Composing Cardinal Direction Relations

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Abstract. We study the recent proposal of Goyal and Egenhofer who presented a model for qualitative spatial reasoning about cardinal directions. Our approach is formal and complements the presentation of Goyal and Egenhofer. We focus our efforts on the operation of composition for two cardinal direction relations. We point out that the only published method to compute the composition does not always work correctly. Then we consider progressively more expressive classes of cardinal direction relations and give composition algorithms for these classes. Our theoretical framework allows us to prove formally that our algorithms are correct. Finally, we demonstrate that in some cases, the binary relation resulting from the composition of two cardinal direction relations cannot be expressed using the relations defined by Goyal and Egenhofer.

1 Introduction

Qualitative spatial reasoning has received a lot of attention in the areas of Geographic Information Systems [3, 5], Artificial Intelligence [2, 4, 16, 17], Databases [14] and Multimedia [18]. Several kinds of useful spatial relations have been studied so far e.g., topological relations [1–3, 16, 17], cardinal direction relations [7, 10] and qualitative distance relations [5, 20].

In this paper we concentrate on *cardinal direction relations* [7, 10, 12]. Cardinal direction relations describe how regions of space are placed relative to one another (e.g., region a is *north of* region b). Our starting point is the recent model of Goyal and Egenhofer [6, 7]. This model is currently one of the most cognitively plausible models for qualitative reasoning with cardinal directions. It works with extended regions and has potential in Multimedia and Geographic Information Systems applications.

In a recent paper [19] we have studied the problem of checking the consistency of a given set of constraints in the model of [6, 7] and its variations. In this paper we concentrate on the problem of *composing* cardinal direction relations.

The composition operation for various kinds of spatial relations has received a lot of attention in the literature [3, 5, 12]. *Composition tables* are typically used as a mechanism for inferring new spatial relations from existing ones. This is very helpful in the following situations:

1. Detecting possible inconsistencies in a given set of spatial relations [10, 17].
2. Preprocessing spatial queries so that inconsistent queries are detected or the search space is pruned [13].

Therefore it is very important to study the composition operation and define appropriate composition tables for interesting models of spatial relations such as [6, 7].

The technical contributions of this paper can be summarized as follows:

- We give formal definitions for the cardinal direction relations that can be expressed in the model of Goyal and Egenhofer [7].
- We use our formal framework to study the composition operation for cardinal direction relations in the model of [7]. Goyal and Egenhofer first studied this operation in [7] but their method does not always work correctly.
- The previous observation leaves us with the task of finding a correct method for computing the composition. To do this, we consider progressively more expressive classes of cardinal direction relations and give composition algorithms for these classes. Our theoretical framework allows us to *prove formally* that our algorithms are correct.
  Finally, when we consider the problem in its generality, we show something very interesting: the binary relation resulting from the composition of some cardinal direction relations *cannot even be expressed* using the relations defined in [7] unless the language is augmented by extra predicates (Section 5).

The rest of the paper is organized as follows. Section 2 presents the formal model. In Sections 3 and 4 we consider two subclasses of cardinal direction relations and give composition algorithms for these classes. In Section 5 we show that the result of the composition of some cardinal direction relations cannot be expressed using the relations defined in [7]. Our conclusions are presented in Section 6.

2 A Formal Model for Cardinal Direction Information

We consider the Euclidean space $\Re^2$. *Regions* are defined as non-empty sets of points in $\Re^2$. A region a is *connected* if for every pair of points in a there exists a line (not necessarily a straight one) joining the two points such that all the points on the line are also in a. The connectedness property prevents a region from having disjoint parts. A region is *disconnected* if it is not connected.

Let a be a region. The *greatest lower bound* or the *infimum* [11] of the *projection* of region a on the x-axis (respectively y-axis) is denoted by $inf_x(a)$ (respectively $inf_y(a)$). The *least upper bound* or the *supremum* of the *projection*

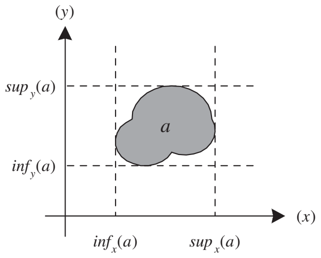


Fig. 1. A region and its bounding box

of region a on the x-axis (respectively y-axis) is denoted by $sup_x(a)$ (respectively $sup_y(a)$).

The *minimum bounding box* of a region a, denoted by $mbb(a)$, is the rectangular region formed by the straight lines $x = inf_x(a)$, $x = sup_x(a)$, $y = inf_y(a)$ and $y = sup_y(a)$ (see Figure 1). Obviously, the projections on the x-axis (respectively y-axis) of a region and its minimum bounding box have the same endpoints.

We will consider throughout the paper the following types of regions:

- Regions that are connected, non-empty subsets of $\Re^2$ and homeomorphic images of two-dimensional disks [7, 15]. The set of these regions will be denoted by REG^{disk}.
- Regions that are formed by finite unions of regions in REG^{disk}. The set of these regions will be denoted by REG^{disk*}. Notice that regions in REG^{disk*} can be disconnected.

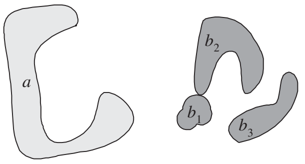


Fig. 2. Regions

In Figure 2, regions a, b_1, b_2 and b_3 are in REG^{disk} (also in REG^{disk*}) and region $b = b_1 \cup b_2 \cup b_3$ is in REG^{disk*}.

Regions in REG^{disk} have been previously studied in [7, 15]. They can be used to model areas in various interesting applications, e.g., land parcels in Geographic Information Systems [3, 5]. In the sequel we will formally define cardinal direction relations for regions in REG^{disk}. To this end, we will need regions in REG^{disk*}.

Let us now consider two arbitrary regions a and b in REG^{disk}. Let region a be related to region b through a cardinal direction relation (e.g., a is north of b). Region b will be called the *reference* region (i.e., the region *to* which the relation is described) while region a will be called the *primary* region (i.e., the region *from* which the relation is described) [7]. The axes forming the minimum bounding box of the reference region b divide the space into 9 tiles (Figure 3a). The peripheral tiles correspond to the eight cardinal direction relations *south*, *southwest*, *west*, *northwest*, *north*, *northeast*, *east* and *southeast*. These tiles will be denoted by $S(b)$, $SW(b)$, $W(b)$, $NW(b)$, $N(b)$, $NE(b)$, $E(b)$ and $SE(b)$ respectively. The central area corresponds to the region's minimum bounding box and is denoted by $B(b)$. By definition each one of these tiles includes the parts of the axes forming it. The union of all 9 tiles is $\Re^2$.

