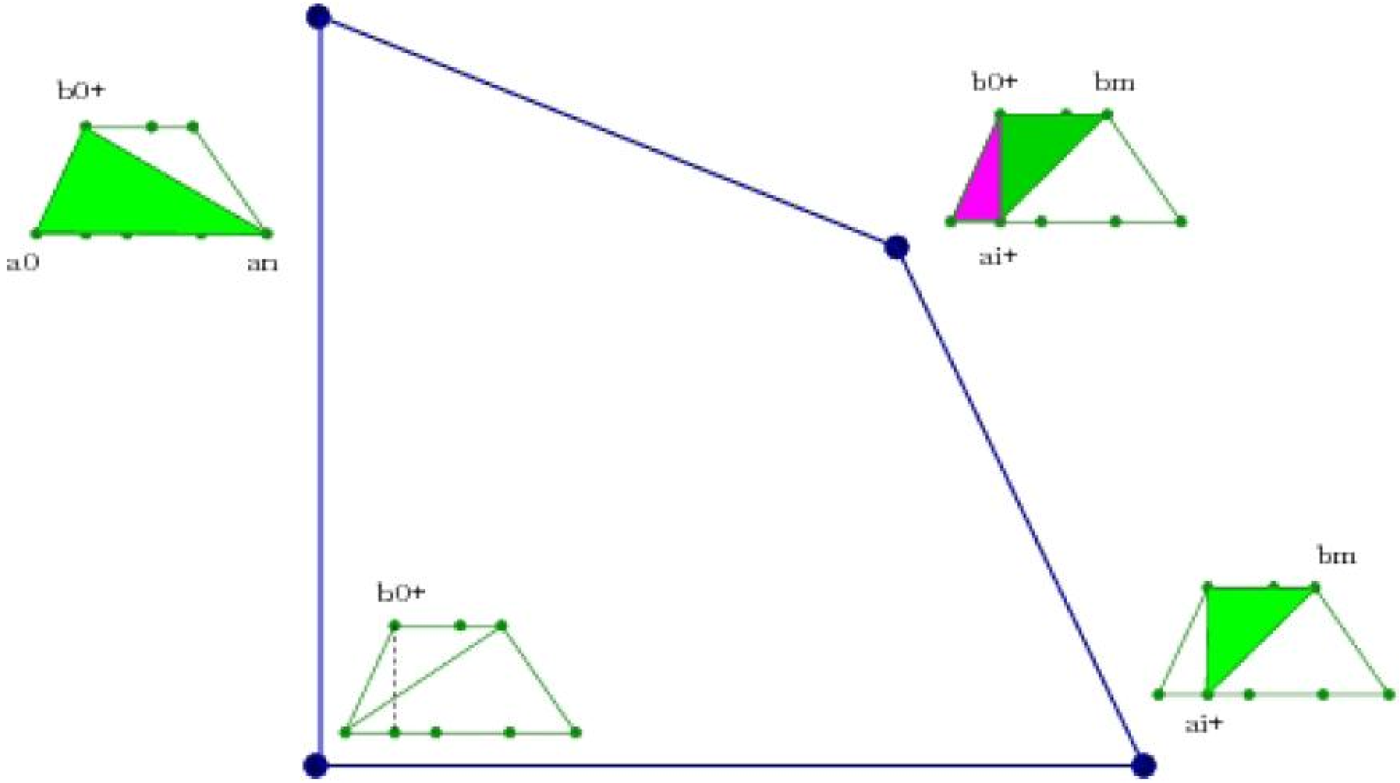
$\exists$ constant term in some $P_i(t) \Rightarrow \exists$ implicit vertex $(0,0)$:



Cor: coef$(x^{b_m}) = \pm(-c_{1m})^{a_n}$, coef$(y^{a_n}) = \pm(-c_{0n})^{b_m}$.

