The densest k-subgraph problem on chordal graphs

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joint work with
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2nd **A**thens **C**olloquium on **A**lgorithms and **C**omplexity **2007**

- **Densest k-subgraph (DkS)**:

  Input: A graph $G = (V, E)$, $|V| = n$, and an integer k, $k \leq n$

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DkS remains **NP-hard** even for:

- bipartite graphs, comparability graphs, chordal graphs

  [Corneil and Perl, Discr. Appl. Math. 1984]
- planar graphs

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- bipartite graphs of maximal degree three

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DkS can be solved in **polynomial** time on:

- cographs, split graphs, k-trees

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- Approximation algorithms

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  4. [Ye and Zang, T.R. U. Iowa 1999]
  5. [Asahiro, Iwama, Tamaki and Tokuyama, J. of Algorithms 2000]
  6. [Feige, Kortsarz and Peleg, Algorithmica 2001]
  7. [Feige and Langberg, J. of Algorithms 2001]
  8. [Han, Ye and Zhang, Math. Progr. 2002]
  9. [Billionnet and Roupin, T.R. Cedric-CNAM 2004]

- Best known approximation ratio $\rightarrow$ $\mathbf{O(n^{\frac{1}{3}})}$ [6]

- Greedy algorithm $\rightarrow$ $\mathbf{O(\frac{n}{k})}$ [5]

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- There is **not PTAS** for the DkS problem [Khot, FOCS 2004]
- There is **not any known inapproximability result** for some approximation ratio
- **PTAS**
  - for dense graphs
    - of minimum degree $\Omega(n)$
    - of $\Omega(n^2)$ edges when k is $\Omega(n)$

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A graph is called **chordal** if every cycle of length strictly greater than three possesses a chord, that is, an edge joining two nonconsecutive vertices of the cycle.

A vertex of a graph G is called **simplicial** if its adjacent vertices induce a complete subgraph in G.

1. All maximal cliques of a chordal graph can be found in polynomial time.
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- $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ chordal graph
- $\mathbf{S}$ subset of $|S| = k$ vertices
- $\mathbf{E}(\mathbf{S})$ set of edges in the subgraph of G induced by S
- $\mathbf{E}(\mathbf{A}, \mathbf{B})$ set of edges between two disjoint subsets $A, B \subseteq V$
- $\mathbf{S}^*$ optimal solution to the DkS problem

Greedy Algorithm:

1. $|C_1| \geq |C_2| \geq \ldots \geq |C_m|$, the maximal cliques of G
2. t the largest integer such that $k > |\bigcup_{i=1}^{t-1} C_i| = k'$
3. $S = \bigcup_{i=1}^{t-1} C_i$ plus $k - k' > 0$ vertices of clique C_t

- $L = |C_t|$
- $G_{I;L}$ consisting of at least $\lceil k/L \rceil$ *independent* cliques all of size L

Lemma

Let S and $S_{I;L}$ be the solutions that the Greedy Algorithm returns for the DkS problem on graphs G and $G_{I;L}$, respectively. It holds that $|E(S)| \geq |E(S_{I;L})| = \frac{k(L-1)-b(L-b)}{2}$, where $b = k \bmod L$.

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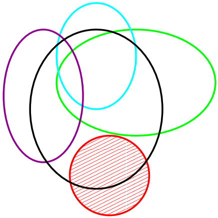
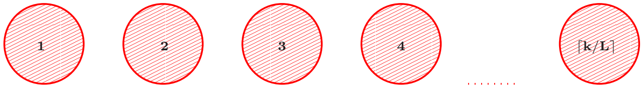


G



$G_{I;L}$

Lemma

Let $c \geq 2$ be the size of a maximum clique of a chordal graph $G = (V, E)$. It holds that $|E| \leq (c-1)(|V| - \frac{c}{2})$ and this bound is the best possible.

proof:

