Edge pricing of multicommodity networks for selfish users with elastic demands

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Abstract

We examine how to induce selfish heterogeneous users in a multicommodity network to reach an equilibrium that minimizes the social cost. In the absence of centralized coordination, we use the classical method of imposing appropriate taxes (tolls) on the edges of the network. We significantly generalize previous work [26, 17, 12] by allowing user demands to be *elastic*. In this setting the demand of a user is not fixed a priori but it is a function of the routing cost experienced, a most natural assumption in traffic and data networks.

1 Introduction

We examine a network environment where uncoordinated users, each with a specified origin-destination pair, select a path to route an infinitesimal amount of their respective commodity. Let f be a flow vector defined on the paths of the network, which describes a given routing according to the standard multicommodity flow conventions. The users are selfish: each wants to choose a path P that minimizes the cost $T_P(f)$. The quantity $T_P(f)$ depends typically on the latency induced on P by the aggregated flow of all users using some edge of the path.

We model the interaction of the selfish users by studying the system in the steady state captured by the classic notion of a *Wardrop equilibrium* [25]. This state is characterized by the following principle: in equilibrium, for every origin-destination pair (s_i, t_i) the cost on every used $s_i - t_i$ path is equal and less than or equal to the cost on any unused path between s_i and t_i. The Wardrop principle states that in equilibrium the users have no incentive to change their chosen route; under some minor technical assumptions the Wardrop equilibrium concept is equivalent to the Nash equilibrium in the underlying game. The literature on traffic equilibria is very large (see, e.g., [2, 10, 8, 1]). The framework is in principle applicable both to transportation and decentralized data networks. In recent years, starting with the work of Roughgarden and Tardos [22], the latter area motivated a fruitful treatment of the topic from a computer science perspective.

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The behavior of uncoordinated selfish users can incur undesirable consequences from the point of view of the system as a whole. The *social cost* function, usually defined as the total user latency, expresses this societal point of view. Since for several function families [22] one cannot hope that the uncoordinated users will reach a traffic pattern which minimizes the social cost, the system designer looks for ways to induce them to do so. A classic approach, which we follow in this paper, is to impose economic disincentives, namely put nonnegative per-unit-of-flow *taxes* (tolls) on the network edges, see e.g. [2, 16, 27]. The tax-related monetary cost will be, together with the load-dependent latency, a component of the cost function $T_P(f)$ experienced by the users, cf. (1) below. As in [4, 26] we consider the users to be *heterogeneous*, i.e., belonging to classes that have different sensitivities towards the monetary cost. This is expressed by multiplying the monetary cost with a factor $a(i)$ for user class i. We call *optimal* the taxes for which any induced user equilibrium flow minimizes the social cost.

The existence of a vector of optimal edge taxes for heterogeneous users in multicommodity networks is not a priori obvious. It has been established for fixed demands in [26, 12, 17]. In this paper we significantly generalize this previous work by allowing user demands to be *elastic.* Elastic demands have been studied extensively in the traffic community (see, e.g., [13, 1, 16]). In this setting the demand d_i of a user class i is not fixed a priori but it is a function $D_i(u)$ of the vector u of routing costs experienced by the various user classes. Demand elasticity is natural in traffic and data networks. People may decide whether to travel based on traffic conditions. Users requesting data from a web server may stop doing so if the server is slow. Even more elaborate scenarios, such as multi-modal traffic, can be implemented via a judicious choice of the demand functions. E.g., suppose that origin-destination pairs 1 and 2 correspond to the same physical origin and destination points but to different modes of transit, such as subway and bus. There is a total amount d of traffic to be split among the two modes. The modeler could prescribe the modal split by following, e.g., the well-studied logit model [1]:

$$D_1(u) = d\frac{e^{\theta u_1 + A_1}}{e^{\theta u_1 + A_1} + e^{\theta u_2 + A_2}}, \quad D_2(u) = d - D_1(u)$$

for given negative constant θ and nonnegative constants A_1 and A_2. Here u_1 (resp. u_2) denotes the routing cost on all used paths of mode 1 (resp. 2).

For the elastic demand setting we show in Section 3 the existence of taxes that induce the selfish users to reach an equilibrium f that minimizes the total latency for the demands vector $D(u)$ reached at that equilibrium. Note that the total user latency in f is not guaranteed to be minimal globally, when compared with minimum-latency flows for other demand vectors. For our result we only require that the vector $D(u)$ of the demand functions is monotone according to Definition 1 below. The functions $D_i(u)$ do not have to be strictly monotone individually, and for some $i \neq j$, $D_i(u)$ can be increasing while $D_j(u)$ can be decreasing on a particular variable (as for example in the logit model mentioned above). The result is stated in Theorem 1 and constitutes the main contribution of this paper. The structure of the proof is explained at the beginning of Section 3.1.

As mentioned, the equilibrium flow in the elastic demand setting satisfies the demand values that materialize at this equilibrium. These values are not known a priori. One might argue that with high taxes, which increase the routing cost, the actual demand routed (which being elastic depends also on the taxes) will be unnaturally low. This argument does not take fully into account the generality of the demand functions $D_i(u)$ which do not even have to be decreasing; even if they do, they do not have to vanish as u increases. Still it is true that the model is indifferent to potential lost benefit due to users who do not participate. Nevertheless, there are settings where users may decide not to participate without incurring any loss to either the system or themselves and these are settings we model in Section 3. Moreover, in many cases the system designer chooses explicitly to regulate the effective use of a resource instead of heeding the individual welfare of selfish users. Charging drivers in order to discourage them from entering historic city cores is an example, among many others, of a social policy of this type.

A more user-friendly agenda is served by the study of a different social cost function which sums total latency and the lost benefit due to the user demand that was not routed [13, 14]. This setting was recently considered in [6] from a price of anarchy [19] perspective. In this case the elasticity of the demands is specified implicitly through a function $\Gamma_i(x)$ (which is assumed nonincreasing in [6]) for every user class i. $\Gamma_i(d_i)$ determines the minimum per-user benefit extracted if d_i users from the class decide to make the trip. Hence $\Gamma_i(d_i)$ also denotes the maximum travel cost that each of the first d_i users (sorted in order of nonincreasing benefit) from class i is willing to tolerate, in order to travel. This model can be easily fitted into the fixed-demand framework of [26, 12, 17]. For the sake of completeness we show the existence of optimal taxes for this model in Section 4. We demonstrate however that for these optimal taxes to exist, participating users must tolerate, in the worst-case, higher travel costs than those specified by their $\Gamma(\cdot)$ function.

In the second part of the paper we start from the fact that optimal taxes exist and study the efficiency of the tax vector from two aspects which we proceed to explain. This study is limited to the case of fixed demands.

Taxation steers the selfish users towards an equilibrium that minimizes the social cost. The latter function does not take into account the monetary cost incurred by the taxes paid. Nevertheless, it is conceivable that taxes imposed by some central authority are considered by the said authority to be part of the social cost. Taxation could incur disutility to the system as a whole, e.g., if the money paid is detracted from socially beneficial investments. How much does the system hurt itself in terms both of total latency and total taxes paid by letting the selfish users reach a traffic pattern instead of imposing them one? To answer this question, in Section 6 we study the price of anarchy [19] of the game with this modified social cost function (total latency plus total taxes paid). The *price of anarchy* is the ratio of the worst social cost achieved at equilibrium over the minimum social cost. We show that for certain families of latency functions the price of anarchy of the network actually decreases when the optimal taxes are introduced as a component of the social cost. We emphasize that this improvement happens not for an arbitrary set of taxes (one could easily give specific taxation values for which the price of anarchy becomes as small as desired), but for the *specific set of optimal taxes* that in addition drive the users to a minimum total latency pattern. Qualitatively, this reinforces the intuition that the users behave in a less anarchic fashion when optimal taxes are present.

Nevertheless, one can still wonder whether the taxation part of the cost experienced by the network users is quite disproportionate to the latency part. In other words, while optimal taxes help us to drive the users towards a behavior that minimizes the total latency, it may be the case that these taxes are huge (and therefore impractical) compared to this total latency. In Section 7 we show that this is *not* the case for homogeneous users and a wide array of latency functions. More specifically, we revert again to the definition of social cost as the total latency, i.e. a definition which does not include the taxes paid. To quantify the overall effect of taxation we wish to upper bound the ratio

$$\frac{\sum_e f_e^*(l_e(f_e^*) + b_e)}{\sum_e \hat{f}_e l_e(\hat{f}_e)}$$

where f^* is any equilibrium flow reached by the tax-conscious users, b is the vector of optimal taxes, $l_e()$ is the latency function on edge e, and $\hat{f}$ is the flow that minimizes the social cost $\sum_e f_e l_e(f_e)$. Under the assumption that the users are homogeneous (in which case it is well known that the taxes can be set equal to the marginal costs) we give upper bounds for several families of latency functions. In particular for strictly increasing linear latency functions we show a tight upper bound of 2.