Laurent-polynomial parameterization

$\{a_0, \ldots, a_n\}, \{b_0, \ldots, b_m\} \subset \mathbb{Z}$, unique selected a_i^+, b_0^+.



Up-right vertex $= (b_m, |a_0|)$ iff $\det \begin{bmatrix} |a_0| & a_n \\ |b_0| & b_m \end{bmatrix} > 0$, $(|b_0|, a_n)$ iff $\det < 0$.

Rational curves, different denominators

$$x_i = \frac{P_i(t)}{Q_i(t)}, \text{gcd}(P_i, Q_i) = 1 \;\to\; f_i = x_i Q_i(t) - P_i(t) \in K[t],\; i = 0, 1,$$

$$\text{where } \text{supp}(f_0) = \{a_0, \ldots, a_n\},\; \text{supp}(f_1) = \{b_0, \ldots, b_m\} \subset \mathbb{N}.$$

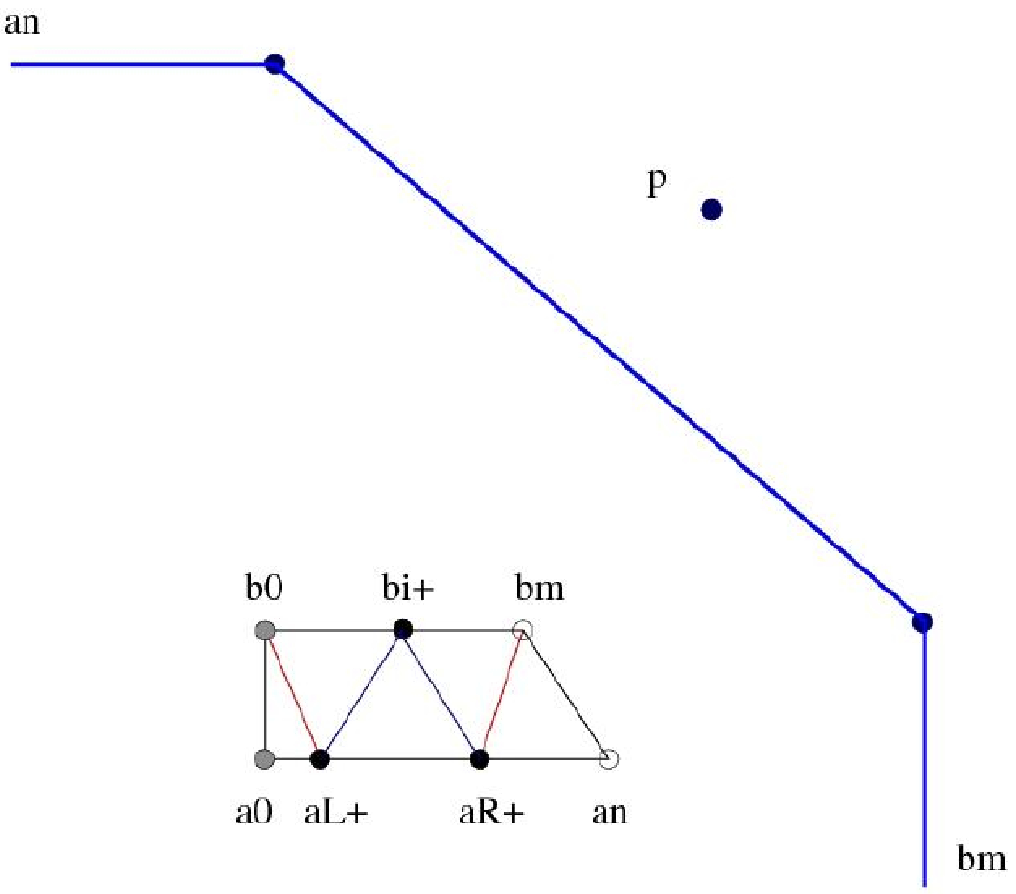
Lemma. Consider direction $(1,1)$. The upper [resp. lower] hull of the Newton polygon has vertices of the form:

$$\left(\sum_{k,l,r} \text{vol}(a_k^+, b_l, b_r), \sum_{l,r,j} \text{vol}(a_l, a_r, b_j^+) \right) \in \mathbb{N}^2,$$

where $a_k^+ \in \text{supp}(Q_0)$ [resp. $\text{supp}(Q_0) - \text{supp}(P_0)$],
and $b_j^+ \in \text{supp}(Q_1)$ [resp. $\text{supp}(Q_1) - \text{supp}(P_1)$]