- (double) induction on c and $|V|$

- a chordal graph, has at least one simplicial vertex

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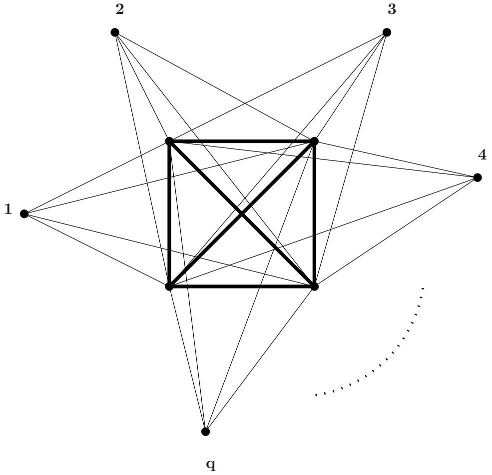
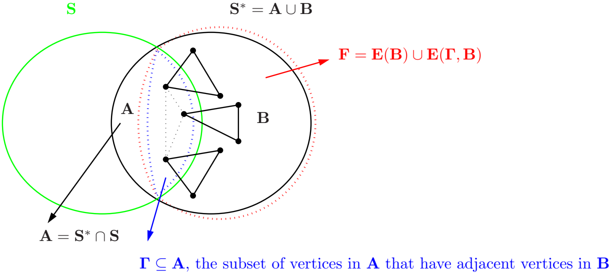
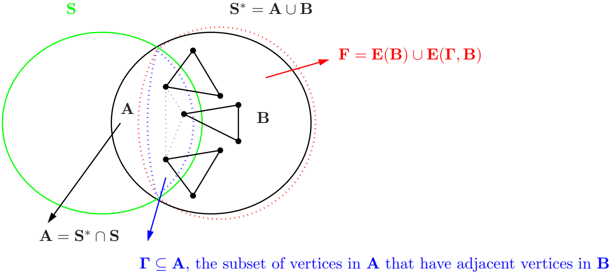


$$c = 5, \quad |V| = q + 4$$



- $G^F = (\Gamma \cup B, F)$ edge-induced subgraph, in general non chordal
- $G_{B \cup \Gamma}$ vertex-induced subgraph, chordal

1. The size of a maximum clique of graph $G^F = (\Gamma \cup B, F)$ is at most L.
2. The degree of a simplicial vertex of the graph $G_{B \cup \Gamma}$ is at most $L - 1$.



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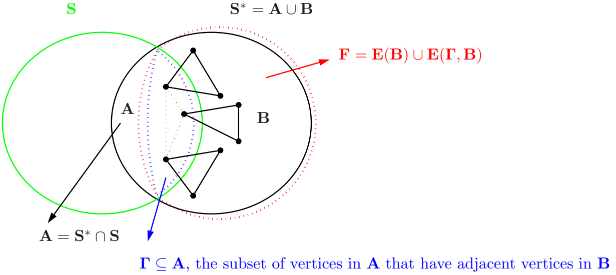

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For the edge-induced graph $G^F = (\Gamma \cup B, F)$ *of* G *it holds that* $|F| \leq (L-1)(|\Gamma \cup B| - \frac{L}{2})$.

proof:

- $|\Gamma \cup B| = |\Gamma| + |B|$
- (double) induction on $|\Gamma|$ and $|B|$
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- $d^F(u) \leq d_{B \cup \Gamma}(w) \leq d_{B \cup \Gamma}(s) \leq L - 1$

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- $|E(S)| \geq \frac{k(L-1)-b(L-b)}{2}$
- $|F| \leq (L-1)(k-\frac{L}{2})$

$$\frac{|E(S^*)|}{|E(S)|} = \frac{|E(A)|+|F|}{|E(S)|} \leq 1 + \frac{|F|}{|E(S)|} \leq 1 + \frac{(L-1)(2k-L)}{k(L-1)-b(L-b)}$$

- $\frac{(L-1)(2k-L)}{k(L-1)-b(L-b)} \leq 2$

Theorem

There is a 3-approximation algorithm for the DkS problem on chordal graphs.

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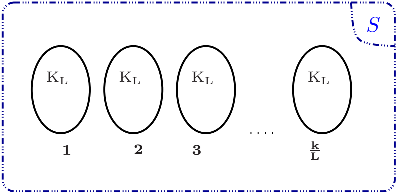
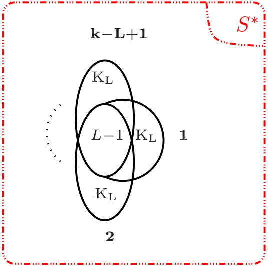


$$E(S) = \frac{k}{L}\frac{L(L-1)}{2} = \frac{k(L-1)}{2}$$



$$E(S^*) = \frac{(L-1)(L-2)}{2} + (k-L+1)(L-1) = (k - \frac{L}{2})(L-1)$$

- What is the complexity of the DkS problem on:
  - (Proper) Interval graphs
  - Permutation graphs

- Approximation algorithms for special graph classes:
  - Bipartite graphs
  - Comparability graphs
  - Planar graphs
  - Bounded degree graphs (even bipartite)
  - Regular graphs

- Better approximation algorithms for arbitrary graphs