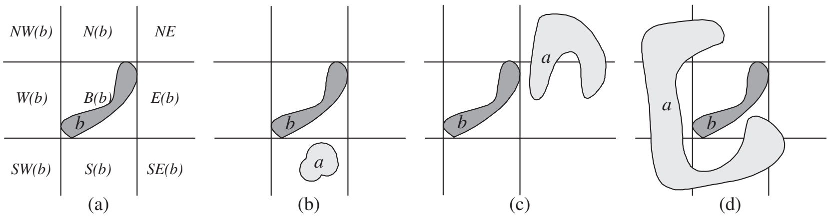


Fig. 3. Reference tiles and relations

If a primary region a is included (in the set-theoretic sense) in tile $S(b)$ of some reference region b (Figure 3b) then we say that *a is south of b* and we write $a\ S\ b$. Similarly, we can define *southwest* (SW), *west* (W), *northwest* (NW), *north* (N), *northeast* (NE), *east* (E), *southeast* (SE) and *bounding box* (B) relations.

If a primary region a lies partly in the area $NE(b)$ and partly in the area $E(b)$ of some reference region b (Figure 3c) then we say that *a is partly northeast and partly east of b* and we write $a\ NE{:}E\ b$.

The general definition of a cardinal direction relation is as follows.

Definition 1. *An* atomic cardinal direction relation *is an element of the set* $\{B, S, SW, W, NW, N, NE, E, SE\}$. *A* basic cardinal direction relation *is an atomic cardinal direction relation or an expression* $R_1{:}\cdots{:}R_k$ *where* $2 \leq k \leq 9$, $R_1, \ldots, R_k \in \{B, S, SW, W, NW, N, NE, E, SE\}$, $R_i \neq R_j$ *for every* i, j *such that* $1 \leq i, j \leq k$ *and* $i \neq j$, *and the tiles* $R_1(b), \ldots, R_k(b)$ *form a region of* REG^{disk} *for any reference region* b.

Example 1. The following are basic cardinal direction relations:

$$S, \quad NE{:}E \quad \text{and} \quad B{:}S{:}SW{:}W{:}NW{:}N{:}E{:}SE.$$

Regions involved in these relations are shown in Figures 3b, 3c and 3d respectively.

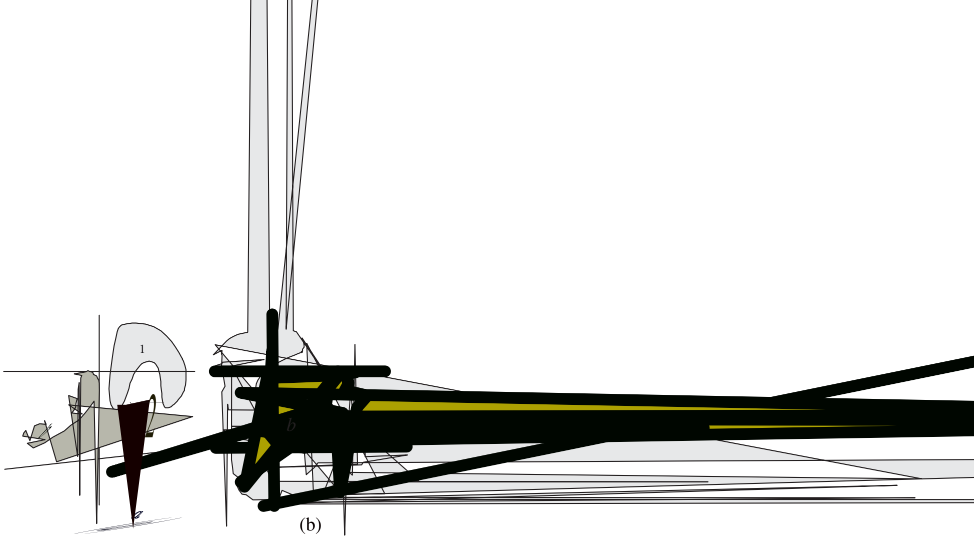


Fig. 4. Relations and ckmponent variables

In order to avoid confusion we will write the atomic elements of a cardinal direction relation according to the following order: $B\delta$ S, SW, W, NW, N, NE, E and SE. Thrs, we always write $B{:}S{:}W$ instead of $W{:}B{:}S$ or $S{:}B{:}W$. The readers should also be aware that for a basic relation such as $B{:}S{:}W$ we will often refer to B, S and W as its *tiles*.

2.1 Defi14ning Cardhnal Direction Relations Formally

Now we can formally define the atomic cardinal direction relations B, S, SW, W, NW, N, NE, E and SE of the model as follows:

$a\ B\ b$ i

$a \ R_1 : \cdot$

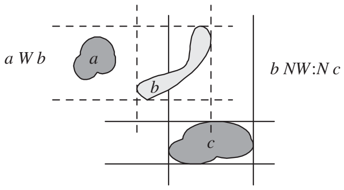


Fig. 5. Composing W with $NW : N$

3 Composing an Atomic with a Basic Cardinal Direction Relation

In Figure 6 we show the composition table for atomic cardinal direction relations [7]. The table uses the function symbol δ as a shortcut. For arbitrary atomic cardinal direction relations $R_1, \ldots, R_k$, the notation $\delta(R_1, \ldots, R_k)$ is a shortcut for the disjunction of all valid *basic* cardinal direction relations that can be constructed by combining atomic relations $R_1, \ldots, R_k$. For instance, $\delta(SW, W, NW)$ stands for the disjunctive relation:

$$\{SW,\ W,\ NW,\ SW{:}W,\ W{:}NW,\ SW{:}W{:}NW\}.$$

Moreover, we define:

$$\delta(\ \delta(R_{11}, \ldots, R_{1k_1}), \delta(R_{21}, \ldots, R_{2k_2}), \ldots, \delta(R_{m1}, \ldots, R_{mk_m})\ ) =$$

$$\delta(R_{11}, \ldots, R_{1k_1}, R_{21}, \ldots, R_{2k_2}, \ldots, R_{m1}, \ldots, R_{mk_m}).$$

Application of the operator δ as it has been defined, suffices for our needs in this paper.

As usual U_{dir} stands for the universal cardinal direction relation. The correctness of the transitivity table can easily be verified using the definitions of Section 2 and the definition of composition (Definition 2).