Proof. The significant resultant coefficients are the following:
the monomials $c_i x_i$ and the binomials $c_i x_i + c_i' \in K[x_i]$
[resp. the monomials $c_i x_i \in K[x_i]$].

Upper-right corner



$$x_{\text{max}} = b_m,$$

$$y_{\text{max}} = (a_R^+ - a_L^+) + \mathcal{X}(b_m^+) \cdot (a_n - a_R^+)$$

if $a_0^-, b_0^-, a_n^-, b_m^-$ then:

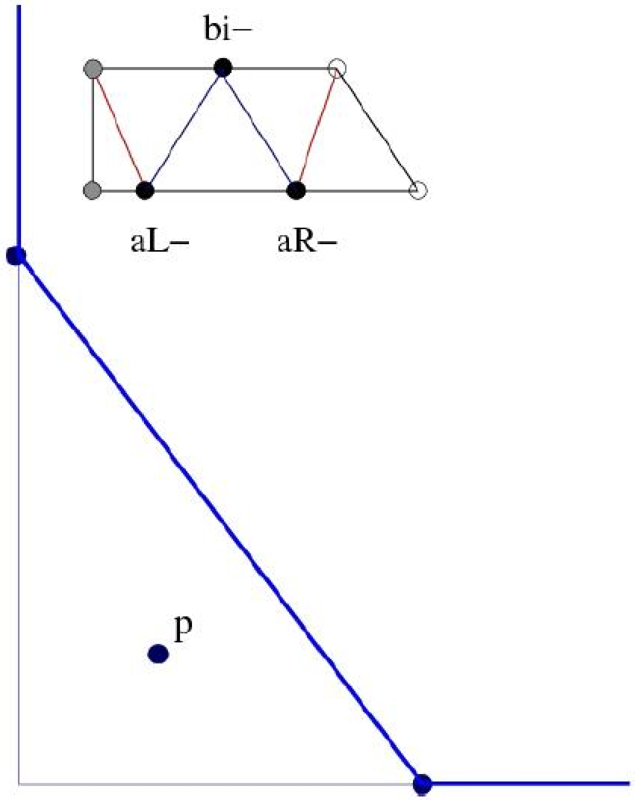
$$\exists\ p = (b_R^+, a_n - a_L^+) \Leftrightarrow$$

$$\det \begin{bmatrix} a_n - a_R^+ & a_L^+ \\ b_L^+ & b_m - b_R^+ \end{bmatrix} > 0,$$

$$p = (b_m - b_L^+, a_R^+) \Leftrightarrow \det < 0$$

Selected $a_k^+ \in \text{supp}(Q_0), b_j^+ \in \text{supp}(Q_1)$, not selected a_k^-, b_j^-.

Lower-left corner



$$x_{\min} = 0,$$

$$y_{\min} = \mathcal{X}(b_0^+)a_L^+ + \\ +\mathcal{X}(b_m^+)(a_n - a_R^-)$$

if $a_0^+, b_0^+, a_n^+, b_m^+$ then:

$$\exists\ p = (b_L^-, a_n - a_R^-) \Leftrightarrow$$

$$\det \begin{bmatrix} a_n - a_R^- & a_L^- \\ b_m - b_R^- & b_L^- \end{bmatrix} < 0,$$

$$p = (b_m - b_R^-, a_L^-) \Leftrightarrow \det > 0$$

Selected $a_k^+ \in \text{supp}(Q_0) - \text{supp}(P_0)$, b_j^+, not selected a_k^-, b_j^-.

