

On the Parameter Identification Problem in the Plane and the Polar Fractal Interpolation Functions

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Fractal interpolation functions provide a new means for fitting experimental data and their graphs can be used to approximate natural scenes. We first determine the conditions that a vertical scaling factor must obey to model effectively an arbitrary function. We then introduce polar fractal interpolation functions as one fractal interpolation method of a non-affine character. Thus, this method may be suitable for a wider range of applications than that of the affine case. The interpolation takes place in polar coordinates and then with an inverse non-affine transformation a simple closed curve arises as an attractor which interpolates the data in the usual plane coordinates. Finally, we prove that this attractor has the same Hausdorff dimension as the polar one. © 1999 Academic Press

1. INTRODUCTION

Based on a theorem of J. E. Hutchinson ([12], p. 731) and using iterated-function-systems (IFS) theory, M. F. Barnsley introduced a class of functions in [3] which he called *fractal interpolation functions* or FIF's for short. He worked basically with *affine* FIF's, in the sense that they are obtained using affine transformations. More general transformations than the affine ones are discussed in [9] and in [10] but there is no evidence

therein that they may be used as an interpolation model. The affine FIF's have in common with elementary functions that they are of a geometrical character and that they can be computed rapidly. The main difference is their fractal character since their graphs usually have non-integral dimension. The graphs of these functions can be used to approximate image components such as the profiles of mountain ranges, the tops of clouds and horizons over forests, to name but a few. Recent applications of this theory include modeling of discrete sequences as in [16], modeling of speech signals as in [17] and compression of static images as in [1].

M. F. Barnsley claims in [3], p. 308, that his model has an attractor (the graph G of the FIF) in a bigger compact metric space than Hutchinson's [12]. Although he defines the fractal interpolation so as to take place in the compact metric space $K = I \times [a, b]$, where I and $[a, b]$ are real closed intervals, he doesn't mention anything about the conditions that the vertical scaling factors must obey, so that G remains within the rectangle K . One possible explanation of this is that he takes $[a, b]$ to be "sufficiently large," for example this interval in [4] and in [6] coincides eventually with \mathbf{R} itself so that G finally fits in K . In this way, one interpolates addressing the issue independently of any frontiers to the space he is working in.

The first aim of our paper is to fill in this gap and make the affine fractal interpolation more flexible to handle situations where the original frame is given and our data must be interpolated within this frame. This becomes feasible by proving that the vertical scaling factors depend solely on the data in hand and so can be automatically generated. In this way when the vertical scaling factors are appropriately chosen, one can fix *a priori* the overall approximate shape of the FIF. Thus, we should be able to construct FIF's whose overall shapes and dimensions can be prescribed. This result is very useful in the parameter identification problem as far as the calculation of the contractivity factors of the FIF is concerned. Then we introduce the Polar FIF's which can handle a wider range of data because of their non-affine character which is based on the polar transformation. Next, we present a theorem which shows that if we transfer our polar model to the plane we get a *simple closed* (i.e., *Jordan*) curve which interpolates our original data. Finally, we prove that the Hausdorff dimension of our interpolation function's graph in the plane coordinates is the same as that of our transformed-in-polar-coordinates function's graph.

2. ITERATED FUNCTION SYSTEMS

Within Fractal Geometry, the method of iterated function systems introduced by J. E. Hutchinson in [12] and popularized by M. F. Barnsley and S. Demko in [5], is a relatively easy way to generate fractal images.

A function $f: X \rightarrow Y$ is called a *Hölder function of exponent a* if

$$|f(x) - f(y)| \leq c |x - y|^a$$

for $x, y \in X$, $a \geq 0$ and for some constant c . Note that, if $a > 1$, the functions are constants. Obviously, $c \geq 0$. The function f is called a *Lipschitz function* if a may be taken to be equal to 1. A Lipschitz function is a *contraction* with *contractivity factor* c , if $c < 1$. An *iterated function system* or IFS for short, is a collection of a complete metric space (X, ρ) together with a finite set of continuous mappings $w_n: X \rightarrow X$, $n = 1, 2, \dots, N$, where ρ is a distance between elements of X . It is often convenient to write an IFS formally as $\{X; w_1, w_2, \dots, w_N\}$ or, somewhat more briefly, as $\{X; w_{1-N}\}$.