We now turn our attention to the problem of the composition of an atomic cardinal direction relation with some “well-behaved” basic cardinal direction relations. We will need the following definition.

Definition 3. *A basic cardinal direction relation R is called* rectangular *iff there exist two rectangles (with sides parallel to the x- and y-axes) a and b such that $a\ R\ b$ is satisfied; otherwise it is called* non-rectangular.

Example 2. All atomic relations are rectangular. Relations $B{:}N$ and $B{:}S{:}SW{:}W$ are rectangular while relations $B{:}S{:}SW$ and $B{:}S{:}N{:}SE$ are non-rectangular.

The set of rectangular cardinal direction relations contains the following 36 relations:

$R_1 \setminus R_2$	N	NE	E	SE
N	N	NE	$\delta(NE, E)$	$\delta(NE, E, SE)$
NE	$\delta(N, NE)$	NE	$\delta(NE, E)$	$\delta(NE, E, SE)$
E	$\delta(N, NE$	NE	$\delta(E, B)$	SE
SE	$\delta(N, NE, E, SE, S, B)$	$\delta(NE, E, SE)$	$\delta(E, SE)$	SE
S	$\delta(N, S, B)$	$\delta(NE, E, SE)$	$\delta(E, SE)$	SE
SW	$\delta(S, SW, W, NW, N, B)$	U_{dir}	$\delta(E, SE, S, SW, W, B)$	$\delta(SE, S, SW)$
W	$\delta(NW, N$	$\delta(NW, N, NE)$	$\delta(E, W, B)$	$\delta(SE, S, SW)$
NW	$\delta(NW, N)$	$\delta(NW, N, NE)$	$\delta(W, NW, N, NE, E, B)$	U_{dir}
B	N	NE	E	SE

$R_1 \setminus R_2$	S	SW	W	NW	B
N	$\delta(S, N, B)$	$\delta(SW, W, NW)$	$\delta(W, NW)$	NW	$\delta(N, B)$
NE	$\delta(N, NE, E, SE, S, B)$	U_{dir}	$\delta(W, NW, N, NE, E, B)$	$\delta(NW, N, NE)$	$\delta(B, N, NE, E)$
E	$\delta(SE, S)$	$\delta(SE, S, SW)$	$\delta(W, E, B)$	$\delta(NW, N, NE)$	E
SE	$\delta(SE, S)$	$\delta(SE, S, SW)$	$\delta(E, SE, S, SW, W, B)$	U_{dir}	$\delta(B, S, E, SE)$
S	S	SW	$\delta(SW, W)$	$\delta(SW, W, NW)$	$\delta(S, B)$
SW	$\delta(S, SW)$	SW	$\delta(SW, W)$	$\delta(SW, W, NW)$	$\delta(B, S, SW, W)$
W	$\delta(S, SW)$	SW	W	NW	$\delta(W, B)$
NW	$\delta(S, SW, W, NW, N, B)$	$\delta(SW, W, NW)$	$\delta(W, NW)$	NW	$\delta(B, W, NW, N)$
B	S	SW	W	NW	B

Fig. 6. The composition $R_1 \circ R_2$ of atomic relations R_1 and R_2

$$
\begin{gathered}
\{B,\ S,\ SW,\ W,\ NW,\ N,\ NE,\ E,\ SE, \\
S{:}SW,\ B{:}W,\ NW{:}N,\ N{:}NE,\ B{:}E,\ S{:}SE,\ SW{:}W,\ B{:}S,\ E{:}SE,\ W{:}NW,\ B{:}N,\ NE{:}E, \\
S{:}SW{:}SE,\ NW{:}N{:}NE,\ B{:}W{:}E,\ B{:}S{:}N,\ SW{:}W{:}NW,\ NE{:}E{:}SE, \\
B{:}S{:}SW{:}W,\ B{:}W{:}NW{:}N,\ B{:}S{:}E{:}SE,\ B{:}N{:}NE{:}E, \\
B{:}S{:}SW{:}W{:}NW{:}N,\ B{:}S{:}N{:}NE{:}E{:}SE,\ B{:}S{:}SW{:}W{:}E{:}SE,\ B{:}W{:}NW{:}N{:}NE{:}E, \\
B{:}S{:}SW{:}W{:}NW{:}N{:}NE{:}E{:}SE\}.
\end{gathered}
$$

Lemma 1. *Let R_1 be an atomic and $R_{21}{:}\cdots{:}R_{2l}$ be a basic cardinal direction relation (atomic or non-atomic). The composition of R_1 with $R_{21}{:}\cdots{:}R_{2l}$ can be computed using the formula*

$$R_1 \circ (R_{21}{:}\cdots{:}R_{2l}) = \delta(R_1 \circ R_{21}, \ldots, R_1 \circ R_{2l}) \tag{1}$$

in any of the following cases:

- *(i) if $R_1 \in \{W,\ E\}$ and $R_{21}{:}\cdots{:}R_{2l} \in \{B,\ S,\ SW,\ W,\ NW,\ N,\ NE,\ E,\ SE,\ W{:}NW,\ B{:}N,\ NE{:}E,\ SW{:}W,\ B{:}S,\ E{:}SE,\ NW{:}W{:}SW,\ N{:}B{:}S,\ SE{:}E{:}NE\}$.*
- *(ii) if $R_1 \in \{S,\ N\}$ and $R_{21}{:}\cdots{:}R_{2l} \in \{B,\ S,\ SW,\ W,\ NW,\ N,\ NE,\ E,\ SE,\ NW{:}N,\ N{:}NE,\ B{:}W,\ B{:}E,\ SW{:}S,\ S{:}SE,\ NW{:}N{:}NE,\ W{:}B{:}E,\ S{:}SW{:}SE\}$.*
- *(iii) if $R_1 \in \{NW,\ SW,\ NE,\ SE\}$ and $R_{21}{:}\cdots{:}R_{2l}$ is any atomic cardinal direction relation in $\mathcal{D}^{disk}$.*
- *(iv) if $R_1 = B$ and $R_{21}{:}\cdots{:}R_{2l}$ is any rectangular cardinal direction relation in $\mathcal{D}^{disk}$.*

Proof. For Case (i), let us first assume that $R_1 = W$. We can prove, by a simple case analysis, that for any $R_{21}{:}\cdots{:}R_{2l} \in \{B,\ S,\ SW,\ W,\ NW,\ N,\ NE,\ E,\ SE,\ W{:}NW,\ B{:}N,\ NE{:}E,\ SW{:}W,\ B{:}S,\ E{:}SE,\ NW{:}W{:}SW,\ N{:}B{:}S,\ SE{:}E{:}NE\}$ Equation 1 holds. For instance, if $R_{21}{:}\cdots{:}R_{2l} = SW{:}W$, Equation 1 gives:

$$W \circ SW{:}W = \delta(W \circ SW, W \circ W) = \delta(SW, W) = \{SW,\ W,\ SW{:}W\}$$

This result can be easily verified (see also Figure 7). The case where $R_1 = E$ is symmetric.