Rational curves, common denominator

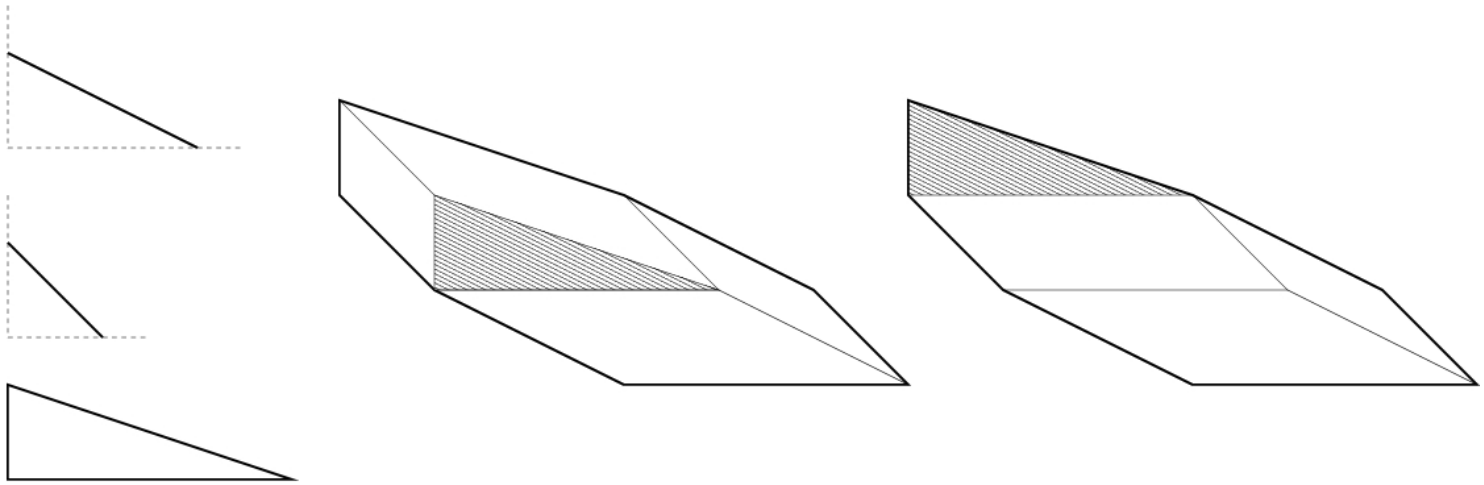
$$x_0 = \frac{P_0(t)}{Q(t)},\ x_1 = \frac{P_1(t)}{Q(t)},\ \mathrm{gcd}(P_i, Q) = 1,$$

$$B_i = \mathrm{supp}(P_i) = \{b_{iL}, \ldots, b_{iR}\}, i = 0, 1,\ B_2 = \mathrm{supp}(Q) = \{b_{2L}, \ldots, b_{2R}\} \subset \mathbb{N}$$

Let $f_0 = x_0 r - P_0(t),\ f_1 = x_1 r - P_1(t),\ f_2 = r - Q(t) \in K[t, r]$,

$$A_i = \mathrm{supp}(f_i) = \{a_{i0} = (0, 1), a_{iL} = (b_{iL}, 0), \ldots, a_{iR} = (b_{iR}, 0)\}, i = 0, 1, 2$$

Example: Folium of Descartes: $x = \frac{3t^2}{1+t^3}, y = \frac{3t}{1+t^3}$.