Some related work of ours has previously appeared in conference proceedings. The results for fixed demands published in [17] are a corollary of the much more general Theorem 1 of the present paper. The results in Sections 6 and 7 were published in preliminary form in [18].

2 Preliminaries

The model: Let $G = (V, E)$ be a directed network (possibly with parallel edges but with no self-loops), and a set of *users*, each with an infinitesimal amount of traffic (flow) to be routed from an origin node to a destination node of G. Moreover, each user α has a positive *tax-sensitivity* factor $a(\alpha) > 0$. We will assume that the tax-sensitivity factors for all users come from a finite set of possible positive values. We can bunch together into a single *user class* all the users with the same origin-destination pair and with the same tax-sensitivity factor; let k be the number of different such classes. We denote by $\mathcal{P}_i, a(i)$ the flow paths that can be used by class i, and the tax-sensitivity of class i, for all $i = 1, \ldots, k$ respectively. We will also use the term 'commodity i' for class i. Set $\mathcal{P} \doteq \cup_{i=1,\ldots,k} \mathcal{P}_i$. Each edge $e \in E$ is assigned a *latency function* $l_e(f_e)$ which gives the latency experienced by any user that uses e due to congestion caused by the total flow f_e that passes through e. In other words, as in [4], we assume the additive model in which for any path $P \in \mathcal{P}$ the latency is $l_P(f) = \sum_{e \in P} l_e(f_e)$, where $f_e = \sum_{e \in P} f_P$ and f_P is the flow through path P. If every edge is assigned a per-unit-of-flow tax $b_e \geq 0$, a selfish user in class i that uses a path $P \in \mathcal{P}_i$ experiences total cost $T_P(f)$ equal to

$$\sum_{e \in P} l_e(f_e) + a(i) \sum_{e \in P} b_e \tag{1}$$

hence the name 'tax-sensitivity' for the $a(i)$'s: they quantify the importance each user assigns to the taxation of a path.

A function $g : \mathbb{R}^n \to \mathbb{R}^m$ is *positive* if $g(x) > 0$ when $x > 0$. We assume that the functions l_e are strictly increasing, i.e., $x > y \geq 0$ implies $l_e(x) > l_e(y)$, and that $l_e(0) \geq 0$. This implies that $l_e(f_e) > 0$ when $f_e > 0$, i.e., the function l_e is positive.

Definition 1 *Let* $f : K \to \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$. *The function* f *is* monotone *on* K *if* $(x - y)^T(f(x) - f(y)) \geq 0$, $\forall x \in K, y \in K$. *The function* f *is* strictly monotone *if the previous inequality is strict when* $x \neq y$.

In what follows we will use heavily the notion of a nonlinear complementarity problem. Let $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))$ be a vector-valued function from the n-dimensional space $\mathbb{R}^n$ into itself. Then the nonlinear complementarity problem of mathematical programming is to find a vector x that satisfies the following system:

$$x^T F(x) = 0, \quad x \geq 0, \quad F(x) \geq 0.$$

The price of anarchy is formally defined where it is used, i.e., in Section 6.

3 The elastic demand problem

In this section the social cost function is defined as the total latency $\sum_e f_e l_e(f_e)$. We set up the problem in the appropriate mathematical programming framework and formulate the main result for this model in Theorem 1.

The traffic (or Wardrop) equilibria for a network can be described as the solutions of the following mathematical program (see [1] p. 216):

$$\begin{aligned} (T_P(f) - u_i) f_P = 0 &\quad \forall P \in \mathcal{P}_i, i = 1 \ldots k \\ T_P(f) - u_i \geq 0 &\quad \forall P \in \mathcal{P}_i, i = 1 \ldots k \\ \sum_{P \in \mathcal{P}_i} f_P - D_i(u) = 0 &\quad \forall i = 1 \ldots k \\ f, u \geq 0 & \end{aligned}$$

where T_P is the cost of a user that uses path P, f_P is the flow through path P, and $u = (u_1, \ldots, u_k)$ is the vector of shortest travel times (or generalized costs) for the commodities. The first two equations model Wardrop's principle by requiring that for any origin-destination pair i the travel cost for all paths in $\mathcal{P}_i$ with nonzero flow is the same and equal to u_i. The remaining equations ensure that the demands are met and that the variables are nonnegative. Note that the formulation above is very general: every path $P \in \mathcal{P}_i$ for every commodity i has its own T_P (even if two commodities share the same path P, each may have its own T_P).

If the path cost functions T_P are positive and the $D_i(\cdot)$ functions take nonnegative values, [1] shows that the system above is equivalent to the following nonlinear complementarity problem (Proposition 4.1 in [1]):

$$\begin{aligned} (T_P(f) - u_i) f_P = 0 &\quad \forall i, \ \forall P \in \mathcal{P}_i \\ T_P(f) - u_i \geq 0 &\quad \forall i, \ \forall P \in \mathcal{P}_i \\ u_i\Big(\sum_{P \in \mathcal{P}_i} f_P - D_i(u)\Big) = 0 &\quad \forall i \\ \sum_{P \in \mathcal{P}_i} f_P - D_i(u) \geq 0 &\quad \forall i \\ f, u \geq 0 & \end{aligned} \tag{CPE}$$

In our case the costs T_P are defined as $\sum_{e \in P} l_e(f_e) + a(i) \sum_{e \in P} b_e$, $\forall i, \ \forall P \in \mathcal{P}_i$, where b_e is the per-unit-of-flow tax for edge e, and $a(i)$ is the tax sensitivity of commodity i. In fact, as in [17], it will be more convenient to define T_P slightly differently:

$$T_P(f) := \frac{l_P(f)}{a(i)} + \sum_{e \in P} b_e, \quad \forall i,\ \forall P \in \mathcal{P}_i.$$

The special case where $D_i(u)$ is constant for all i, was treated in [26, 17, 12]. The main complication in the general setting is that the minimum-latency flow $\hat{f}$ cannot be considered a priori given before some selfish routing game starts. At an equilibrium the u_i achieve some concrete value which in turn fixes the demands. These demands will then determine the corresponding minimum-latency flow $\hat{f}$. At the same time, the corresponding minimum-latency flow affects the taxes we impose and this, in turn, affects the demands. The outlined sequence of events serves only to ease the description. In fact the equilibrium parameters materialize simultaneously. We should not model the two flows (optimal and equilibrium) as a two-level mathematical program, because there is not the notion of leader-follower here. Instead we use a complementarity problem as done in [1].

Suppose that we are given a vector u^* of generalized costs. Then the social optimum $\hat{f}^*$ for the particular demands $D_i(u^*)$ is the solution of the following mathematical program:

$$\begin{aligned} &\min \sum_{e \in E} l_e(\hat{f}_e)\hat{f}_e && \text{s.t.} && \text{(MP)} \\ &\sum_{P \in \mathcal{P}_i} \hat{f}_P \geq D_i(u^*) && \forall i \\ &\hat{f}_e = \sum_{P \in \mathcal{P}: e \in P} \hat{f}_P && \forall e \in E \\ &\hat{f}_P \geq 0 && \forall P \end{aligned}$$

Under the assumption that the functions $xl_e(x)$ are continuously differentiable and convex, it is well known that $\hat{f}^*$ solves (MP) iff $(\hat{f}^*, \mu^*)$ solves the following pair of primal-dual linear programs (see, e.g., [11, pp. 9–13]):

$$\begin{aligned} &\min \sum_{e \in E} \left(l_e(\hat{f}^*_e) + \hat{f}^*_e \frac{\partial l_e}{\partial f_e}(\hat{f}^*_e) \right) \hat{f}_e \ \text{s.t.} && \text{(LP2)} \\ &\sum_{P \in \mathcal{P}_i} \hat{f}_P \geq D_i(u^*), && \forall i \\ &\hat{f}_e = \sum_{P \in \mathcal{P}: e \in P} \hat{f}_P, && \forall e \in E \\ &\hat{f}_P \geq 0, && \forall P \end{aligned}$$

$$\begin{aligned} &\max \sum_i D_i(u^*)\mu_i && \text{s.t.} && \text{(DP2)} \\ &\mu_i \leq \sum_{e \in P} \left(l_e(\hat{f}^*_e) + \hat{f}^*_e \frac{\partial l_e}{\partial f_e}(\hat{f}^*_e) \right) && \forall i, \forall P \in \mathcal{P}_i \\ &\mu_i \geq 0 && \forall i \end{aligned}$$

Let the functions $D_i(u)$ be bounded and set

$$K_{\hat{f}} := \max_i \max_{u \geq 0} \{D_i(u)\} + 1.$$

Let n denote $|V|$. By optimality, and since functions $l_e(\cdot)$ are positive and strictly increasing, the solutions $\hat{f}^*$, μ^* of (LP2), (DP2) are upper bounded as follows:

$$\hat{f}^*_P \leq \sum_{P \in \mathcal{P}_i} \hat{f}^*_P = D_i(u^*) < K_{\hat{f}}, \ \forall P \in \mathcal{P}_i$$

$$\mu_i \leq \sum_{e\in P}\left(l_e(\hat{f}_e^*) + \hat{f}_e^*\frac{\partial l_e}{\partial f_e}(\hat{f}_e^*)\right) < n \cdot \max_{e\in E}\max_{0\leq x\leq k\cdot K_{\hat{f}}}\{l_e(x) + x\frac{\partial l_e}{\partial f_e}(x)\},\ \forall i$$

It is important to note that these upper bounds are *independent of* u^*.