Case (ii) is symmetric to Case (i). Finally Cases (iii) and (iv) can trivially be verified. □

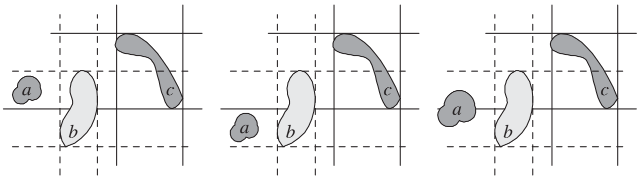


Fig. 7. Composing W with $SW{:}W$

Goyal and Egenhofer [7] apply Equation 1 of Lemma 1 to *all* pairs of atomic and basic cardinal direction relations. Unfortunately, as we have already seen, this is not correct.

We now turn our attention to the composition of an atomic with an arbitrary basic cardinal direction relation. We will first need a few definitions and lemmas before we present the main theorem (Theorem 1).

Definition 4. *Let $R_1 = R_{11}{:}\cdots{:}R_{1k}$ and $R_2 = R_{21}{:}\cdots{:}R_{2l}$ be two cardinal direction relations. R_1* includes *R_2 iff $\{R_{21}, \ldots, R_{2l}\} \subseteq \{R_{11}, \ldots, R_{1k}\}$ holds.*

Example 3. The basic cardinal direction relation $B{:}S{:}SW{:}W$ includes relation $B{:}S{:}SW$.

Definition 5. *Let R be a basic cardinal direction relation. The* bounding relation *of R, denoted by $\mathcal{B}r(R)$ is the smallest rectangular relation (with respect to the number of tiles) that includes R.*

Example 4. The bounding relation of the basic cardinal direction relation $B{:}S{:}SW$ is relation $B{:}S{:}SW{:}W$.

Definition 6. *Let R be a rectangular cardinal direction relation. We will denote the rectangular relation formed by the westernmost tiles of a relation R by $\mathcal{M}ost(W, R)$. Similarly, we can define the rectangular relations $\mathcal{M}ost(S, R)$, $\mathcal{M}ost(N, R)$ and $\mathcal{M}ost(E, R)$. Moreover, we will denote the atomic relation formed by the southwesternmost tiles of a relation R by $\mathcal{M}ost(SW, R)$. Similarly, we can define the atomic relations $\mathcal{M}ost(SE, R)$, $\mathcal{M}ost(NW, R)$ and $\mathcal{M}ost(NE, R)$. Finally, as a special case, we define $\mathcal{M}ost(B, R) = R$.*

Example 5. Let us consider the rectangular relation $B{:}S{:}SW{:}W$. Then according to Definition 6 we have:

$$\begin{array}{rr} \mathcal{M}ost(W, B{:}S{:}SW{:}W) = SW{:}W, & \mathcal{M}ost(S, B{:}S{:}SW{:}W) = S{:}SW, \\ \mathcal{M}ost(N, B{:}S{:}SW{:}W) = B{:}W, & \mathcal{M}ost(E, B{:}S{:}SW{:}W) = B{:}S, \\ \mathcal{M}ost(SW, B{:}S{:}SW{:}W) = SW, & \mathcal{M}ost(SE, B{:}S{:}SW{:}W) = S, \\ \mathcal{M}ost(NW, B{:}S{:}SW{:}W) = W, & \mathcal{M}ost(NE, B{:}S{:}SW{:}W) = B, \end{array}$$
$$\mathcal{M}ost(B, B{:}S{:}SW{:}W) = B{:}S{:}SW{:}W.$$

The following lemma expresses an important property of operator $\mathcal{M}ost$.

Lemma 2. *Let R_1 be an atomic and R_2 be a rectangular cardinal direction relation. Assume that relation $\mathcal{M}ost(R_1, R_2)$ is $Q_1{:}\cdots{:}Q_t$. Then, the composition of R_1 with $\mathcal{M}ost(R_1, R_2)$ can be computed using the formula:*

$$R_1 \circ \mathcal{M}ost(R_1, R_2) = \delta(R_1 \circ Q_1, \ldots, R_1 \circ Q_t) \tag{2}$$

Proof. The proof is by case analysis. Let us first assume that $R_1 = W$. For any $R_2 \in \mathcal{D}^{disk}$, the relation $\mathcal{M}ost(W, R_2)$ will be in one of the following relations:

- *one-tile (atomic!) relations:* B, S, SW, W, NW, N, NE, E, SE.
- *two-tile relations:* $W{:}NW$, $B{:}N$, $NE{:}E$, $SW{:}W$, $B{:}S$, $E{:}SE$.
- *three-tile relations:* $NW{:}W{:}SW$, $N{:}B{:}S$, $SE{:}E{:}NE$.

We can now see that this is a case where the conditions of Lemma 1 hold, so the composition of W with $\mathcal{M}ost(W, R_2)$ can be calculated using Equation 1 of Lemma 1. This is exactly what Equation 2 does.

Similarly, we can prove that Equation 2 holds for any other atomic cardinal direction relation $R_1 \in \{B, S, SW, NW, N, NE, E, SE\}$. In all cases Lemma 1 is used. $\square$

The following is one more useful lemma.