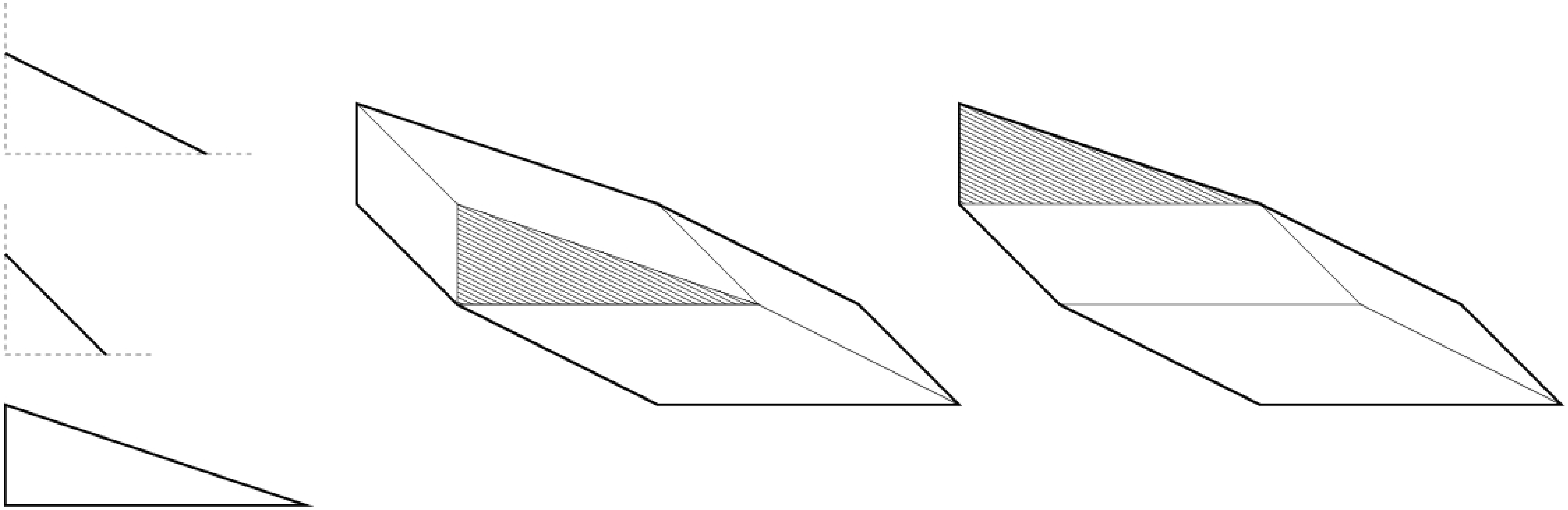


Mixed subdivisions

Lemma. In Minkowski sum $\mathrm{CH}(A_0)+\mathrm{CH}(A_1)+\mathrm{CH}(A_2)$, all a_{i0}-mixed cells are as follows, for $\{i,j,k\}=\{0,1,2\}$:

$$a_{i0}+E_j+E_k, \text{ where edge } E_j=(a_{j0},a_{jt})\subset A_j,$$

and $E_k\subset A_k$ is either non-horizontal (a_{k0},a_{km}), or horizontal (a_{kl},a_{km}).



Cor. It suffices to consider subdivisions of segment $((0,2),(u,2))$.

Implicit equation

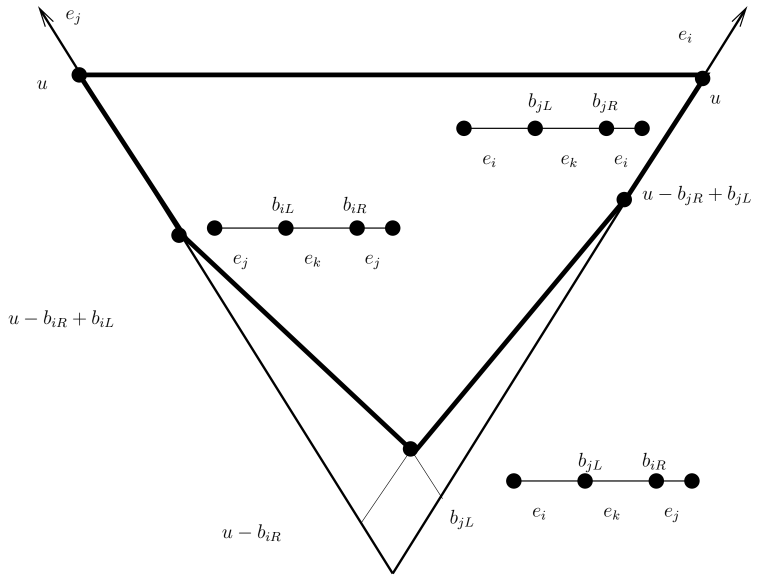
Lemma. Implicit equation $\phi \in K[x_0, x_1]$ has