We introduce the associated map of subsets $W: \mathcal{H}(X) \rightarrow \mathcal{H}(X)$, given by

$$W(E) = \bigcup_{n=1}^N w_n(E) \quad \text{for all } E \in \mathcal{H}(X),$$

where $\mathcal{H}(X)$ is the metric space of all nonempty compact subsets of X with respect to the Hausdorff distance. The map W is called the *collage map* to alert us to the fact that $W(E)$ is formed as a union or collage of sets. Sometimes $\mathcal{H}(X)$ is referred to as the “space of fractals in X ” (but note that not all members of X are fractals).

If w_n are contractions with corresponding contractivity factors s_n for $n = 1, 2, \dots, N$, the IFS is termed *hyperbolic* and the map W itself is then a contraction with contractivity factor $s = \max\{s_1, s_2, \dots, s_N\}$ ([4], Theorem 7.1, p. 81). In what follows we abbreviate by f^k the k -fold composition $f \circ f \circ \dots \circ f$.

The *attractor* of a hyperbolic IFS is the unique set \mathcal{A} for which $\lim_{k \rightarrow \infty} W^k(E_0) = \mathcal{A}$ for every starting set E_0 . The term attractor is chosen to suggest the movement of E_0 towards \mathcal{A} under successive applications of W . \mathcal{A} is also the unique set in $\mathcal{H}(X)$ which is not changed by W , so $W(\mathcal{A}) = \mathcal{A}$, and from this important perspective it is often called the *invariant set* of the IFS.

A transformation w is *affine* if it may be represented by a matrix A and translation \mathbf{t} as $w(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$, or (if $X = \mathbf{R}^2$)

$$w \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d \\ e \end{bmatrix}. \tag{1}$$

The *code* of w is the 6-tuple (a, b, c, s, d, e) , and the *code of an IFS* is a table whose rows are the codes of w_1, w_2, \dots, w_N . We refer the interested reader to [4] or [11].

3. FRACTAL INTERPOLATION FUNCTIONS REVISITED

Let the continuous function f be defined on a real closed interval $I = [x_0, x_N]$ and with range a complete metric space (Y, ρ_Y) , where x_0, x_1, \dots, x_N be $N+1$ distinct points with $x_0 < x_1 < \dots < x_N$. It is not assumed that these points are equidistant. The function f is called an *interpolation function* corresponding to the generalized set of data

$$\{(x_i, y_i) \in I \times Y : i = 0, 1, \dots, N\},$$

if $f(x_i) = y_i$ for all $i = 0, 1, \dots, N$. We shall write for brevity $f(x_i) = f_i$, $i = 0, 1, \dots, N$. The points (x_i, f_i) are called the *interpolation points*. We say that the function f *interpolates* the data and that (the graph of) f passes through the interpolation points. In this section we focus on the existence, construction and properties of such functions whose graphs $G = \{(x, f(x)) : x \in I\}$ are attractors of IFS. Throughout this section we will work in the complete metric space $K = I \times Y$ with respect to the Euclidean, or to some other equivalent, metric.

Set $I_n = [x_{n-1}, x_n]$ and let $L_n: I \rightarrow I_n$ for $n = 1, 2, \dots, N$, be contractive homeomorphisms such that

$$L_n(x_0) = x_{n-1}, \quad L_n(x_N) = x_n, \quad (2)$$

$$|L_n(b_1) - L_n(b_2)| \leq l |b_1 - b_2| \quad (3)$$

whenever $b_1, b_2 \in I$, for some $l \in [0, 1)$. Clearly, $\{x_0, x_1, \dots, x_N\}$ is a partition of the compact interval I . Note also that $\{I; L_{1-N}\}$ is a hyperbolic IFS whose unique attractor is

$$I = \bigcup_{n=1}^N L_n(I) = \bigcup_{n=1}^N I_n,$$

where

$$I_n^0 \cap I_m^0 = \emptyset \quad \text{when } n \neq m, \quad \text{for } n, m = 1, 2, \dots, N.$$

In other words, L_n satisfy the *open set condition* for $n = 1, 2, \dots, N$ and also

$$I_n \cap I_{n+1} = x_n, \quad \text{for } n = 1, 2, \dots, N-1.$$

Furthermore, let mappings $M_n: K \rightarrow Y$ for $n = 1, 2, \dots, N$ be continuous such that

$$M_n(x_0, f_0) = f_{n-1}, \quad M_n(x_N, f_N) = f_n, \quad (4)$$

$$\rho_Y(M_n(x, b_1), M_n(x, b_2)) \leq s \rho_Y(b_1, b_2), \quad (5)$$

for all $x \in I, b_1, b_2 \in Y$ and for some $s \in [0, 1)$. Condition (5) means that M_n are contractive in the second variable, for $n = 1, 2, \dots, N$.