Lemma 3. *Let R_1 be an atomic relation and R_2 be a basic cardinal direction relation. The following implications hold:*

(i) $(\forall a, b, c \in REG^{disk})(\ a\ R_1\ b \wedge b\ R_2\ c \rightarrow$
$$(\exists d \in REG^{disk})(a\ R_1\ d \wedge d\ \mathcal{B}r(R_2)\ c)\ )$$

(ii) $(\forall a, b, c \in REG^{disk})(\ a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c \rightarrow$
$$(\exists d \in REG^{disk})(a\ R_1\ d \wedge d\ R_2\ c)\ )$$

(iii) $(\forall a, b, c \in REG^{disk})(\ a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c \rightarrow$
$$(\exists d \in REG^{disk})(a\ R_1\ d \wedge d\ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c)\ )$$

(iv) $(\forall a, b, c \in REG^{disk})(\ a\ R_1\ b \wedge b\ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c \rightarrow$
$$(\exists d \in REG^{disk})(a\ R_1\ d \wedge d\ \mathcal{B}r(R_2)\ c)\ )$$

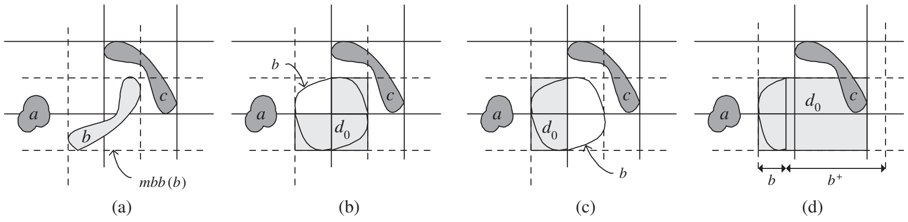


Fig. 8. Proving Lemma 3

Proof. (i) This is easy to see. If $a\ R_1\ b \wedge b\ R_2\ c$ holds, then $a\ R_1\ mbb(b) \wedge mbb(b)\ \mathcal{B}r(R_2)\ c$ also holds.
For instance, if $a\ W\ b \wedge b\ B{:}S{:}SW\ c$ holds (Figure 8a), then $a\ W\ mbb(b) \wedge mbb(b)\ B{:}S{:}SW{:}W\ c$ also holds, where $B{:}S{:}SW{:}W = \mathcal{B}r(B{:}S{:}SW)$.

(ii) Assume that R_2 is $R_{21}{:}\cdots{:}R_{2k}$ where $R_{21}, \ldots, R_{2k}$ are atomic cardinal direction relations. Let a, b, c be regions in REG^{disk} and $a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c$ holds. Consider the tiles $R_{21}(c), \ldots, R_{2k}(c)$ and form the region $d_0 = mbb(b) \cap (R_{21}(c) \cup \cdots \cup R_{2k}(c))$. Then, $a\ R_1\ d_0$ holds because d_0 and b have the same minimum bounding box. Also $d_0\ R_2\ c$ holds by the construction of d_0.
For instance, if $a\ W\ b \wedge b\ \mathcal{B}r(B{:}S{:}SW)\ c$ holds (Figure 8b), then $a\ W\ d_0 \wedge d_0\ B{:}S{:}SW\ c$ also holds, where $d_0 = mbb(b) \cap (B(c) \cup S(c) \cup SW(c))$ (the light grey area of Figure 8b).

(iii) The proof is by case analysis. Let us first assume that $R_1 = W$. Assume also that $\mathcal{M}ost(W, \mathcal{B}r(R_2))$ is $Q_1{:}\cdots{:}Q_t$, where $Q_1, \ldots, Q_t$ are atomic cardinal direction relations. Let a, b, c be regions in REG^{disk} and $a\ R_1\ b \wedge b\ \mathcal{B}r(R_2)\ c$ holds. Let us form region $d_0 = mbb(b) \cap (Q_1(c) \cup \cdots \cup Q_t(c))$. Then, $a\ W\ d_0 \wedge d_0\ \mathcal{M}ost(W, \mathcal{B}r(R_2))\ c$ holds by the definition of $\mathcal{M}ost(W, \mathcal{B}r(R_2))$ (Definition 6) and the construction of d_0. Thus, the implication holds.
For instance, if $a\ W\ b \wedge b\ \mathcal{B}r(B{:}S{:}SW)\ c$ holds (Figure 8c), then $a\ W\ d_0 \wedge d_0\ \mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW))\ c$ also holds, where $SW{:}W = \mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW))$ and $d_0 = mbb(b) \cap (SW(c) \cup W(c))$ (the light grey area of Figure 8b).
Similarly, the lemma holds for any other basic cardinal direction relation $R_1 \in \{B, S, SW, NW, N, NE, E, SE\}$.

(iv) This proof is by case analysis. Let us first assume that $R_1 = W$. Assume also that $\mathcal{B}r(R_2)$ is $Q_1{:}\cdots{:}Q_t$, where $Q_1, \cdots, Q_t$ are atomic cardinal direction relations. Let a, b, c be regions in REG^{disk} and $a\ R_1\ b \wedge b\ \mathcal{M}ost(W, \mathcal{B}r(R_2))\ c$ holds. Let us form a box $b^+ \in REG^{disk}$ such that:

$$sup_y(b^+) = sup_y(b), \quad inf_y(b^+) = inf_y(b)$$
$$inf_x(b^+) = sup_x(b), \quad sup_x(b^+) > sup_x(c)$$

and region $d_0 = (mbb(b) \cup b^+) \cap (Q_1(c) \cup \cdots \cup Q_t(c))$. Then, $a\ W\ d_0 \wedge d_0\ \mathcal{B}r(R_2)\ c$ holds by the construction of d_0. Thus, the implication holds.
For instance, if $a\ W\ b \wedge b\ \mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW))\ c$ holds (Figure 8d), then $a\ W\ d_0 \wedge d_0\ \mathcal{B}r(B{:}S{:}SW))\ c$ also holds, where $B{:}S{:}SW{:}W = \mathcal{B}r(B{:}S{:}SW)$

and $d_0 = (mbb(b) \cup b^+) \cap (B(c) \cup S(c) \cup SW(c) \cup W(c))$ (the light grey area of Figure 8d).

Similarly, the lemma holds for any other basic cardinal direction relation $R_1 \in \{B, S, SW, NW, N, NE, E, SE\}$. The proofs for these cases only vary in the way they construct b^+.

□

Now, after all the necessary definitions and lemmas, we can present our result.

Theorem 1. *Let R_1 be an atomic cardinal direction relation and R_2 be a basic cardinal direction relation. Then*

$$R_1 \circ R_2 = R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2)). \tag{3}$$

Proof. Let $a, c \in REG^{disk}$ and $a\ R_1 \circ R_2\ c$. Then, according to Definition 2 there exists a region $b \in REG^{disk}$ such that:

$$a\ R_1\ b\ \wedge\ b\ R_2\ c.$$

Then, according to Lemma 3(i) there exists a region $e \in REG^{disk}$ such that:

$$a\ R_1\ e\ \wedge\ e\ \mathcal{B}r(R_2)\ c.$$

Then, according to Lemma 3(iii) there exists a region $g \in REG^{disk}$ such that:

$$a\ R_1\ g\ \wedge\ g\ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c.$$

Therefore, we have $a\ R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c$.