$$\deg \phi = \mathrm{vol}[\mathrm{CH}(\cup_i \mathrm{supp}(f_i))] = \max_i\{b_{iR}\} - \min_i\{b_{iL}\} = u - 0.$$

Cor. $\mathrm{Res}_{t,r}(f_0, f_1, f_2)$ is homogeneous, of degree u, in the 3 coefficients corresponding to the a_{00}, a_{10}, a_{20}.
$\Phi :=$ specialization of $\mathrm{Res}_{t,r}(f_0, f_1, f_2)$ to polynomial in the 3 coeffs:
$N(\Phi) \subset \mathbb{R}^3$ projects bijectively to $N(\phi) \subset \mathbb{R}^2$.

Thm. For computing the vertices of $N(\phi)$ (or $N(\Phi)$), it suffices to consider subdivisions defined by the vertices of the $\mathrm{CH}(A_i), i = 0, 1, 2$, i.e. the relevant liftings of each A_i are linear in $\mathbb{R}^2$.

Implicit vertices (A)



Thm. If all CH$(B_i \cup B_j) = [0, u]$, then $N(\phi) =$CH$((0,0),(0,u),(u,0))$. Otherwise, if $\exists!\, B_k = [0, u]$, then the (e_i, e_j)-vertices of $N(\phi)$ lie in

$$\{(u, 0), (0, u), (0, u - b_{iR} + b_{iL}), (b_{jL}, u - b_{iR}), (u - b_{jR} + b_{jL}, 0)\},$$

where $\{i, j, k\} = \{0, 1, 2\}$, and $b_{iL}(u - b_{jR}) \geq b_{jL}(u - b_{iR})$.

Implicit vertices (B)

Thm. If $\forall\, B_t \neq [0, u]$, then choose $\{i, j, k\} = \{0, 1, 2\}$ s.t.:

$$0 < b_{iL} \le b_{iR} = u, 0 = b_{jL} \le b_{jR} < u, 0 \le b_{kL} \le b_{kR} < u.$$

If $b_{kL} > 0$, the (e_i, e_j)-vertices lie in

$$\{(b_{jR}, 0), (b_{kR}, u - b_{kR}), (b_{kL}, u - b_{kL}), (0, u - b_{0L}), (0, 0)\}.$$

If $b_{kL} = 0$, the 3rd and 4th vertices are replaced by $(0, u)$.

Implicit polygon cuts

Corollary. Start with a triangle or quadrilateral that has a vertex at $(0,0)$ and incident edges which lie on the axes.

- Polynomial parameterizations:

Take a right triangle, apply at most one corner cut excluding the origin.

- Rational parameterizations with equal denominators:

Take a right triangle, apply at most two cuts (same or different corners).

- Rational parameterizations with different denominators:

Take a quadrilateral, apply at most two cuts (same or different corners).

Conclusions

Inverse. If $\phi(x,y)$ admits a polynomial parameterization, then $N(\phi)$ has one edge on its upper hull wrt $(1,1)$.
Then, if $N(\phi) =$ segment, it contains no interior lattice points.

Future:

- Specify genericity conditions, determine extremal coefficients.
- Polytope of implicit Surfaces.
- When are linear liftings sufficient? Pyramids? Separated variables?
- Project resultant polytope to low dimension.
- Numerical implicitization.