Now define functions $w_n: K \rightarrow K$ by

$$w_n(x, y) = (L_n(x), M_n(x, y)) \tag{6}$$

for all $(x, y) \in K$ and $n = 1, 2, \dots, N$. If $Y = [a, b]$, the next theorem has been proven in [3].

THEOREM 1. *The IFS $\{K; w_{1-N}\}$ defined above has a unique attractor $G \in \mathcal{H}(K)$. Furthermore, G is the graph of a continuous function $f: I \rightarrow Y$ which obeys*

$$f(x_i) = f_i, \quad i = 0, 1, \dots, N.$$

DEFINITION 1. The function f whose graph is the attractor of an IFS as described in Theorem 1, is called a *fractal interpolation function* or FIF for short.

Notice that, generally, the IFS $\{K; w_{1-N}\}$ may not be hyperbolic. To construct a hyperbolic IFS whose attractor is the graph of a function, it is assumed that the mappings $M_n, n = 1, 2, \dots, N$ not only satisfy Condition (5) but also

$$\rho_Y(M_n(b_1, y), M_n(b_2, y)) \leq c |b_1 - b_2| \tag{7}$$

for all $y \in Y, b_1, b_2 \in I, n = 1, 2, \dots, N$ and for some $c > 0$. This condition means that M_n are uniformly Lipschitz in the first variable, for $n = 1, 2, \dots, N$.

Since the completeness depends on the choice of metric we have the following

THEOREM 2. *There is a metric ρ_ϕ on K , equivalent to the Euclidean metric, such that the IFS $\{K; w_{1-N}\}$ is hyperbolic with respect to ρ_ϕ .*

The formal proof of the above theorem can be found in [6].

4. AFFINE FRACTAL INTERPOLATION FUNCTIONS AND THE DETERMINATION OF THEIR VERTICAL SCALING FACTORS

Now, we will restrict our attention to affine transformations. Let N be a positive integer greater than 1 and $Y = \mathbf{R}$. Define $L_n: I \rightarrow I_n$ by

$$L_n(x) = a_n x + d_n,$$

where the real numbers a_n, d_n , for $n = 1, 2, \dots, N$, are chosen to ensure that Condition (2) holds, i.e. $L_n(I) = I_n$. Thus, for $n = 1, 2, \dots, N$,

$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0},$$

$$d_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}.$$

Since $N \geq 2$, $|a_n| < 1$, so L_n are contractive homeomorphisms, for $n = 1, 2, \dots, N$, as they obey Condition (3) with $l = \max\{|a_n| : n = 1, 2, \dots, N\}$.

Now define $M_n: K \rightarrow Y$ by

$$M_n(x, y) = c_n x + s_n y + e_n$$

where the real constants c_n and e_n , depending on the adjustable real parameter s_n , are chosen to ensure that Condition (4) holds. That is, $s_n \in (-1, 1)$ is chosen and then

$$c_n = \frac{f_n - f_{n-1}}{x_N - x_0} - s_n \frac{f_N - f_0}{x_N - x_0},$$

$$e_n = \frac{x_N f_{n-1} - x_0 f_n}{x_N - x_0} - s_n \frac{x_N f_0 - x_0 f_N}{x_N - x_0},$$

for $n = 1, 2, \dots, N$. The mappings M_n , $n = 1, 2, \dots, N$ obey Condition (5) with $s = \max\{|s_n| : n = 1, 2, \dots, N\}$ and Condition (7) with $c = \max\{|c_n| : n = 1, 2, \dots, N\}$.