Conversely, let $a, c \in REG^{disk}$ and $a\ R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c$. Then, according to Definition 2 there exists a region $b \in REG^{disk}$ such that:

$$a\ R_1\ b\ \wedge\ b\ \mathcal{M}ost(R_1, \mathcal{B}r(R_2))\ c.$$

Then, according to Lemma 3(iv) there exists a region $e \in REG^{disk}$ such that:

$$a\ R_1\ e\ \wedge\ e\ \mathcal{B}r(R_2)\ c.$$

Then, according to Lemma 3(ii) there exists a region $g \in REG^{disk}$ such that:

$$a\ R_1\ g\ \wedge\ g\ R_2\ c.$$

Therefore, we have $a\ R_1 \circ R_2\ c$ and thus the theorem holds. □

The above theorem give us a method to compute the composition $R_1 \circ R_2$ of an atomic cardinal direction relation R_1 with a basic cardinal direction relation R_2. First we have to calculate the relation $\mathcal{M}ost(R_1, \mathcal{B}r(R_2))$. Then we use Lemma 2 and the table of Figure 6 to compute $R_1 \circ R_2$.

We illustrate the above procedure in the following example.

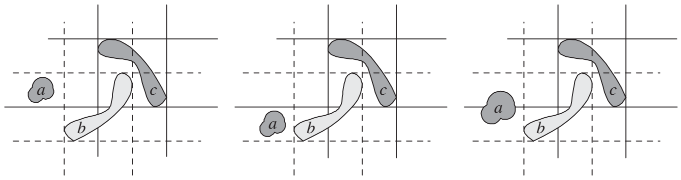


Fig. 9. Composing an atomic with a basic cardinal direction relation

Example 6. Let $R_1 = W$ be an atomic and $R_2 = B{:}S{:}SW$ be a basic cardinal direction relation. Then

$$\mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW)) = SW{:}W.$$

Thus, the composition $W \circ B{:}S{:}SW{:}W$ can be calculated using Theorem 1 as follows:

$$W \circ B{:}S{:}SW{:}W = W \circ SW{:}W.$$

Using Lemma 2 we have:

$$W \circ B{:}S{:}SW{:}W = \delta(W \circ SW, W \circ W).$$

Using the table of Figure 6 we equivalently have:

$$W \circ B{:}S{:}SW{:}W = \delta(SW, W).$$

Finally, expanding operator δ we have:

$$W \circ B{:}S{:}SW{:}W = \{SW,\ W,\ SW{:}W\}.$$

The above equation can be easily verified (see also Figure 9).

4 Composing a Rectangular with a Basic Cardinal Direction Relation

In this section we will study the composition of a rectangular with a basic cardinal direction relation. We will need the following definition.

Definition 7. *Let R_1 and R_2 be two basic cardinal direction relations. The* tile-union *of R_1 and R_2, denoted by tile-union(R_1, R_2), is a relation formed from the union of tiles of R_1 and R_2.*

For instance, if $R_1 = B{:}S{:}SW$ and $R_2 = S{:}SW{:}W$ then *tile-union*$(R_1, R_2) = B{:}S{:}SW{:}W$. Note that the result of *tile-union* is not always a valid cardinal direction relation. For instance, if $R_1 = W$ and $R_2 = E$ then *tile-union*$(R_1, R_2) = W{:}E \notin \mathcal{D}^{disk}$.

Theorem 2. *Let $R_1 = R_{11}{:}\cdots{:}R_{1k}$ be a rectangular and R_2 be a basic cardinal direction relation, where $R_{11}, \ldots, R_{1k}$ are atomic cardinal direction relations. Then*

$$\begin{aligned} R_1 \circ R_2 = \{Q \in \mathcal{D}^{disk} : Q = \textit{tile-union}(s_1, \ldots, s_k) \wedge \\ s_1 \in R_{11} \circ R_2 \wedge \cdots \wedge s_k \in R_{1k} \circ R_2\}. \end{aligned}$$

Proof. The composition $R_1 \circ R_2$ is defined (Definition 2) as follows:

$$R_1 \circ R_2 = \{Q \in \mathcal{D}^{disk} : (\forall a, c)(\ a\ Q\ c \leftrightarrow (\exists b)(a\ R_1\ b \wedge b\ R_2\ c)\ )\}$$

Let $\mathcal{C}_{R_1,R_2} = \{Q \in \mathcal{D}^{disk} : Q = \textit{tile-union}(s_1, \ldots, s_k) \wedge s_1 \in R_{11} \circ R_2 \wedge \cdots \wedge s_k \in R_{1k} \circ R_2\}$. We will prove the following equation:

$$R_1 \circ R_2 = \mathcal{C}_{R_1,R_2} \tag{4}$$

Let us assume that R_1 is $R_{11}{:}\cdots{:}R_{1k}$, where $R_{11}, \ldots, R_{1k}$ are atomic cardinal direction relations. We distinguish two cases.

Case 1 assumes that $k = 1$. Then $R_1 = R_{11}$, i.e., R_1 is an atomic cardinal direction relation and the result is trivial.

Case 2 assumes that $k > 1$. Let $a, c \in REG^{disk}$ and $a\ R_1 \circ R_2\ c$. Then, according to Definition 2 there exists a region $b \in REG^{disk}$ such that

$$a\ R_1\ b\ \wedge\ b\ R_2\ c.$$

Because $a\ R_1\ b$, there exist regions $a_1, \ldots, a_k \in REG^{disk}$ such that $a = a_1 \cup \cdots \cup a_k$ and

$$a_1\ R_{11}\ b \wedge \cdots \wedge a_k\ R_{1k}\ b$$

Taking into account $b\ R_2\ c$, we have:

$$a_1\ R_{11} \circ R_2\ c \wedge \cdots \wedge a_k\ R_{1k} \circ R_2\ c.$$

Therefore, there exist relations $s_1 \in R_{11} \circ R_2, \ldots, s_k \in R_{1k} \circ R_2$ such that:

$$a_1\ s_1\ c \wedge \cdots \wedge a_k\ s_k\ c.$$

Since $a = a_1 \cup \cdots \cup a_k$ holds, it follows that $\textit{tile-union}(s_1, \ldots, s_k)$ is a relation in $\mathcal{D}^{disk}$ and

$$a\ \textit{tile-union}(s_1, \ldots, s_k)\ c.$$

Therefore, we have $a\ \mathcal{C}_{R_1,R_2}\ c$.

The converse is done by case analysis. The first case that we will deal with assumes that $R_1 = SW{:}W$. Let $a, c \in REG^{disk}$ and $a\ \mathcal{C}_{SW:W,R_2}\ c$. We will prove that $a\ SW{:}W \circ R_2\ c$ holds.