Secondary polytopes

Secondary polytope

Consider the graph of regular triangulations of point-set $C \subset \mathbb{Z}^d$, where edges correspond to (bistellar) flips.

Theorem [Gelfand-Kapranov-Zelevinsky, Billera-Sturmfels]
If C affinely spans $\mathbb{R}^d$, then the graph can be embedded in $\mathbb{R}^{|C|-d-1}$ as the secondary polytope $\Sigma(C)$. For triangulation T,

$$(v_T)_i = \sum_{i \in \mathrm{vtx}(\sigma) : \sigma \in T} \mathrm{vol}(\sigma), \qquad i = 1, \ldots, |C|,$$

where $\mathrm{vtx}(\sigma)$ are the vertices of simplex σ.

E.g. $C \subset \mathbb{Z}^2$, $|C| = 4$:

Circuits

A circuit $Z = \{c_1, \ldots, c_t\}$ is a minimal affinely-dependent subset of C, satisfying $\lambda_1 c_1 + \cdots + \lambda_t c_t = 0$, where $\lambda_i \neq 0$, $\sum_i \lambda_i = 0$.

Z admits triangulations $Z^+ = \{Z \setminus \{c_i\} \mid \lambda_i > 0\}$, $Z^- = \{Z \setminus \{c_i\} \mid \lambda_i < 0\}$. Each flip $T \leftrightarrow T'$ corresponds to precisely one circuit Z s.t.

$$T' \simeq T \setminus Z^+ \cup Z^-$$

E.g. $Z = C$,

$$-\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$Z^+ = T =$

$= T' = Z^-$

Projecting $\Sigma(C)$ to $\mathbb{R}$

Project to 1st coordinate, corresponding to $c_1 \in C \subset \mathbb{Z}^d$ (point set).

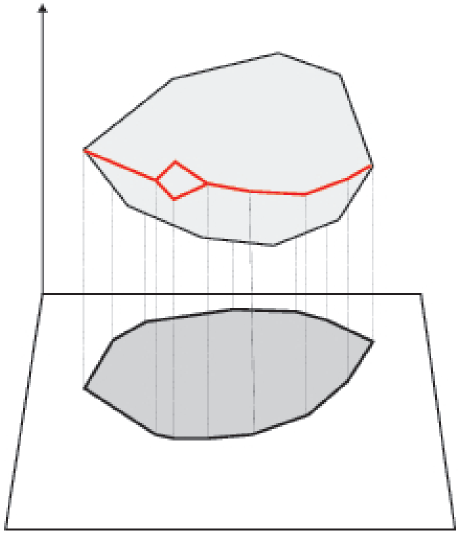
- Let T be a regular triangulation, and consider flip $T \leftrightarrow T'$. Then c_1 is a vertex of every new simplex iff $(v_T)_1 < (v_{T'})_1$.

- Let Z_j be the flips that make $(v)_1$ increase, and σ_j the unique simplex vanishing with Z_j not containing c_1. Then, the triangulation T maximizing $(v_T)_1$ is s.t. the volume of simplices containing σ_j is max.

- Hence $(v)_1 \uparrow$; if strictly$\uparrow$ then min-path.

[E-Konaxis-Palios'07]

Projecting $\Sigma(C)$ to $\mathbb{R}^k, k \geq 2$



- Complexity: Time $= O^*(s^2m)$LP(dim Σ, s), Space $= O(ns)$,

$s =$ max #any-dim simplices$= O(k^n)$, $m =$ #mixed-cell config's.

- Gift-wrapping, CCW$(u, v, w) =$ det $\begin{bmatrix} 1 & u_1 & u_2 \\ 1 & v_1 - u_1 & v_2 - u_2 \\ 1 & w_1 - v_1 & w_2 - v_2 \end{bmatrix}$.

Goal: complexity proportional to #silhouette-points.