Define functions w_n as in Eq. (6). Then the IFS is of the form $\{K; w_{1-N}\}$, where the maps are affine transformations as in (1) and, in particular, of the special structure

$$w_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_n & 0 \\ c_n & s_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_n \\ e_n \end{bmatrix}$$

where a_n, c_n, s_n, d_n , and e_n are real numbers for $n = 1, 2, \dots, N$. The transformations w_n are *shear* transformations, where s_n are their *vertical scaling factors*. These transformations, constrained by Conditions (2) and (4), are giving

$$w_n \begin{bmatrix} x_0 \\ f_0 \end{bmatrix} = \begin{bmatrix} x_{n-1} \\ f_{n-1} \end{bmatrix} \quad \text{and} \quad w_n \begin{bmatrix} x_N \\ f_N \end{bmatrix} = \begin{bmatrix} x_n \\ f_n \end{bmatrix}, \quad \text{for } n = 1, 2, \dots, N.$$

By choosing $s_n \in (-1, 1)$ to be the free parameter, we are able to specify the vertical scaling produced by the transformation.

Combining Theorems 1 and 2 with the IFS $\{K; w_{1-N}\}$ defined above, we have the following

DEFINITION 2. The function f whose graph is the attractor of an IFS as described above, is called an *affine fractal interpolation function* or AFIF for short.

In most of the cases, e.g., image compression, a “smaller” space than $Y = \mathbf{R}$ need only be given and we can then interpolate our data within a given rectangle. Often Y is considered as a real closed interval $[a, b]$ which is compact and so a complete metric space. To ensure that the graph of the AFIF is contained in the rectangle K some conditions about the adjustable parameters s_n must be specified. The following theorem which we introduce here helps us in the determination of the vertical scaling factors needed to model an arbitrary function.

THEOREM 3. *The graph of an AFIF remains within a given rectangle $K = I \times [a, b]$ if and only if the vertical scaling factors s_n obey*

$$s_n^{\min} \leq s_n \leq s_n^{\max}$$

and $|s_n| < 1$, where

$$s_n^{\max} = \min \left\{ \frac{b - f_n}{b - f_N}, \frac{b - f_{n-1}}{b - f_0}, \frac{a - f_n}{a - f_N}, \frac{a - f_{n-1}}{a - f_0} \right\},$$

$$s_n^{\min} = \max \left\{ \frac{a - f_{n-1}}{b - f_0}, \frac{a - f_n}{b - f_N}, \frac{b - f_{n-1}}{a - f_0}, \frac{b - f_n}{a - f_N} \right\},$$

for $n = 1, 2, \dots, N$. These bounds are the best possible.

Proof. The maximum and minimum values of

$$M_n(x, y) = c_n x + s_n y + e_n$$

on the rectangle $[x_0, x_N] \times [a, b]$ are achieved at the extreme points (x_0, a) , (x_0, b) , (x_N, a) , (x_N, b) (see, for example, [13]). While we require $M_n(x, y) \in [a, b]$, we select the parameters s_n such that $M_n(x_0, a)$, $M_n(x_0, b)$, $M_n(x_N, a)$, $M_n(x_N, b) \in [a, b]$. We can easily deduce that the s_n obey the required inequalities. In case of a zero denominator we avoid the corresponding term. The demonstration of the converse is obvious. ■

Another thing that follows from the theorem is the fact that s_n can be automatically adjusted and defined via the consecutive interpolation

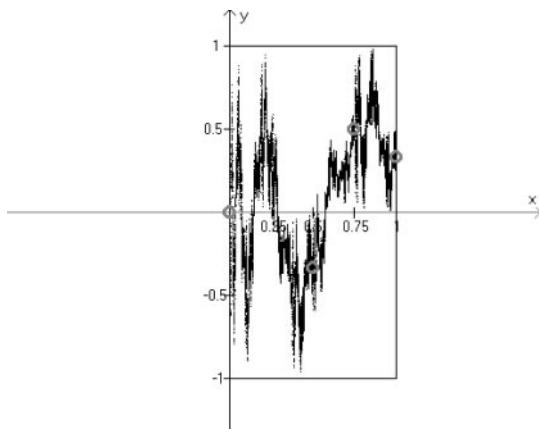


FIG. 1. The graph of an AFIF.

points f_{n-1} and f_n , the endpoints f_0 and f_N and the given coordinates a and b .