Because $a\ \mathcal{C}_{SW:W,R_2}\ c$, there exist relations s_1 and s_2 such that $s_1 \in SW \circ R_2$, $s_2 \in W \circ R_2$ and $a\ \textit{tile-union}(s_1, s_2)\ c$. According to Theorem 1 we have $SW \circ R_2 = SW \circ \mathcal{M}ost(SW, \mathcal{B}r(R_2))$ and $W \circ R_2 = W \circ \mathcal{M}ost(W, \mathcal{B}r(R_2))$. The following tables show all the possible values of $\mathcal{B}r(R_2)$. For each of these values it also presents the values of $\mathcal{M}ost(SW, \mathcal{B}r(R_2))$ (column SW) and $\mathcal{M}ost(W, \mathcal{B}r(R_2))$ (column W).

$\mathcal{B}r(R_2)$	SW	W
B	B	B
S	S	S
SW	SW	SW
W	W	W
NW	NW	NW
N	N	N
NE	NE	NE
E	E	E
SE	SE	SE
$S{:}SW$	SW	SW
$B{:}W$	W	W
$NW{:}N$	NW	NW
$N{:}NE$	N	N
$B{:}E$	B	B
$S{:}SE$	S	S
$SW{:}W$	SW	$SW{:}W$
$B{:}S$	S	$B{:}S$
$E{:}SE$	SE	$E{:}SE$

$\mathcal{B}r(R_2)$	SW	W
$W{:}NW$	W	$W{:}NW$
$B{:}N$	B	$B{:}N$
$NE{:}E$	E	$NE{:}E$
$S{:}SW{:}SE$	SW	SW
$NW{:}N{:}NE$	NW	NW
$B{:}W{:}E$	W	W
$B{:}S{:}N$	S	$B{:}S{:}N$
$SW{:}W{:}NW$	SW	$SW{:}W{:}NW$
$NE{:}E{:}SE$	SE	$NE{:}E{:}SE$
$B{:}S{:}SW{:}W$	SW	$SW{:}W$
$B{:}W{:}NW{:}N$	W	$W{:}NW$
$B{:}S{:}E{:}SE$	S	$B{:}S$
$B{:}N{:}NE{:}E$	B	$B{:}N$
$B{:}S{:}SW{:}W{:}NW{:}N$	SW	$SW{:}W{:}NW$
$B{:}S{:}N{:}NE{:}E{:}SE$	S	$B{:}S{:}N$
$B{:}S{:}SW{:}W{:}E{:}SE$	SW	$SW{:}W$
$B{:}W{:}NW{:}N{:}NE{:}E$	W	$W{:}NW$
$B{:}S{:}SW{:}W{:}NW{:}N{:}NE{:}E{:}SE$	SW	$SW{:}W{:}NW$

Thus, the possible values of expressions $SW \circ R_2$ and $W \circ R_2$ are given by the following tables.

$SW \circ R_2$	$W \circ R_2$
$SW \circ B$	$W \circ B$
$SW \circ S$	$W \circ S$
$SW \circ SW$	$W \circ SW$
$SW \circ W$	$W \circ W$
$SW \circ NW$	$W \circ NW$
$SW \circ N$	$W \circ N$
$SW \circ NE$	$W \circ NE$
$SW \circ E$	$W \circ E$

$SW \circ R_2$	$W \circ R_2$
$SW \circ SE$	$W \circ SE$
$SW \circ SW$	$W \circ SW$
$SW \circ W$	$W \circ W$
$SW \circ NW$	$W \circ NW$
$SW \circ N$	$W \circ N$
$SW \circ B$	$W \circ B$
$SW \circ S$	$W \circ S$
$SW \circ SW$	$W \circ SW{:}W$

$SW \circ R_2$	$W \circ R_2$
$SW \circ S$	$W \circ B{:}S$
$SW \circ SE$	$W \circ E{:}SE$
$SW \circ W$	$W \circ W{:}NW$
$SW \circ B$	$W \circ B{:}N$
$SW \circ E$	$W \circ NE{:}E$
$SW \circ S$	$W \circ B{:}S{:}N$
$SW \circ SW$	$W \circ SW{:}W{:}NW$
$SW \circ SE$	$W \circ NE{:}E{:}SE$

By an enumeration of all these possible values, we can now prove that $a\ SW{:}W \circ R_2\ c$ holds. For instance, assume that $R_2 = B{:}S{:}SW$. Then $SW \circ R_2$ is $SW \circ SW$ and $W \circ R_2$ is $W \circ SW{:}W$ (this case corresponds 16th pair of the above table). We will prove that if $a\ \mathcal{C}_{SW:W,B:S:SW}\ c$ holds then $a\ (SW{:}W \circ B{:}S{:}SW)\ c$ also holds.

If $a\ \mathcal{C}_{SW:W,B:S:SW}\ c$ holds then there exist relations s_1, s_2 such that

$$s_1 \in SW \circ SW = \{SW\}, \quad s_2 \in W \circ SW{:}W = \{SW, W, SW{:}W\}$$

and a *tile-union*(s_1, s_2) c. Thus $s_1 = SW$ and $s_2 \in \{SW, W, SW{:}W\}$. Let us then assume that $s_2 = SW{:}W$ (all other cases can be proved similarly). This situation is illustrated in Figure 10. Using line $y = m$, where $m < inf_x(c)$, we can divide region a into two subregions a_1 and a_2 such that

$$a_1 \ SW \ c \wedge a_2 \ SW{:}W \ c.$$

Let us now form a box $b^+ \in REG^{disk}$ such that:

$$\begin{array}{rr} inf_x(b^+) = max\{sup_x(a_1), sup_x(a_2)\}, & sup_x(b^+) > sup_x(c) \\ inf_y(b^+) = m, & sup_y(b^+) > sup_y(c) \end{array}$$

and region $d_0 = b^+ \cap (B(c) \cup S(c) \cup SW(c))$ (the light grey area of Figure 10). Then $a \ SW{:}W \ d_0 \wedge d_0 \ B{:}S{:}SW \ c$ holds by the construction of d_0. Thus $a \ (SW{:}W \circ B{:}S{:}SW) \ c$ holds.