We wish to find a tax vector b that will steer the edge flow solution of (CPE) towards $\hat{f}$. Similarly to [17] we add this requirement as a constraint to (CPE): for every edge e we require that $f_e \leq \hat{f}_e$. By adding also the complementary slackness conditions for (LP2)-(DP2) that ensure the optimality of $(\hat{f}, \mu)$ for (MP), we obtain the following complementarity problem:

$$\begin{aligned}
f_P(T_P(f) - u_i) &= 0 \quad \forall i, \forall P \in \mathcal{P}_i \qquad\qquad \text{(GENERAL CP)}\\
T_P(f) &\geq u_i \quad \forall i, \forall P \in \mathcal{P}_i\\
u_i(\sum_{P\in\mathcal{P}_i} f_P - D_i(u)) &= 0 \quad \forall i\\
\sum_{P\in\mathcal{P}_i} f_P - D_i(u) &\geq 0 \quad \forall i\\
b_e(f_e - \hat{f}_e) &= 0 \quad \forall e \in E\\
f_e &\leq \hat{f}_e \quad \forall e \in E\\
(\sum_{e\in P}(l_e(\hat{f}_e) + \hat{f}_e\frac{\partial l_e}{\partial f_e}(\hat{f}_e)) - \mu_i)\hat{f}_P &= 0 \quad \forall i, \forall P \in \mathcal{P}_i\\
\sum_{e\in P}(l_e(\hat{f}_e) + \hat{f}_e\frac{\partial l_e}{\partial f_e}(\hat{f}_e)) - \mu_i &\geq 0 \quad \forall i, \forall P \in \mathcal{P}_i\\
\mu_i(\sum_{P\in\mathcal{P}_i} \hat{f}_P - D_i(u)) &= 0 \quad \forall i\\
\sum_{P\in\mathcal{P}_i} \hat{f}_P - D_i(u) &\geq 0 \quad \forall i\\
f_P, b_e, u_i, \hat{f}_P, \mu_i &\geq 0
\end{aligned}$$

where $f_e = \sum_{P\ni e} f_P$, $\hat{f}_e = \sum_{P\ni e}\hat{f}_P$.

The users should be steered towards $\hat{f}$ without being conscious of the constraints $f_e \leq \hat{f}_e$; the latter should be felt only implicitly, i.e., through the corresponding tax b_e. Our main result is expressed in the following theorem. For convenience, we view $D_i(u)$ as the ith coordinate of a vector-valued function $D : \mathbb{R}^k \to \mathbb{R}^k$. The monotonicity of $-D(\cdot)$ mentioned in the ensuing theorem follows Definition 1.

Theorem 1 *Consider the selfish routing game with the latency function seen by the users in class i being*

$$T_P(f) := \sum_{e\in P} l_e(f_e) + a(i)\sum_{e\in P} b_e,\ \forall i,\ \forall P \in \mathcal{P}_i.$$

If (i) for every edge $e \in E$, $l_e(\cdot)$ is a strictly increasing continuous function with $l_e(0) \geq 0$ such that $xl_e(x)$ is convex and continuously differentiable and (ii) D_i are continuous functions bounded from above for all i such that $D(\cdot)$ is positive and $-D(\cdot)$ is monotone then there is a vector of per-unit taxes $b \in \mathbb{R}_+^{|E|}$ such that, if $\bar{f}$ is a traffic equilibrium for this game, $\bar{f}_e = \hat{f}_e,\ \forall e \in E$,. Therefore $\bar{f}$ minimizes the social cost $\sum_{e\in E} f_e l_e(f_e)$.

The main results of [26, 17, 12] follow now as a corollary of Theorem 1 as the special case of constant (positive) demands:

Corollary 1 [26, 17, 12] *Consider the selfish routing game with the latency function seen by the users in class i being*

$$T_P(f) := \sum_{e \in P} l_e(f_e) + a(i) \sum_{e \in P} b_e, \ \forall i, \ \forall P \in \mathcal{P}_i.$$

If for every edge $e \in E$, $l_e(\cdot)$ is a strictly increasing continuous function with $l_e(0) \geq 0$ such that $xl_e(x)$ is convex and continuously differentiable and D_i are fixed constants then there is a vector of per-unit taxes $b \in \mathbb{R}_+^{|E|}$ such that, if $\bar{f}$ is a traffic equilibrium for this game, $\bar{f}_e = \hat{f}_e, \ \forall e \in E,$. Therefore $\bar{f}$ minimizes the social cost $\sum_{e \in E} f_e l_e(f_e)$.

3.1 Proof of Theorem 1

The structure of our proof for Theorem 1 is as follows. First we give two basic Lemmata 1 and 2. We then argue that the two lemmata together with a proof that a solution to (GENERAL CP) exists imply Theorem 1. We establish that such a solution for (GENERAL CP) exists in Theorem 2. The proof of the latter theorem uses the fixed-point method of [24] and arguments from linear programming duality. To improve readability, we give separately the proofs of Lemmata 2,3 and 4 in Section 3.2.

The following result of [1], can be easily extended to our case:

Lemma 1 (Theorem 6.2 in [1]) *Assume that the $l_e(\cdot)$ functions are strictly increasing for all $e \in E$, $D(\cdot)$ is positive and $-D(\cdot)$ is monotone. Then if more than one solutions (f, u) exist for (CPE), u is unique and f induces a unique edge flow.*

In Section 3.2 we prove the following

Lemma 2 *Let $(f^*, b^*, u^*, \hat{f}^*, \mu^*)$ be any solution of (GENERAL CP). Then $\sum_{P \in \mathcal{P}_i} f_P^* = D_i(u^*), \ \forall i$ and $f_e^* = \hat{f}_e^*, \ \forall e \in E$.*

Lemma 2 implies that any *equilibrium flow* $\bar{f}$ for the selfish routing game where the users are conscious of the modified latency

$$T_P(f) := \frac{l_P(f)}{a(i)} + \sum_{e \in P} b_e^*, \ \ \forall i, \ \forall P \in \mathcal{P}_i,$$

is a minimum-latency solution for *the demand vector reached in the same equilibrium.* Therefore the b^* vector would be the vector of the optimal taxes. To complete the proof of Theorem 1 it suffices to show the existence of (at least) one solution to (GENERAL CP). This is established in the following theorem whose proof takes up the rest of Section 3.1.

Theorem 2 *If $f_e l_e(f_e)$ are continuous, convex, strictly monotone functions for all $e \in E$, and $D_i(\cdot)$ are nonnegative continuous functions bounded from above for all i, then (GENERAL CP) has a solution.*

Proof: We can show that (GENERAL CP) is equivalent (in terms of solutions) to the following CP:

$$\begin{aligned}
f_P(T_P(\hat{f}) - u_i) &= 0 \quad \forall i, \forall P \in \mathcal{P}_i \qquad \text{(GENERAL CP}'\text{)}\\
T_P(\hat{f}) &\geq u_i \quad \forall i, \forall P \in \mathcal{P}_i\\
u_i(\sum_{P\in\mathcal{P}_i} f_P - D_i(u)) &= 0 \quad \forall i\\
\sum_{P\in\mathcal{P}_i} f_P - D_i(u) &\geq 0 \quad \forall i\\
b_e(f_e - \hat{f}_e) &= 0 \quad \forall e \in E\\
f_e &\leq \hat{f}_e \quad \forall e \in E\\
(\sum_{e\in P}(l_e(\hat{f}_e) + \hat{f}_e \frac{\partial l_e}{\partial f_e}(\hat{f}_e)) - \mu_i)\hat{f}_P &= 0 \quad \forall i, \forall P \in \mathcal{P}_i\\
\sum_{e\in P}(l_e(\hat{f}_e) + \hat{f}_e \frac{\partial l_e}{\partial f_e}(\hat{f}_e)) - \mu_i &\geq 0 \quad \forall i, \forall P \in \mathcal{P}_i\\
\mu_i(\sum_{P\in\mathcal{P}_i} \hat{f}_P - D_i(u)) &= 0 \quad \forall i\\
\sum_{P\in\mathcal{P}_i} \hat{f}_P - D_i(u) &\geq 0 \quad \forall i\\
f_P, b_e, u_i, \hat{f}_P, \mu_i &\geq 0
\end{aligned}$$

The only difference between (GENERAL CP) and (GENERAL CP′) is that $T_P(f) = \sum_{e\in P}(\frac{l_e(f_e)}{a(i)} + b_e)$ is replaced by $T_P(\hat{f}) = \sum_{e\in P}(\frac{l_e(\hat{f}_e)}{a(i)} + b_e)$ in the first two constraints.