EXAMPLE 1. Let $I = [0, 1]$, $Y = [-1, 1]$ and let

$$\{(0, 0), (1/2, -1/3), (3/4, 1/2), (1, 1/3)\}$$

be a given set of data. Then, the graph of the AFIF remains within the rectangle $K = I \times Y$, if we choose the s_n , $n = 1, 2, 3$ such that: $-1 < s_1 \leq 1/2$, $-3/8 \leq s_2 \leq 2/3$ and $-1/2 \leq s_3 \leq 1/2$. Figure 1 displays this graph with $s_1 = -0.99$, $s_2 = -3/8$ and $s_3 = -1/2$.

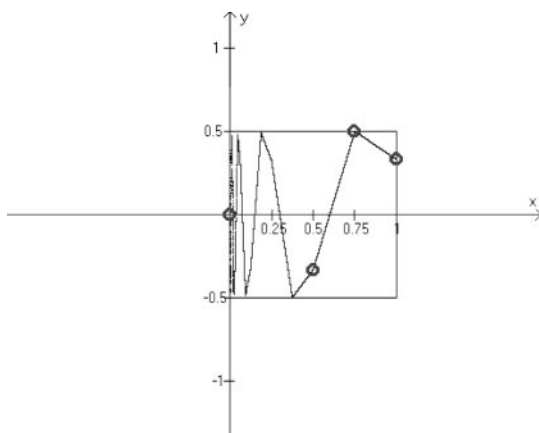


FIG. 2. The graph of an AFIF.

EXAMPLE 2. Let $I = [0, 1]$, $Y = [-1/2, 1/2]$ and let

$$\{(0, 0), (1/2, -1/3), (3/4, 1/2), (1, 1/3)\}$$

be a given set of data. Then, the graph of the AFIF remains within the rectangle $K = I \times Y$, if we choose the s_n , $n = 1, 2, 3$ such that: $-1 < s_1 \leq 1/2$ and $s_2 = s_3 = 0$. Figure 2 displays this graph with $s_1 = -0.99$ and $s_2 = s_3 = 0$.

From Theorem 4.3 we have the following useful corollaries.

COROLLARY 1. *The graph of an AFIF remains within a given rectangle $K = I \times [a, b]$ with $f_0 = f_N$ if and only if the vertical scaling factors s_n obey*

$$s_n^{\min} \leq s_n \leq s_n^{\max}$$

and $|s_n| < 1$, where

$$s_n^{\max} = \min \left\{ \frac{b - \max\{f_{n-1}, f_n\}}{b - f_0}, \frac{a - \max\{f_{n-1}, f_n\}}{a - f_0} \right\},$$

$$s_n^{\min} = \max \left\{ \frac{a - \min\{f_{n-1}, f_n\}}{b - f_0}, \frac{b - \min\{f_{n-1}, f_n\}}{a - f_0} \right\},$$

for $n = 1, 2, \dots, N$. These bounds are the best possible.

COROLLARY 2. *The graph of an AFIF remains within a given strip $K = I \times [a, \infty)$ with $f_0 = f_N$ if and only if the vertical scaling factors s_n obey*

$$0 \leq s_n \leq s_n^{\max},$$

and $|s_n| < 1$, where

$$s_n^{\max} = \frac{a - \max\{f_{n-1}, f_n\}}{a - f_0},$$

for $n = 1, 2, \dots, N$. These bounds are the best possible.

5. POLAR FRACTAL INTERPOLATION FUNCTIONS AND THEIR HAUSDORFF DIMENSION

Certain implicit functional relations can be transformed into another types of variables or coordinates. One such transformation is into *polar coordinates* in which we take

$$E = \{(\theta, r) : 0 \leq \theta < 2\pi, r > 0\}$$

and we let $\mathbf{g} = (g_1, g_2)$ be the *coordinate transformation* (for its definition see [2]) which maps each point (θ, r) in E onto the point $(x, y) \in \mathbf{R}^2 \setminus \{(0, 0)\}$ by the familiar formulas

$$x = g_1(\theta, r) = r \cos \theta, \quad y = g_2(\theta, r) = r \sin \theta.$$

