Stable Husbands

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Abstract. Suppose n boys and n girls rank each other at random. We show that any particular girl has at least $(\frac{1}{2} - \epsilon) \ln n$ and at most $(1 + \epsilon) \ln n$ different husbands in the set of all Gale/Shapley stable matchings defined by these rankings, with probability approaching 1 as $n \to \infty$, if ϵ is any positive constant. The proof emphasizes general methods that appear to be useful for the analysis of many other combinatorial algorithms.

1. Introduction. This is a tale of n girls and n boys who play a game called "stable matching," invented by Gale and Shapley [3]. Each player ranks each player of the opposite sex according to preference; thus, there are n permutations of the set of boys, representing the preferences of the individual girls, and there are n permutations of the set of girls, representing the preferences of the individual boys. The object of the game is for the boys and girls to match up so as to obtain n marriages that are *stable*, in the sense that no girl and boy prefer each other to their current partners.

For example, suppose Alice, Brigitte, Cindy, and Debra play with Wilfred, Xavier, Yuri, and Zeke; and suppose their preference rankings are as follows, from favorite to least desired:

Alice likes	$Y > X > Z > W$	Wilfred likes	$A > B > D > C$
Brigitte likes	$X > W > Y > Z$	Xavier likes	$C > A > D > B$
Cindy likes	$W > Y > X > Z$	Yuri likes	$B > D > A > C$
Debra likes	$X > W > Z > Y$	Zeke likes	$B > A > C > D$

The matching (AW, BX, CY, DZ) is unstable, because for example A prefers Z to W and at the same time Z prefers A to D. But the matching (AZ, BW, CX, DY) is stable; most of the players are matched with a person other than their first choice, but the objects of their affections don't want to change. The given preferences also admit another stable matching, namely (AY, BW, CX, DZ). In this example only two matchings are stable.

The *stable husbands* of a girl are the boys she can be married to in at least one stable matching. Thus, Alice's stable husbands in the example are Yuri and Zeke. Brigitte has only one stable husband, namely Wilfred; she likes Xavier better, but he can't stand her.

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2. An algorithm. Gale and Shapley [3] gave a procedure to find a stable matching, given any set of preferences; McVitie and Wilson [10] extended the method so that all stable matchings would be found. More recently, Gusfield [5] exploited the interesting lattice structure of stable matchings to construct an elegant algorithm that simultaneously determines the stable husbands of all girls in $O(n^2)$ steps. For our purposes in the present paper it suffices to consider a simplified variant of these procedures, which finds the stable husbands of just one given girl G.

The basic idea is to maintain partial matchings in which each boy who currently has a partner is paired with his best possible choice among all stable matchings of a certain class. One of the boys who doesn't have a current partner is temporarily called P; he will *propose* to one of the girls, and she will then decide whether to accept or to reject his proposal (at least for the time being). The role of P passes from boy to boy according to the following simple rules:

A0. Initially all boys and girls are unpaired.

A1. If at least one boy has no current partner, let P be one such boy. Otherwise all boys and girls are already paired, and we have a stable matching; output G's partner S as one of her stable husbands, then remove the pair GS from the current matching and let $P = S$. (Henceforth we will consider only stable matchings in which G is not married to S.)

A2. If P has already proposed to all the girls, terminate the algorithm. Otherwise let H be the girl P likes best among all those he hasn't approached so far; P now proposes to H.

A3. If girl H has already been proposed to by a boy she prefers to P, she rejects P's proposal. Otherwise, she accepts P, and they become paired in the current matching; her previous partner (if any) now assumes the role of P. If she had no previous partner, the algorithm continues at step A1, otherwise it continues at A2. ☺