Lemma 3 *(GENERAL CP) is equivalent to (GENERAL CP′).*

To show that (GENERAL CP′) has a solution, we will follow a classic proof method by Todd [24] that reduces the solution of a complementarity problem to a fixed-point problem. In what follows, let $[x]^+ := \max\{0, x\}$. If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x))$ is a function with components $\phi_1, \ldots, \phi_n$ defined as

$$\phi_i(x) = [x_i - F_i(x)]^+,$$

then $\hat{x}$ is a fixed point to ϕ iff $\hat{x}$ solves the complementarity problem

$$x^T F(x) = 0, F(x) \geq 0, x \geq 0.$$

We would like to apply Brouwer's fixed-point theorem (Proposition 6.6 in [3]) with ϕ mapping a compact and convex set to itself. Unfortunately, ϕ is defined on a convex set but it does not necessarily map a compact set into itself. To circumvent this difficulty we follow [1]: we will restrict the domain of ϕ to a large cube, clearly a compact and convex set, with an artificial boundary. We will also force all values of ϕ to fall in this cube. We will show that the fixed points of this restricted version of ϕ are fixed points of the original ϕ by showing that no such fixed point falls on the artificial boundary of the cube.

Note that for (GENERAL CP$'$) $x = (f, u, b, \hat{f}, \mu)$. We start by defining the cube which will contain x. Let

$$K_f := K_{\hat{f}},$$
$$K_\mu := n \cdot \max_{e\in E} \max_{0\le x\le k\cdot K_{\hat{f}}} \{l_e(x) + x\frac{\partial l_e}{\partial f_e}(x)\}$$

Let S be the maximum possible entry of the inverse of any ± 1 matrix of dimension at most $(k+m)\times(k+m)$, where m denotes $|E|$ (note that S depends only on $(k+m)$.) Also, let $a_{max} = \max_i\{1/a(i)\}$ and $l_{max} = \max_e\{l_e(k\cdot K_f)\}$. Then define

$$K_b := (k+m)Sma_{max}l_{max} + 1,$$
$$K_u := n \cdot \left(\max_{e\in E, i\in\{1,\ldots,k\}} \left\{\frac{l_e(k\cdot K_f)}{a(i)}\right\} + K_b\right) + 1$$

We allow x to take values from the cube $\{0 \le f_P \le K_f, P\in\mathcal{P}\}, \{0\le u_i \le K_u, i = 1,\ldots,k\}, \{0\le b_e\le K_b, e\in E\}, \{0 \le \hat{f}_P \le K_{\hat{f}}, P\in\mathcal{P}\}, \{0\le \mu_i \le K_\mu, i = 1,\ldots,k\}$. We define $\phi = (\{\phi_P : P\in\mathcal{P}\}, \{\phi_i : i = 1,\ldots,k\}, \{\phi_e : e\in E\}, \{\phi_{\hat{P}} : P\in\mathcal{P}\}, \{\phi_{\hat{i}} : i = 1,\ldots,k\})$ with $|\mathcal{P}|+k+m+|\mathcal{P}|+k$ components as follows:

$$\phi_P(f,u,b,\hat{f},\mu) = \min\{K_f, [f_P + u_i - T_P(\hat{f})]^+\} \qquad \forall i, \forall P\in\mathcal{P}_i$$
$$\phi_i(f,u,b,\hat{f},\mu) = \min\{K_u, [u_i + D_i(u) - \sum_{P\in\mathcal{P}_i} f_P]^+\} \qquad i = 1,\ldots,k$$
$$\phi_e(f,u,b,\hat{f},\mu) = \min\{K_b, [b_e + f_e - \hat{f}_e]^+\} \qquad \forall e\in E$$
$$\phi_{\hat{P}}(f,u,b,\hat{f},\mu) = \min\{K_{\hat{f}}, [\hat{f}_P + \mu_i - \sum_{e\in P}(l_e(\hat{f}_e) + \hat{f}_e\frac{\partial l_e}{\partial f_e}(\hat{f}_e))]^+\} \qquad \forall i, \forall P\in\mathcal{P}_i$$
$$\phi_{\hat{i}}(f,u,b,\hat{f},\mu) = \min\{K_{\hat{i}}, [\mu_i + D_i(u) - \sum_{P\in\mathcal{P}_i} \hat{f}_P]^+\} \qquad i = 1,\ldots,k$$

where $f_e = \sum_{P\ni e} f_P$, $\hat{f}_e = \sum_{P\ni e}\hat{f}_P$. By Brouwer's fixed-point theorem, there is a fixed point x^* in the cube defined above, i.e., $x^* = \phi(x^*)$. In particular we have that $f^*_P = \phi_P(x^*), u^*_i = \phi_i(x^*), b^*_e = \phi_e(x^*), \hat{f}^*_P = \phi_{\hat{P}}(x^*), \mu^*_i = \phi_{\hat{i}}(x^*)$ for all $P, \hat{P}\in\mathcal{P}, i = 1,\ldots,k$, $e\in E$.

Following the proof of Theorem 5.3 of [1] we can show that

$$\hat{f}^*_P = [\hat{f}^*_P + \mu^*_i - \sum_{e\in P}(l_e(\hat{f}^*_e) + \hat{f}^*_e\frac{\partial l_e}{\partial f_e}(\hat{f}^*_e))]^+, \ \forall i, \forall P\in\mathcal{P}_i$$
$$\mu^*_i = [\mu^*_i + D_i(u^*) - \sum_{P\in\mathcal{P}_i}\hat{f}^*_P]^+, \ \forall i$$
$$f^*_P = [f^*_P + u^*_i - T_P(\hat{f}^*)]^+, \ \forall i, \forall P\in\mathcal{P}_i. \qquad (2)$$

Note that this implies that $(\hat{f}^*, \mu^*)$ satisfy the complementary slackness conditions for (LP2)-(DP2) for u^*. Here we prove only (2) (the other two are proven in a similar way). Let $f^*_P = K_f$ for some $i, P\in\mathcal{P}_i$ (if $f^*_P < K_f$ then (2) holds). Then $\sum_{P\in\mathcal{P}_i}\hat{f}^*_P > D_i(u^*)$, which implies that $u^*_i + D_i(u^*) - \sum_{P\in\mathcal{P}_i}\hat{f}^*_P < u^*_i$, and therefore by the definition of ϕ_i we have that $u^*_i = 0$. Since $T_P(\hat{f}^*)\ge 0$, this implies that $f^*_P \ge f^*_P + u^*_i - T_P(\hat{f}^*)$. If $T_P(\hat{f}^*) > 0$, the definition of ϕ_P implies that $f^*_P = 0$, a contradiction. Hence it must be the case that $T_P(\hat{f}^*) = 0$, which in turn implies (2).

If there are $i, P \in \mathcal{P}_i$ such that $f^*_P > 0$, then (2) implies that $u^*_i = T_P(\hat{f}^*) = \sum_{e\in P} \frac{l_e(\hat{f}^*_e)}{a(i)} + \sum_{e\in P} b^*_e$. In this case we have that $u^*_i < K_u$, because $u^*_i = K_u \Rightarrow \sum_{e\in P} \frac{l_e(\hat{f}^*_e)}{a(i)} + \sum_{e\in P} b^*_e = n \cdot \left(\max_{e\in E, i\in\{1,\ldots,k\}} \left\{\frac{l_e(K_f)}{a(i)}\right\} + K_b\right) + 1$ which is a contradiction since $b^*_e \leq K_b$. On the other hand, if there are $i, P \in \mathcal{P}_i$ such that $f^*_P = 0$, then (2) implies that $u^*_i \leq T_P(\hat{f}^*)$. Again $u^*_i < K_u$, because if $u^*_i = K_u$ we arrive at the same contradiction. Hence we have that

$$u^*_i = [u^*_i + D_i(u^*) - \sum_{P\in\mathcal{P}_i} f^*_P]^+, \ \forall i. \tag{3}$$

Next, we consider the following primal-dual pair of linear programs:

$$\min \sum_i \sum_{P\in\mathcal{P}_i} f_P \frac{l_P(\hat{f}^*)}{a(i)} \quad \text{s.t.} \qquad (\text{LP}^*)$$

$$\begin{aligned} &\sum_{P\in\mathcal{P}_i} f_P \geq D_i(u^*) && i = 1,\ldots,k \\ &f_e = \sum_{P\in\mathcal{P}:e\in P} f_P && \forall e \in E \\ &f_e \leq \hat{f}^*_e && \forall e \in E \\ &f_P \geq 0 && \forall P \end{aligned}$$

$$\max \sum_i D_i(u^*)u_i - \sum_{e\in E} \hat{f}^*_e b_e \ \text{s.t.} \qquad (\text{DP}^*)$$

$$\begin{aligned} &u_i \leq \frac{l_P(\hat{f}^*)}{a(i)} + \sum_{e\in P} b_e && \forall i, \ \forall P \in \mathcal{P}_i \\ &b_e, u_i \geq 0 && \forall e \in E, \forall i \end{aligned}$$