THEOREM 4. *Let a set of data $\{(x_i, f_i) \in \mathbf{R}^2 \setminus \{(0, 0)\} : i = 0, 1, \dots, N-1\}$ be given with at least one pair (x_i, f_i) in each quadrant and let $\{(\theta_i, r_i) \in [0, 2\pi) \times (0, \infty) : i = 0, 1, \dots, N-1\}$ be its corresponding set of data in the polar coordinates, where $0 = \theta_0 < \theta_1 < \dots < \theta_{N-1} < \theta_N = 2\pi$. Then there exists a simple closed curve $\mathbf{r}: [0, 2\pi] \rightarrow \mathbf{R}^2$ with $\mathbf{r}(\theta_i) = (x_i, f_i)$ for all $i = 0, 1, \dots, N-1$ which arises as attractor of a certain IFS.*

Proof. Let $\hat{I} = [\theta_0, \theta_N]$, where $\theta_0, \theta_1, \dots, \theta_N$ be $N+1$ distinct points and $0 = \theta_0 < \theta_1 < \dots < \theta_{N-1} < \theta_N = 2\pi$. It is not assumed that these points are equidistant. Select $\alpha > 0$ in such a way that $r_i \in \hat{Y}$ for all $i = 0, 1, \dots, N-1$, where $\hat{Y} = [\alpha, \beta]$ (resp. $\hat{Y} = [\alpha, \infty)$). We will work in the complete metric spaces $K = I \times Y$ and $\hat{K} = \hat{I} \times \hat{Y}$ with respect to the Euclidean, or to some other equivalent, metric, where $I = [x_0, x_{N-1}]$ and $Y = [a, b]$.

Construct a function $r: \hat{I} \rightarrow \hat{Y}$ with $r(\theta_i) = r_i$, $i = 0, 1, \dots, N-1$ and $r_N = r_0$ using the same method as described in Theorems 1 and 2. To ensure that $r(\theta) \in \hat{Y}$ we select the vertical scaling factors of our polar model with the help of Corollary 1 (resp. Corollary 2); we will show then that a simple closed curve is formed passing through each interpolation point (x_i, f_i) , for all $i = 0, 1, \dots, N-1$.

The coordinate transformation \mathbf{g} and its inverse \mathbf{g}^{-1} set up a one-to-one correspondence between the compact subsets of E and the compact subsets of $\mathbf{g}(E)$. Since $r(\theta) > 0$, the curve $\mathbf{r}(\theta) = (r(\theta) \cos \theta, r(\theta) \sin \theta)$ is simple for $\theta \in [0, 2\pi]$ and closed because $r(\theta_0) = r(\theta_N)$. It follows that the continuous function $\mathbf{r}: \hat{I} \rightarrow K$, with $\mathbf{r}(\theta) = (r(\theta) \cos \theta, r(\theta) \sin \theta)$, is a well defined function which obeys $\mathbf{r}(\theta_i) = (x_i, f_i)$, for all $i = 0, 1, \dots, N-1$ and whose graph is a simple closed curve. ■

DEFINITION 3. The function \mathbf{r} whose image is the attractor of an IFS as described above, is called a *polar fractal interpolation function* or PFIF for short.

Note that this function is a non-affine *vector-valued* FIF, contrary to the affine ones introduced in [14].

EXAMPLE 3. Let

$$\{(-3/4, -1/2), (1/2, 3/4), (3/4, -1/2), (-3/4, 3/4)\}$$

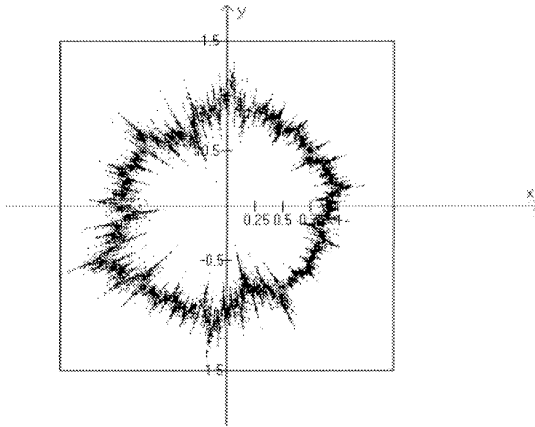


FIG. 3. The graph of a PFIF.

be a given set of data. Figure 3 displays the graph of the PFIF with $s_1 = s_3 = -0.9$, $s_2 = 0.9$, $s_4 = 1/2$ and Figure 4 displays the graph of the same PFIF in polar coordinates.