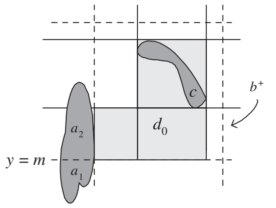


Fig. 10. Proving Theorem 2

Similarly, the theorem holds for any other rectangular and not atomic cardinal direction relation R_1. □

Using Theorem 2 we can easily device Algorithm Compose-Rect-Basic that computes the composition R_3 of a rectangular cardinal direction relation R_1 with a basic cardinal direction relation R_2. Assume that R_1 is $R_{11}{:}\cdots{:}R_{1k}$, where $R_{11}, \ldots, R_{1k}$ are atomic cardinal direction relations. Algorithm Compose-Rect-Basic proceed as follows. Initially the algorithm calculates relations S_i, $1 \leq i \leq k$ as the composition of the atomic relation R_{1i} with the basic cardinal direction relation R_2 (as in Section 2). Subsequently, Algorithm Compose-Rect-Basic forms relations by taking the *tile-union* of an atomic cardinal direction relation s_i, from every cardinal direction relation S_i $(1 \leq i \leq k)$. Finally, the algorithm checks whether the result of the union corresponds to a valid cardinal direction relation. If it does then this relation is added to the result R_3; otherwise it is discarded.

We have implemented Algorithm Compose-Rect-Basic and generated the compositions $R_1 \circ R_2$ for every rectangular cardinal direction relation R_1 and basic cardinal direction relation R_2. The results and the code are available from the authors.

The following is an example of Algorithm Compose-Rect-Basic in operation.

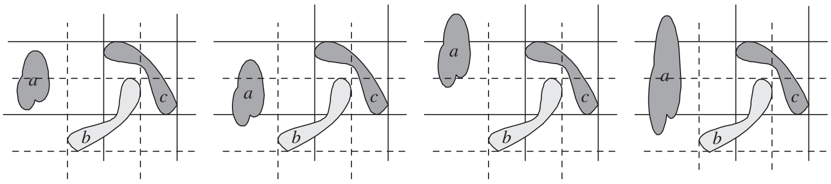


Fig. 11. Composing a rectangular with a basic cardinal direction relation

Example 7. Assume that we want to calculate the composition of the rectangular relation $W{:}NW$ with the basic relation $B{:}S{:}SW$ (Figure 11). We have:

$$S_1 = W \circ B{:}S{:}SW = \delta(W \circ SW, W \circ SW) = \delta(W, SW) = \{SW, W, SW{:}W\}$$
$$S_2 = NW \circ B{:}S{:}SW = NW \circ W = \{W, NW, W{:}NW\}.$$

Now we construct all relations formed by the union of one relation from S_1 and one relation from S_2. These relations are: $SW{:}W$, $SW{:}NW$, $SW{:}W{:}NW$, W, $W{:}NW$, $W{:}NW$, $SW{:}W$, $SW{:}W{:}NW$ and $SW{:}W{:}NW$. From the above relations only the following are valid cardinal direction relations:

$$SW{:}W,\ SW{:}W{:}NW,\ W,\ W{:}NW,\ SW{:}W{:}NW.$$

Therefore, we have:

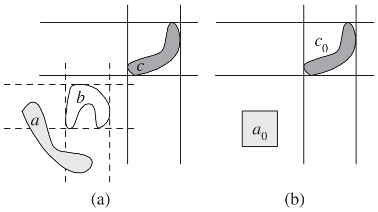
$$W{:}NW{:}N \circ W{:}N = \{W,\ SW{:}W,\ W{:}NW,\ SW{:}W{:}NW\}.$$

The above equation can be easily verified (see also Figure 11).

5 Composing Basic Cardinal Direction Relations

Let us now consider the general question of composing two *non-rectangular* basic cardinal direction relations. For this case we have a very interesting result: the language of cardinal direction constraints (as defined in Section 2) *is not expressive enough* to capture the binary relation which is the result of the composition of non-rectangular basic cardinal direction relations. This is illustrated by the following example.

Example 8. Let us consider region variables a, b, c and cardinal direction constraints $a\ S{:}SW{:}W\ b$ and $b\ SW\ c$ (see Figure 12a). The only cardinal direction constraint implied by these two constraints is $a\ SW\ c$. Thus, someone would be tempted to conclude that $(S{:}SW{:}W \circ SW) = SW$. If this equality was correct then for each pair of regions a_0, c_0 such that $a_0\ SW\ c_0$, there exists a region b_0 such that $a_0\ S{:}SW{:}W\ b_0$ and $b_0\ SW\ c_0$. However Figure 12b shows two such regions a_0 and c_0 such that $a_0\ SW\ c_0$ and it is impossible to find a region b_0 such that $a_0\ S{:}SW{:}W\ b_0$.



(a) (b)

Fig. 12. Illustration of Example 8

If we consider this example carefully, we will notice that the given constraint on a and b implies the following constraint on a: the *area* covered by each region substituted for a cannot be rectangular; it should extend so that it covers tiles $S(b), SW(b)$ and $W(b)$ for any region b. Obviously this constraint is not expressible in the language of cardinal direction relations presented in Section 2.

It is an open question to define an appropriate set of predicates that could be used to augment the constraint language of Section 2 so that the constraints needed to define the result of a composition operation are expressible in all cases.

It is important to point out that the above non-expressibility result should not deter spatial database practitioners who would like to consider adding the cardinal direction relations described in this paper to their system. The discussion of the introduction (i.e., using the inferences of a composition table for spatial relations in order to prune the search space during optimisation of certain queries) still applies but now one has to be careful to say that she is using a constraint propagation mechanism and not a composition table!

Unfortunately, we cannot be as positive about using the cardinal direction relations defined in this paper in the constraint databases frameworks of [8] or [9]. In these frameworks, the class of constraints involved must be closed under the operation of *variable elimination*. Example 8 above demonstrates that this is not true for the class of cardinal direction constraints examined in this paper. For example, if we have constraints

$$a \; S{:}SW{:}W \; b, \quad b \; SW \; c$$

and we eliminate variable b, the result of the elimination is not expressible in the constraint language we started with! So the language of Section 2 needs to be modified in order to be used in a constraint database model. We currently working on extending this language to remove this limitation.

6 Conclusions

In this paper we gave a formal presentation of the cardinal direction model of Goyal and Egenhofer [7]. We used our formal framework to study the composition operation for cardinal direction relations in this model. We considered progressively more expressive classes of cardinal direction relations and gave composition algorithms for these classes. Our theoretical framework allowed us to prove formally that our algorithms are correct. Finally, we have demonstrated that in some cases, the binary relation resulting from the composition of two cardinal direction relations cannot even be expressed using the vocabulary defined in [7].

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