From the above, it is clear that $\hat{f}^*$ is a feasible solution for (LP*), and (u^*, b^*) is a feasible solution for (DP*). Moreover, since the objective function of (LP*) is bounded from below by 0, (DP*) has at least one bounded optimal solution as well. The following lemma is folklore,[1] but we give a proof in Section 3.2 for the sake of completeness. We show that there is an optimal solution $(\hat{u}, \hat{b})$ of (DP*) such that all the $\hat{b}_e$'s are suitably upper bounded:

Lemma 4 (folklore) *There is an optimal solution $(\hat{u}, \hat{b})$ of (DP*) such that $\hat{b}_e \leq K_b - 1, \ \forall e \in E$.*

Let $\hat{f}$ be an optimal primal solution of (LP*) that corresponds to the optimal dual solution $(\hat{u}, \hat{b})$ of (DP*). Let

$$L(f, u, b) = \sum_i \sum_{P\in\mathcal{P}_i} f_P \frac{l_P(\hat{f}^*)}{a(i)} + \sum_{e\in E} b_e(f_e - \hat{f}^*_e) + \sum_i u_i(D_i(u^*) - \sum_{P\in\mathcal{P}_i} f_P) \tag{4}$$

be the *Lagrangian* of (LP*)-(DP*). Then it is well known that $(\hat{f}, \hat{u}, \hat{b})$ is a *saddle point* for the Lagrangian (see e.g. [21]), i.e.,

$$L(\hat{f}, u, b) \leq L(\hat{f}, \hat{u}, \hat{b}) \leq L(f, \hat{u}, \hat{b}), \ \forall f, u, b. \tag{5}$$

Because $\hat{f}$ is optimal for (LP*) and functions $l_e(\cdot)$ are positive, we have that $\sum_{P\in\mathcal{P}_i} \hat{f}_P = D_i(u^*), \forall i$. Since $\hat{f}$ satisfies the assumptions of Claim 1 in the proof of Lemma 2, we also

[1] In fact, [12] in their Theorem 2 show an even stronger bound for b_e, but here we don't need this stronger result.

obtain that $\hat{f}_e = \hat{f}^*_e, \forall e$. Therefore $L(\hat{f}, \hat{u}, \hat{b}) = \sum_i \sum_{P \in \mathcal{P}_i} \hat{f}_P \frac{l_P(\hat{f}^*)}{a(i)}$. Equation (3) implies that for all i, $D_i(u^*) - \sum_{P \in \mathcal{P}_i} f^*_P \leq 0$, hence

$$L(f^*, \hat{u}, \hat{b}) \leq \sum_i \sum_{P \in \mathcal{P}_i} f^*_P \frac{l_P(\hat{f}^*)}{a(i)} + \sum_{e \in E} (f^*_e - \hat{f}^*_e) \hat{b}_e. \tag{6}$$

Going back to (u^*, b^*) which is feasible for (DP*), we get from weak duality that

$$\sum_i D_i(u^*) u^*_i - \sum_{e \in E} \hat{f}^*_e b^*_e \leq \sum_i \sum_{P \in \mathcal{P}_i} \hat{f}_P \frac{l_P(\hat{f}^*)}{a(i)}. \tag{7}$$

By (2), for all i and $P \in \mathcal{P}_i$, if $f^*_P > 0$, then $u^*_i = T_P(\hat{f}^*) = \sum_{e \in P} \frac{l_e(\hat{f}^*_e)}{a(i)} + \sum_{e \in P} b^*_e$. Also (3) yields that $\sum_{P \in \mathcal{P}_i} f^*_P = D_i(u^*)$ for all i with $u^*_i > 0$. Therefore

$$\sum_i D_i(u^*) u^*_i - \sum_{e \in E} \hat{f}^*_e b^*_e = \sum_i \sum_{P \in \mathcal{P}_i} (f^*_P u^*_i) - \sum_{e \in E} \hat{f}^*_e b^*_e = \sum_i \sum_{P \in \mathcal{P}_i} f^*_P \frac{l_P(\hat{f}^*)}{a(i)} + \sum_{e \in E} (f^*_e - \hat{f}^*_e) b^*_e$$

and then (7) implies that

$$\sum_i \sum_{P \in \mathcal{P}_i} f^*_P \frac{l_P(\hat{f}^*)}{a(i)} + \sum_{e \in E} (f^*_e - \hat{f}^*_e) b^*_e \leq \sum_i \sum_{P \in \mathcal{P}_i} \hat{f}_P \frac{l_P(\hat{f}^*)}{a(i)} = L(\hat{f}, \hat{u}, \hat{b}). \tag{8}$$

If for some edge $e \in E$ $b^*_e = [b^*_e + f^*_e - \hat{f}^*_e]^+$, we have that if $b^*_e > 0$ then $f^*_e = \hat{f}^*_e$, and if $b^*_e = 0$, then $f^*_e \leq \hat{f}^*_e$. If $b^*_e = K_b$ and $b^*_e < [b^*_e + f^*_e - \hat{f}^*_e]^+$, then $f^*_e > \hat{f}^*_e$. Assume that there is at least one edge e such that $b^*_e = K_b$ and $b^*_e < [b^*_e + f^*_e - \hat{f}^*_e]^+$. Then because of Lemma 4 we have that

$$\sum_{e \in E} (f^*_e - \hat{f}^*_e) \hat{b}_e < \sum_{e \in E} (f^*_e - \hat{f}^*_e) b^*_e, \tag{9}$$

which in turn implies that

$$L(f^*, \hat{u}, \hat{b}) \overset{(6),(9)}{<} \sum_i \sum_{P \in \mathcal{P}_i} f^*_P \frac{l_P(\hat{f}^*)}{a(i)} + \sum_{e \in E} (f^*_e - \hat{f}^*_e) b^*_e \overset{(8)}{\leq} L(\hat{f}, \hat{u}, \hat{b})$$

But from the second inequality of (5) we have that $L(\hat{f}, \hat{u}, \hat{b}) \leq L(f^*, \hat{u}, \hat{b})$, a contradiction.

Hence it cannot be the case $b^*_e = K_b$ and $b^*_e > [b^*_e + f^*_e - \hat{f}^*_e]^+$ for any edge e, therefore

$$b^*_e = [b^*_e + f^*_e - \hat{f}^*_e]^+, \ \forall e \in E. \tag{10}$$

Equations (2),(3),(10) imply that $(f^*, u^*, b^*, \hat{f}^*, \mu^*)$ is indeed a solution of (GENERAL CP′), and therefore a solution to (GENERAL CP). This completes the proof of Theorem 2. □

3.2 Proofs of Lemmata 2, 3 and 4

Proof of Lemma 2 The proof of the first part of the lemma is essentially the same as the proof by contradiction of Proposition 4.1 in [1]. Suppose that $\sum_{P \in \mathcal{P}_i} f^*_P > D_i(u^*) \geq 0$ for some i. Then $u^*_i(\sum_{P \in \mathcal{P}_i} f^*_P - D_i(u^*)) = 0 \Rightarrow u^*_i = 0$ and there is a path $P \in \mathcal{P}_i$ such that $f^*_P > 0$. Since $f^*_P \neq 0$ and the $T_P()$ function is positive, $T_P(f^*) > 0 = u^*_i$. Because $(f^*, b^*, u^*, \hat{f}^*, \mu^*)$ is a solution of (GENERAL CP), we have that $f^*_P(T_P(f^*) - u^*_i) = 0 \Rightarrow f^*_P = 0$, a contradiction. Hence $\sum_{P \in \mathcal{P}_i} f^*_P = D_i(u^*),\ \forall i$.

Since f^* is part of a solution for (GENERAL CP), $f^*_e \leq \hat{f}^*_e,\ \forall e \in E$. The following claim is a result of the special nature of $\hat{f}^*$ as a minimizer of the social cost:

Claim 1 *Let $\bar{f}$ be a flow that satisfies the following set of constraints:*

$$\begin{aligned} \sum_{P \in \mathcal{P}_i} f_P \geq D_i(u^*) \qquad & \forall i \in \{1, \ldots, k\} \\ f_e = \sum_{P \in \mathcal{P}: e \in P} f_P \quad & \forall e \in E \\ f_e \leq \hat{f}^*_e \qquad & \forall e \in E \\ f_P \geq 0 \qquad & \forall P \in \mathcal{P} \end{aligned}$$

Then $\bar{f}_e = \hat{f}^*_e,\ \forall e \in E$.