The last thing that remains to be proven is that our new curve has the same Hausdorff dimension \dim_H as the respective one in polar coordinates. We first need the following

PROPOSITION 1. *Let $F \subset \mathbf{R}^n$. If $f: F \rightarrow \mathbf{R}^m$ is a Lipschitz function, then*

$$\dim_H f(F) \leq \dim_H F.$$

Proof. See [8], Corollary 2.4(a), p. 30. ■

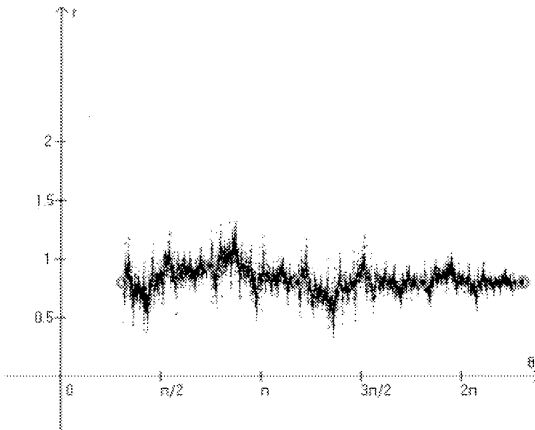


FIG. 4. The graph of a PFIF in polar coordinates.

THEOREM 5. *Let \hat{G} and G denote the graph of r and \mathbf{r} , respectively. Then $\dim_H \hat{G} = \dim_H G$.*

Proof. Let us consider \mathbf{R}^2 with respect to the box metric

$$\|x - y\|_1 = |x_1 - y_1| + |x_2 - y_2|, \quad x, y \in \mathbf{R}^2.$$

We first prove that $\mathbf{g}: \hat{G} \rightarrow G$ is a Lipschitz function. Let $(\theta_1, r_1), (\theta_2, r_2) \in \hat{G}$. Then

$$\begin{aligned} \|\mathbf{g}(\theta_1, r_1) - \mathbf{g}(\theta_2, r_2)\|_1 &\leq \|\mathbf{g}(\theta_1, r_1) - \mathbf{g}(\theta_1, r_2)\|_1 + \|\mathbf{g}(\theta_1, r_2) - \mathbf{g}(\theta_2, r_2)\|_1 \\ &= |r_1 - r_2| |\cos \theta_1| + |r_1 - r_2| |\sin \theta_1| \\ &\quad + r_2 |\cos \theta_1 - \cos \theta_2| + r_2 |\sin \theta_1 - \sin \theta_2| \\ &\leq 2 |r_1 - r_2| + 2r_2 |\theta_1 - \theta_2| \\ &\leq 2 |r_1 - r_2| + 2b |\theta_1 - \theta_2| \\ &\leq c \|(\theta_1, r_1) - (\theta_2, r_2)\|_1, \end{aligned}$$

where $c = \max\{2, 2b\}$. Using Proposition 1 one has $\dim_H G = \dim_H \mathbf{g}(\hat{G}) \leq \dim_H \hat{G}$.

Let $G^{(j)}$ be the intersection of G with the j -quadrant so that $G = \bigcup_{j=1}^4 G^{(j)}$. From the construction of G we have that, if $(x, y) \in G^{(1)}$, then $x + y \geq a$, hence the convex hull $S = \text{conv}(G^{(1)})$ is a convex and compact set which does not contain the origin $(0, 0)$. Then, there exists $\mathbf{g}^{-1}: S \rightarrow \mathbf{R}^2$ with $\mathbf{g}^{-1} = (h_1, h_2)$ and we have that all the partial derivatives $\partial h_i / \partial x, \partial h_i / \partial y, i = 1, 2$ are continuous on S . Thus, they are bounded on S and let $M > 0$ be a bound of them. Let $(x_1, y_1), (x_2, y_2) \in G^{(1)}$. By the Mean-Value Theorem we have

$$\begin{aligned} \|\mathbf{g}^{-1}(x_1, y_1) - \mathbf{g}^{-1}(x_2, y_2)\|_1 &= |h_1(x_1, y_1) - h_1(x_2, y_2)| + |h_2(x_1, y_1) - h_2(x_2, y_2)| \\ &= |dh_1(a_1, a_2)(x_1 - x_2, y_1 - y_2)| + |dh_2(b_1, b_2)(x_1 - x_2, y_1 - y_2)| \end{aligned}$$