Proof of Claim: Vector $\hat{f}^*$ solves optimally (MP) hence

$$\sum_{e \in E} \hat{f}^*_e l_e(\hat{f}^*_e) \leq \sum_{e \in E} \bar{f}_e l_e(\bar{f}_e) \tag{11}$$

Since $\bar{f}_e \leq \hat{f}^*_e$ and $l_e(\cdot)$ is increasing we obtain that $l_e(\bar{f}_e) \leq l_e(\hat{f}^*_e)$. Since $l_e(\cdot)$ is nonnegative for all $e \in E$, we obtain that

$$\bar{f}_e l_e(\bar{f}_e) \leq \hat{f}^*_e l_e(\hat{f}^*_e),\ \forall e \in E.$$

If for some e, $\bar{f}_e < \hat{f}^*_e$, then because $l_e(\cdot)$ is increasing $0 \leq l_e(\bar{f}_e) \leq l_e(\hat{f}^*_e)$; because $l_e(\cdot)$ is positive $l_e(\hat{f}^*_e) \neq 0$. From these two facts

$$\bar{f}_e l_e(\bar{f}_e) < \hat{f}^*_e l_e(\hat{f}^*_e)$$

and then $\sum_{e \in E} \bar{f}_e l_e(\bar{f}_e) < \sum_{e \in E} \hat{f}^*_e l_e(\hat{f}^*_e)$ which contradicts (11). □

Since f^* satisfies the constraints of Claim 1, we have that $f^*_e = \hat{f}^*_e,\ \forall e \in E$. □

Proof of Lemma 3 If $(f^*, b^*, u^*, \hat{f}^*, \mu^*)$ is a solution of (GENERAL CP), then it is also a solution of (GENERAL CP′) due to Lemma 2. Conversely, if $(\bar{f}, \bar{b}, \bar{u}, \bar{\hat{f}}, \bar{\mu})$ is a solution of (GENERAL CP′), then we can prove the analogue of Lemma 2 for (GENERAL CP′) (we only have to notice that $l_e(\bar{\hat{f}}_e) \geq l_e(\bar{f}_e) > 0$ whenever $\bar{f}_e > 0$, hence if there is $P \in \mathcal{P}_i$ such that $\bar{f}_P > 0$, then $\sum_{e \in P} \frac{l_e(\bar{\hat{f}}_e)}{a(i)} + \sum_{e \in P} \bar{b}_e > 0$). □

Proof of Lemma 4 The proof is almost the same as the proof of Theorem 2 in [12]. Let $(\hat{u}, \hat{b})$ be an optimal basic feasible solution of (DP*). Then the solution $(\hat{u}, \hat{b})$ can be partitioned into two components $(\hat{u}_B, \hat{b}_B), (\hat{u}_N, \hat{b}_N)$, with

$$(\hat{u}_N, \hat{b}_N) = 0, \ (\hat{u}_B, \hat{b}_B) = A_B^{-1} t$$

where A, t are the coefficient matrices for the constraints of (DP*) (i.e., $A[u \ b]^T \leq t$ in (DP*)), B is the set of A rows in the basis of $(\hat{u}, \hat{b})$, and N the rest of the A rows. By observing that A_B is of dimension at most $(k+m) \times (k+m)$ and its entries are all ± 1, we can conclude that the entries of A_B^{-1} are upper-bounded by S, the maximum possible entry of any inverse of any ± 1 matrix of dimension at most $(k+m) \times (k+m)$ (note that S depends only on $(k+m)$.) Recall that $a_{max} = \max_i\{1/a(i)\}$ and $l_{max} = \max_e\{l_e(k \cdot K_f)\}$. Then $t_P = \frac{l_P(\hat{f}^*)}{a(i)} \leq a_{max} m l_{max}, \ \forall i, \forall P \in \mathcal{P}_i$, and therefore

$$\hat{b}_e \leq (k+m) S m a_{max} l_{max} = K_b - 1, \ \forall e \in E.$$

□

4 Optimal taxes for elastic-demand users with participation benefits

In this section the social cost function is defined as the total latency of the participating users plus the lost benefit due to the non-participating users as in [13, 14, 6]. Again we examine the general case of heterogeneous users. We explain first the meaning of equilibrium in this new setting. Let the benefit distribution be $\Gamma_i(x)$: a strictly decreasing[2], continuous function with domain $[0, G_i]$ for $i = 1, \ldots, k$. The quantity G_i is the maximum potential demand for commodity i. Due to elasticity, demand $g_i \leq G_i$ will be actually routed. Think of the users of class i as points on the interval $[0, G_i]$, and assume that they are sorted in order of decreasing benefit. In a user equilibrium we require that $u_i = \Gamma_i(g_i)$, i.e. all participating users have a benefit at least equal to their travel cost. See [13, 14, 6] for further discussion.

Now we modify accordingly the social cost. We define the social optimum $(\hat{f}, \hat{g})$ as the the solution of the following optimization problem:

$$\min_{f,g} \left\{ \sum_{e \in E} f_e l_e(f_e) + \sum_{i=1}^{k} \int_{g_i}^{G_i} \Gamma_i(x) dx : f \text{ is a flow satisfying demands } g_i, 0 \leq g_i \leq G_i \right\}$$

which is also a solution of the following optimization problem after a simple change of variables:

$$\min_{f,g} \left\{ \sum_{e \in E} f_e l_e(f_e) + \sum_{i=1}^{k} \int_{0}^{G_i - g_i} \Gamma_i(G_i - z) dz : f \text{ is a flow satisfying demands } g_i, 0 \leq g_i \leq G_i \right\} \tag{12}$$

[2] In case the $\Gamma_i(\cdot)$ functions are nonincreasing, we can find taxes for which some (instead of any) equilibrium induces an optimal flow. See [17] for details.

We assume that such a solution $(\hat{f}, \hat{g})$ exists.

This model can be reduced [14] to the classic Wardrop setting with fixed demands: we add an imaginary new edge $e_i = (s_i, t_i)$ connecting the origin-destination pair of commodity i. By imaginary we mean that this new edge is not actually seen by the users and is not included in the set $\mathcal{P}_i$. We think of the unrouted demand $G_i - g_i$ as being sent along this edge. The cost of this new edge is set to $\Gamma_i(\sum_{P\in\mathcal{P}_i} f_P)$, it is therefore a function of the demand actually routed. The reason is that we try as in [14] to exploit Wardrop's principle: at equilibrium f the users along paths in $\mathcal{P}_i$ and the excess demand along the imaginary edge e_i should "feel" the same latency u_i. Without taxes this latency would be the same for all paths and therefore equal to $\Gamma_i(\sum_{P\in\mathcal{P}_i} f_P)$. By the monotonicity of Γ_i all users along the paths in $\mathcal{P}_i$ have benefit greater that or equal to $\Gamma_i(\sum_{P\in\mathcal{P}_i} f_P)$ and therefore find it indeed advantageous to travel.

Consider now the following complementarity problem. The new variables f_{e_i}, v_i correspond to the flow and the tax on the imaginary edge e_i, $\forall i$. According to the preceding discussion one would ideally want v_i to be zero.

$$
\begin{aligned}
f_P\Big(\frac{l_P(f)}{a(i)} + \sum_{e\in P} b_e - u_i\Big) &= 0 && \forall i, \forall P \in \mathcal{P}_i && \text{(ELASTIC CP)}\\
f_{e_i}(\Gamma_i(G_i - f_{e_i}) + v_i - u_i) &= 0 && \forall i\\
\frac{l_P(f)}{a(i)} + \sum_{e\in P} b_e &\geq u_i && \forall i, \forall P \in \mathcal{P}_i\\
\Gamma_i(G_i - f_{e_i}) + v_i &\geq u_i && \forall i\\
u_i\Big(\sum_{P\in\mathcal{P}_i} f_P + f_{e_i} - G_i\Big) &= 0 && \forall i\\
\sum_{P\in\mathcal{P}_i} f_P + f_{e_i} &\geq G_i && \forall i\\
v_i(f_{e_i} - (G_i - \hat{g}_i)) &= 0 && \forall i\\
f_{e_i} &\leq G_i - \hat{g}_i && \forall i\\
b_e(f_e - \hat{f}_e) &= 0 && \forall e \in E\\
f_e &\leq \hat{f}_e && \forall e \in E\\
f_P, b_e, u_i, v_i &\geq 0 && \forall i, \forall P, \forall e
\end{aligned}
$$

The objective function of (12) is a sum of terms, one for every edge (real or imaginary). Crucially, every term is an increasing function of the corresponding edge flow. Therefore, in any solution (f^*, u^*, b^*) of (ELASTIC CP) we have that $f^*_{e_i} = G_i - \hat{g}_i$, $\sum_{P\in\mathcal{P}_i} f^*_P = \hat{g}_i$ for all i, and $f^*_e = \hat{f}_e, \forall e \in E$. If any of these equalities is a strict inequality then $(\hat{f}, \hat{g})$ is not an optimal solution of (12) for reasons similar to the argument in Lemma 2. Corollary 1 [26, 12, 17] implies the existence of optimal taxes (b^*, v^*) that will induce each user i to send flow $\hat{g}_i$ through the original network incurring flow $\hat{f}_e$ on every edge $e \in E$.

For the case of homogeneous users ($a(i) = 1,\ i = 1, \ldots, k$), it is well known [2, 14] that the Karush-Kuhn-Tucker conditions for (12) imply that setting the tax b_e to $\hat{f}_e \frac{\partial l_e}{\partial f_e}(\hat{f}_e)$ and $v_i = 0, \forall i$ a solution of (ELASTIC CP) is achieved. For this particular solution, the (common) cost of any path $P \in \mathcal{P}_i$ used by commodity i is indeed equal to the benefit $\Gamma_i(\hat{g}_i)$ as originally required.

In the general case of heterogeneous users though, some v_i may have to be non-zero, which implies that users in some class i will have to tolerate travel cost (latency plus taxation) equal to $v_i + \Gamma_i(\hat{g}_i) > \Gamma_i(\hat{g}_i)$. In other words we steer the selfish users through taxation to the optimal flow pattern $\hat{f}$ but we ask from the $\hat{g}_i$ participants from class i to tolerate cost that exceeds the benefit $\Gamma_i(\hat{g}_i)$.

Unfortunately, this is unavoidable, i.e., the taxation model for heterogeneous users is not fully compatible in the general case with the notion of participation benefits. Intuitively the reason is that we now prescribe both the edge flow $\hat{f}$ and the generalized costs u_i of the equilibrium. An infinite family of counterexamples can be easily constructed. For example, consider the simple network with two nodes s, t and a single edge $e = (s,t)$ with latency function $l_e(x) = x$. We have two players (commodities) with $a(1) = 1, a(2) = 2, G_1 = G_2 = 1$ and $\Gamma_1(x) = 8(1-x), \Gamma_2(x) = 4(1-x)$. The *unique* flow that optimizes (12) sends amounts $\hat{f}_1 = 5/7, \hat{f}_2 = 3/7$ of flow for the first and second players respectively through e, and $\Gamma_1(5/7) = 16/7, \Gamma_2(3/7) = 16/7$. But there is no b_e such that $l_e(\hat{f}_1 + \hat{f}_2) + a(1)b_e = l_e(\hat{f}_1 + \hat{f}_2) + a(2)b_e = 16/7$, therefore v_1, v_2 cannot be both 0.

5 Computability of optimal taxes

The results of Sections 3 and 4 imply the existence of optimal taxes for the setting of elastic demands both without and with penalties for non-participation. For the latter case, modeled by (ELASTIC CP), it has been shown [12, 17, 26] that given an optimum solution $(\hat{f}, \hat{g})$ the solution of this complementarity problem is reduced to the solution of a linear program, hence the optimal taxes can be computed in polynomial time. On the other hand, the complementarity problem (GENERAL CP′) can be hard to solve, even with our assumption that functions $xl_e(x)$ are convex, due to the generality of the $D_i(u)$ functions.

6 The price of anarchy with taxes

In this section we turn our attention to the study of the efficiency of the optimal taxes, under the assumption that demands are fixed, i.e., $D_i(u) = d_i$ for all i. We first study the price of anarchy of the new game resulting from the imposition of taxation.

Definition 2 *Let $K := \{f \in \mathbb{R}^{|\mathcal{P}|} : 0 \leq f_P, \forall P \ \wedge\ \sum_{P \in \mathcal{P}_i} f_P = d_i, \forall i\}$ be the set of all flows that satisfy the users' demands. A flow f is called* feasible *if $f \in K$.*

For $P \in \mathcal{P}_i$, let

$$T_P(f) = \sum_{e \in P} l_e(f_e) + a(i) \sum_{e \in P} b_e \tag{13}$$

where b is the optimal tax vector whose existence was established by Corollary 1. Another way to define a Wardrop user equilibrium for the game described in Corollary 1 is as a feasible flow $f^* \in K$ such that

$$\langle T(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K. \tag{14}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product (see, for example, [8]).

For the purposes of this section only, we modify our notion of social cost. For flow f define $C(f) := \sum_P f_P T_P(f)$. The *price of anarchy* or *coordination ratio* ρ for a selfish routing game with social cost function $\mathcal{C}()$ was defined by Koutsoupias and Papadimitriou [19] as

$$\rho := \sup_{f^*} \frac{\mathcal{C}(f^*)}{\mathcal{C}(\bar{f})}$$

where f^* is a user equilibrium and $\bar{f} = \arg\min_{f \in K} \mathcal{C}(f)$. The higher the price of anarchy the more the system may suffer from the lack of coordination of the selfish users. In the upcoming theorem we estimate the price of anarchy when $C()$ is the social cost function.

There is extensive literature on the price of anarchy for the case where the social cost function is defined as $\sum_e f_e l_e(f_e)$ (e.g., [22, 7]). The bounds vary based on the mathematical family in which the function falls (e.g., based on the degree in case of polynomials and so on). The fact that the optimal taxes induce an equilibrium with minimum total latency does not readily imply a new bound on the price of anarchy under $C()$. We show in Theorem 3 that for several families the price of anarchy actually decreases when the social cost is defined as $C(f)$, i.e., it includes the optimal taxes component. Qualitatively this implies that having to pay optimal taxes not only steers the users towards a minimum latency solution as shown in Corollary 1 but it also makes them behave less selfishly in the sense of [19].

We proceed with the technical statements. Correa et al. [7] define the following quantity β for a continuous nondecreasing latency function $l : \mathbb{R}^+ \to \mathbb{R}^+$ and every value $u \geq 0$:

$$\beta(u, l) := \frac{1}{ul(u)} \max_{x \geq 0} \{x(l(u) - l(x))\}, \tag{15}$$

where by convention $0/0 = 0$. In addition, they define $\beta(l) := \sup_{u \geq 0} \beta(u, l)$ and $\beta(\mathcal{L}) := \sup_{l \in \mathcal{L}} \beta(l)$, where $\mathcal{L}$ is a family of latency functions. Note that $\beta(l) \geq \beta(u, l)$, $\forall u, l$, and $\beta(\mathcal{L}) \geq \beta(l)$, $\forall l \in \mathcal{L}$. Also $0 \leq \beta(\mathcal{L}) < 1$. Recall that $\hat{f}$ is a flow that achieves minimum total latency, i.e., is a solution to (MP).

Theorem 3 *Let the demands be fixed and the latency functions l_e belong to a family $\mathcal{L}$. If f^* is any traffic equilibrium for the game of Corollary 1 and $\bar{f}$ is a flow that minimizes $C(f)$ then*

$$\frac{C(f^*)}{C(\bar{f})} \leq 1 + \beta(\mathcal{L}).$$

Proof: First we note that $C(f^*) = \langle T(f^*), f^* \rangle$. Also, because of Corollary 1 we have that $f_e^* = \hat{f}_e$. Then, for any feasible flow $\bar{f} \in K$ we have

$$\begin{aligned}
\langle T(f^*), \bar{f} \rangle &= \sum_i \sum_{P \in \mathcal{P}_i} \bar{f}_P \left(l_P(f^*) + a(i) \sum_{e \in P} b_e \right) \\
&= \sum_{e \in E} \bar{f}_e l_e(f_e^*) + \sum_i \sum_{P \in \mathcal{P}_i} \bar{f}_P a(i) \sum_{e \in P} b_e \\
&= \sum_{e \in E} \bar{f}_e l_e(\hat{f}_e) + \sum_i \sum_{P \in \mathcal{P}_i} \bar{f}_P a(i) \sum_{e \in P} b_e
\end{aligned} \tag{16}$$

where the second equality is due to the additive model.