for some $(a_1, a_2), (b_1, b_2) \in [(x_1, y_1), (x_2, y_2)] \subset S$. But

$$\begin{aligned} |dh_i(x, y)(x_1 - x_2, y_1 - y_2)| &= \left| \frac{\partial h_i(x, y)}{\partial x} (x_1 - x_2) + \frac{\partial h_i(x, y)}{\partial y} (y_1 - y_2) \right| \\ &\leq M(|x_1 - x_2| + |y_1 - y_2|) \\ &= M \| (x_1, x_2) - (y_1, y_2) \|_1 \end{aligned}$$

for every $(x, y) \in S$ and $i = 1, 2$. Thus,

$$\|\mathbf{g}^{-1}(x_1, y_1) - \mathbf{g}^{-1}(x_2, y_2)\|_1 \leq 2M \|(x_1, x_2) - (y_1, y_2)\|_1.$$

In other words, \mathbf{g}^{-1} is a Lipschitz function on S . Therefore, $\dim_H \mathbf{g}^{-1}(G^{(1)}) \leq \dim_H G^{(1)} \leq \dim_H G$. Similarly, $\dim_H \mathbf{g}^{-1}(G^{(j)}) \leq \dim_H G$, $j = 2, 3, 4$. Proposition 1 implies that $\dim_H \hat{G} = \dim_H \mathbf{g}^{-1}G = \dim_H \mathbf{g}^{-1}(\cup_{j=1}^4 G^{(j)}) \leq \dim_H G$. This completes the proof. ■

The following theorem, the formal proof of which can be found in [6] or in [15], helps us to tell more about the fractal dimension of the PFIF.

THEOREM 6. *With the same notations as above, if $\sum_{n=1}^N |s_n| > 1$ and the interpolation points do not all lie on a single straight line (i.e., r_i not all equal), the fractal dimension of \hat{G} (and also of G) is the unique real solution D of*

$$\sum_{n=1}^N a_n^{D-1} |s_n| = 1;$$

otherwise the fractal dimension of \hat{G} (and also of G) is 1.

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REFERENCES

1. M. Ali and T. G. Clarkson, Using linear fractal interpolation functions to compress video images, *Fractals* **2** (1994), 417–421.
2. T. M. Apostol, “Mathematical Analysis,” 2nd ed., Addison–Wesley, Massachusetts, 1974.
3. M. F. Barnsley, Fractal functions and interpolation, *Constr. Approx.* **2** (1986), 303–329.
4. M. F. Barnsley, “Fractals Everywhere,” 2nd ed., Academic Press Professional, San Diego, CA, 1993.
5. M. F. Barnsley and S. Demko, Iterated function systems and the global construction of fractals, *Proc. Roy. Soc. London, Ser. A* **399** (1985), 243–275.
6. M. F. Barnsley, J. H. Elton, D. Hardin, and P. Massopust, Hidden variable fractal interpolation functions, *SIAM J. Math. Anal.* **20** (1989), 1218–1242.
7. Deleted in proof.
8. K. J. Falconer, “Fractal Geometry: Mathematical Foundations and Applications,” Wiley, Chichester, 1990.
9. M. Frame and M. Angers, Some nonlinear iterated function systems, *Comput. & Graphics* **18** (1994), 119–125.

10. E. Gröller, Modeling and rendering of nonlinear iterated function systems, *Comput. & Graphics* **18** (1994), 739–748.
11. S. G. Hoggar, “Mathematics for Computer Graphics,” Cambridge Univ. Press, London/New York, 1992.
12. J. E. Hutchinson, Fractals and self similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.
13. D. G. Luenberger, “Linear and Nonlinear Programming,” 2nd ed., Addison–Wesley, Reading, Massachusetts, 1984.
14. P. R. Massopust, Vector-valued fractal interpolation functions and their box dimension, *Aequ. Math.* **42** (1991), 1–22.
15. P. R. Massopust, “Fractal Functions, Fractal Surfaces and Wavelets,” Academic Press, San Diego, CA, 1994.
16. D. S. Mazel and M. H. Hayes, Using iterated function systems to model discrete sequences, *IEEE Trans. Signal Process.* **40** (1992), 1724–1734.
17. J. L. Véhel, K. Daoudi, and E. Lutton, Fractal modeling of speech signals, *Fractals* **2** (1994), 379–382.