We also have by definition that $\beta(\hat{f}_e, l_e) := \frac{1}{\hat{f}_e l_e(\hat{f}_e)} \max_{x \geq 0}\{x(l_e(\hat{f}_e) - l_e(x))\}$, from which we get for $x := \bar{f}_e$ that $\beta(\hat{f}_e, l_e)\hat{f}_e l_e(\hat{f}_e) \geq \bar{f}_e l_e(\hat{f}_e) - \bar{f}_e l_e(\bar{f}_e)$ or, by the definition of $\beta(\mathcal{L})$,

$$\beta(\mathcal{L})\hat{f}_e l_e(\hat{f}_e) \geq \bar{f}_e l_e(\hat{f}_e) - \bar{f}_e l_e(\bar{f}_e). \tag{17}$$

By the definition of $\hat{f}$ we know that $\sum_{e \in E} \hat{f}_e l_e(\hat{f}_e) \leq \sum_{e \in E} \bar{f}_e l_e(\bar{f}_e)$, therefore (17) implies that

$$(1 + \beta(\mathcal{L})) \cdot \sum_{e \in E} \bar{f}_e l_e(\bar{f}_e) \geq \sum_{e \in E} \bar{f}_e l_e(\hat{f}_e),$$

and this, in turn, together with (16) implies that

$$\begin{aligned} \langle T(f^*), \bar{f} \rangle &\leq (1 + \beta(\mathcal{L})) \cdot \sum_{e \in E} \bar{f}_e l_e(\bar{f}_e) + \sum_i \sum_{P \in \mathcal{P}_i} \bar{f}_P a(i) \sum_{e \in P} b_e \\ &\leq (1 + \beta(\mathcal{L})) \cdot \sum_i \sum_{P \in \mathcal{P}_i} \bar{f}_P \left(l_P(\bar{f}) + a(i) \sum_{e \in P} b_e \right) \\ &= (1 + \beta(\mathcal{L})) \cdot C(\bar{f}). \end{aligned} \tag{18}$$

Since f^* is an equilibrium, (14) implies that

$$\langle T(f^*), f \rangle \geq \langle T(f^*), f^* \rangle, \quad \forall f \in K,$$

which, together with (18) and for $f := \bar{f}$, implies that

$$(1 + \beta(\mathcal{L})) \cdot C(\bar{f}) \geq C(f^*).$$

□

Correa et al. [7] give upper bounds for $\beta(\mathcal{L})$ on several function families $\mathcal{L}$.

Proposition 1 [7] *If the set $\mathcal{L}$ of continuous and nondecreasing latency functions is contained in the set $\{l(\cdot) : l(cx) \geq c^n l(x)$ for $c \in [0, 1]\}$ for some positive number n, then*

$$\beta(\mathcal{L}) \leq \frac{n}{(n+1)^{1+1/n}}.$$

Therefore the worst case bound of Theorem 3 for linear functions is 5/4 which is better than the tight 4/3 bound [22] for general linear latency functions. For $n = 2, 3$ and 4 the upper bound becomes $1.385, 1.472$ and 1.535 respectively, as opposed to $1.626, 1.896$ and 2.151 respectively when b is not used [7]. We emphasize the fact that this improved price of anarchy value refers to the very special set of *optimal* taxes, hence it is an added bonus to the usage of such taxes. If we wanted to use taxation merely for the improvement of the price of anarchy, then other values for b_e may give an even better result. For example, in the simple case of two nodes connected by parallel links, the use of values $b_e := B$ for a huge constant B can cause the taxation cost to dwarf the latency costs, and the price of anarchy will become almost 1. In this paper we are foremost interested in preserving the optimality of the taxes; we get the improvement of the price of anarchy as an interesting by-product. Also note that as n increases, the bound goes to 2. In fact for any family of continuous nondecreasing latency functions it is easy to see that $\beta(\mathcal{L}) \leq 1$ [7]. For any continuous strictly increasing latency function a vector of optimal taxes exists [12, 17]. Therefore we obtain:

Corollary 2 *Let the demands be fixed and the latency functions l_e be continuous and strictly increasing. If f^* is any traffic equilibrium for the game of Corollary 1 and $\bar{f}$ is a flow that minimizes $C(f)$ then*

$$\frac{C(f^*)}{C(\bar{f})} \leq 2.$$

7 Comparison to the original latencies

In general, the b_e values of Corollary 1 can be very big. See [12] for some upper bounds in the case of fixed demands. It may even be the case that the part of the cost $C(f)$ due to the latencies $\sum_e f_e l_e(f_e)$ is negligible compared to the part due to b, which is $\sum_i \sum_P a(i) f_P \sum_{e\in P} b_e$. Therefore the improvement of the coordination ratio which was shown in Section 6 may come at a prohibitive increase to the overall cost. One would like to bound b so that the new overall cost is bounded by a function of the original optimal total latency $\sum_e \hat{f}_e l_e(\hat{f}_e)$.

Unfortunately, we do not know how to bound b for the general $l(\cdot)$ of Corollary 1 effectively. But we can use already known results in the case of *homogeneous* users, i.e., $a(i) = 1,\ \forall i$, to bound the ratio of the worst equilibrium cost when b is used to the original optimal total cost.

It is well known ([2],[10],[23]; see also [5], especially Proposition 3.1) that, for homogeneous networks with differentiable latency functions l_e, one can use the *marginal costs*[3] $\hat{f}_e l'_e(\hat{f}_e)$ as b_e of Corollary 1 to achieve the following classical result:

Theorem 4 [2] *If the demands are fixed and the functions l_e are differentiable, then $\hat{f}$ is an equilibrium for the selfish routing game with $T_P(f) := \sum_{e\in P}(l_e(f_e) + \hat{f}_e l'_e(\hat{f}_e))$.*

Moreover, if we assume that l_e are strictly increasing (as in Corollary 1), then any equilibrium f^* incurs the same edge flow as $\hat{f}$, i.e., $f^*_e = \hat{f}_e,\ \forall\ e$ (Theorem 6.2 in [1]). Let $L_{OPT} := \sum_{e\in E} \hat{f}_e l_e(\hat{f}_e)$.

Theorem 5 *If the demands are fixed, the users are homogeneous, $l_e(f_e) = a_e f_e + \beta_e$ with $a_e > 0, \beta_e \geq 0$ for all $e \in E$, and f^* is an equilibrium for the selfish routing game with taxes $b_e := a_e \hat{f}_e$, then*

$$\frac{C(f^*)}{L_{OPT}} \leq 2.$$

More generally, if the l_e's are polynomials of degree d with positive coefficients, and f^* is an equilibrium for the selfish routing game with taxes $b_e := \hat{f}_e l'_e(\hat{f}_e)$, then

$$\frac{C(f^*)}{L_{OPT}} \leq d + 1.$$

Proof: For the case of linear latency functions, note that these functions are differentiable, therefore Theorem 4 implies that $\hat{f}$ is an equilibrium. Since they are strictly increasing as

[3] To simplify notation, in this section we use $l'_e(\cdot)$ to denote the derivative $\frac{\partial l_e}{\partial f_e}(\cdot)$.

well, we know that $f_e^* = \hat{f}_e,\ \forall e$, for any equilibrium f^*. Hence

$$C(f^*) := \sum_P f_P^* T_P(f^*) = \sum_e f_e^*(a_e f_e^* + \beta_e + a_e \hat{f}_e)$$
$$= \sum_e \hat{f}_e(2a_e\hat{f}_e + \beta_e)$$
$$\leq 2 \cdot L_{OPT}$$

The same argument shows the upper bound for the case of degree d polynomials.

□

Note that the bound above is tight in the case of polynomials with all the coefficients, except for the one of the highest degree factor, being 0.

Theorem 6 *If the demands are fixed, the users are homogeneous, the l_e functions are strictly increasing and continuously differentiable with a convex derivative, and f^* is an equilibrium for the selfish routing game with taxes $b_e := \hat{f}_e l'_e(\hat{f}_e)$, then*

1. *If $l_e(\cdot) \in \{l(\cdot) : l'(cx) \geq cl'(x)$ for $c \in [0,1]\}$, then $\frac{C(f^*)}{L_{OPT}} \leq 3$,*
2. *If $l_e(\cdot) \in \{l(\cdot) : l'(cx) \geq c^n l'(x)$ for $c \in [0,1]\}$, then $\frac{C(f^*)}{L_{OPT}} \leq 2^n + 1$.*

Proof: Theorem 4 implies that $\hat{f}$ is an equilibrium. Since the l_e's are strictly increasing as well, we know that $f_e^* = \hat{f}_e,\ \forall e$, for any equilibrium f^*. Observe that the edges e for which $\hat{f}_e = 0$ do not contribute to $C(f^*)$ therefore we ignore them in the ensuing calculations. It is known that if the l'_e functions are convex and continuous in (γ, δ), then

$$l'_e(x) \leq \frac{1}{2h}\int_{x-h}^{x+h} l'_e(t)dt \tag{19}$$

for $\gamma \leq x - h < x < x + h \leq \delta$ (fact 125, p. 98 of [15]). For $x := f_e/2, h := f_e/2$, inequality (19) becomes

$$f_e l'_e(\frac{f_e}{2}) \leq \int_0^{f_e} l'_e(t)dt. \tag{20}$$

Under the assumptions of Part 1 of the theorem, (20) gives $f_e l'_e(f_e) \leq 2l_e(f_e)$ which, for $f_e := \hat{f}_e$, implies together with an argument similar to the proof of Theorem 5 that

$$\frac{C(f^*)}{\sum_e \hat{f}_e l_e(\hat{f}_e)} \leq 3$$

for any equilibrium flow f^*.

Under the assumptions of Part 2 of the theorem, (20) implies that $f_e l'_e(f_e) \leq 2^n l_e(f_e)$ which, in turn, implies that in this case

$$\frac{C(f^*)}{\sum_e \hat{f}_e l_e(\hat{f}_e)} \leq 2^n + 1$$

for any equilibrium flow f^*. □

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