The D$k$S problem
Clique graphs
Star of cliques
$C_0$ completely in $S^*$
$C_0$ partially in $S^*$
Tree of cliques
Open questions

# The densest $k$-subgraph problem on clique graphs

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joint work with
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- **Densest k-subgraph (D$k$S)**:

  *Input*: A graph $G = (V, E)$, $|V| = n$, and an integer $k$, $k \leq n$

  *Output*: Find a $k$ vertices subgraph of G with the maximum number of edges

- **D$k$S is strongly NP-hard** as generalization of the CLIQUE problem.

**DkS** remains **NP-hard** even for:

- bipartite graphs, comparability graphs, chordal graphs

  [Corneil and Perl, Discr. Appl. Math., 1984]

- planar graphs

  [Keil and Brecht, J. Comb. Math. Comb. Comp., 1991]

- bipartite graphs of maximal degree three

  [Feige and Seltser, Weizmann Inst., 1997]

**DkS** can be solved in **polynomial** time on:

- trees

  [Perl and Shiloach, SIAM J. Alg. Discr. Methods, 1983]
  [Fischetti et al., Networks, 1994]
  [Goldschmidt and Hochbaum, Discr. Appl. Math., 1997]
  [Rader and Woeginger, Oper. Res. Lett., 2002]

- cographs, split graphs, k-trees

  [Corneil and Perl, Discr. Appl. Math., 1984]

- Approximation algorithms

1. [Kortsarz and Peleg, FOCS, 1993]
2. [Feige and Seltser, Weizmann Inst., 1997]
3. [Srivastav and Wolf, APPROX, 1998]
4. [Ye and Zang, T.R. U. Iowa, 1999]
5. [Asahiro et al., J. of Algorithms, 2000]
6. [Feige et. al., Algorithmica, 2001]
7. [Feige and Langberg, J. of Algorithms, 2001]
8. [Han et al., Math. Progr., 2002]
9. [Billionnet and Roupin, Cedric-CNAM, 2004]

- Best known approximation ratio $\rightarrow$ $\mathbf{O(n^{\frac{1}{3}})}$, [6]

- No one of them achieves a constant approximation ratio

- There is **not a PTAS** for the D$k$S problem

  [Khot, FOCS, 2004]

- There is **not known inapproximability result** for some approximation ratio

- A PTAS for dense graphs

  - of minimum degree $\Omega(n)$
  - of $\Omega(n^2)$ edges when $k$ is $\Omega(n)$

  [Arora et al., J. Comput. Syst. Sci., 1999]

- A **clique** of an undirected graph, $G = (V, E)$, is a subset of its vertices inducing a complete subgraph in $G$.

- The **intersection graph** of a family, $F$, of subsets of a set is defined as a graph, $\mathcal{G}$, whose vertices correspond to the subsets in $F$, and there is an edge between two vertices of $\mathcal{G}$ if the corresponding pair of subsets intersect.

- The **clique graph** of a graph $G$ is defined as the intersection graph of the maximal cliques of $G$.

**All maximal cliques, and hence the clique graph, of a chordal graph can be found in polynomial time.**

- **Star (Tree) of cliques** are graphs that having as clique graph a star (tree).

- $\mathbf{S}$ is a subset of $|S| = k$ vertices
- $\mathbf{E(S)}$ is the number of edges in the subgraph induced by $S$
- $\mathbf{S}^*$ is the optimal solution to the D$k$S problem

- $G = (V, E)$ a star of cliques with $\mathbf{m}$ maximal cliques
- $\mathbf{C_0}$ is the **central** clique
- and $\mathbf{C_1, \ldots, C_{m-1}}$ are the **exterior** maximal cliques of the star of cliques

**$C_0$ intersects with each other clique and no other intersection exists**

For each exterior clique $C_i$,

- $\mathbf{a_i = |C_i \cap C_0|}$, i.e., the number of vertices in its intersection with $C_0$
- $\mathbf{b_i = |C_i| - a_i > 0}$, i.e., the number of its vertices outside $C_0$
- $\mathbf{C_0' = C_0 \setminus \bigcup_{i=1}^{m-1} C_i}$, i.e., is the clique consisting of the vertices of $C_0$ not belonging to any other clique

- A clique $C_i,\ 0 \leq i \leq m-1$, is **completely** in a solution $S$ if all its vertices are in $S$.

- The cliques $C_0$ and $C_0'$ are **partially** in a solution $S$ if a non-empty subset of their vertices, but not all, are in $S$.

- An **exterior** clique $C_i,\ 1 \leq i \leq m-1$ is **partially** in $S$ if a non-empty subset of its $C_i \setminus C_0$ vertices, but not all, are in $S$.

**If an exterior clique $C_i$ is partially in $S^*$, then all its $|C_i \cap C_0| = a_i$ vertices are in $S^*$.**

- A clique **is participating** in a solution $S$ if it is either completely or partially in $S$.

## Proposition

*At most one of the cliques $C_0', C_1, \ldots, C_{m-1}$ is partially in an optimal solution.*

## Proposition

*(i) If $C_0$ is the largest clique i.e., $|C_0| > |C_i|,\ 1 \leq i \leq m-1$, then $C_0$ belongs completely to every optimal solution.*

*(ii) If $C_0$ is partially in an optimal solution $S^*$, then $|C_0| \leq |C_i|$ for every clique $C_i$ participating in $S^*$.*

## Lemma

*If clique $C_0$ is completely in the optimal solution, then there is an $O(nk^2)$ dynamic programming algorithm for the DkS problem on a star of cliques.*

$k' = k - |C_0|$ vertices from exterior cliques

For $i = 0, 1, 2, \ldots, k'$ and $j = 2, 3, \ldots, m-1$

$$f(i,j) = \max_{0 \le q \le \min\{i, b_j\}} \left\{ f(i-q, j-1) + q \cdot a_j + \binom{q}{2} \right\}$$

Complexity: $O(nk^2)$

**Notice that if $C_0$ is the largest clique, then, $C_0$ belongs completely to every optimal solution.**



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## Proposition

*If $C_0$ is partially in an optimal solution, then the number of exterior cliques of size at least $|C_0|$ is at most $\sqrt{n}$.*

$\mathbf{r} = \lfloor \frac{k}{|C_0|} \rfloor$, $\delta$ is a fixed number which will be defined later.

If $r < \delta$, then we proceed in an **exhaustive** manner

- examine all the possible sets of $r$ cliques out of cliques of size at least $|C_0|$
- for each one of these sets we compute the $k$ vertices that maximize the number of edges
  - at most one of the cliques in a set of $r$ cliques is partially in $S^*$
  - consider all the $2^r - 1$ subsets of this set
  - the solutions are all the possible $k$-vertex solutions for the set
- the optimal solution is the one with the maximum number of edges.

**Lemma**

*For the case $r < \delta$, $\delta$ be a fixed number, an optimal solution for the DkS problem in a star of cliques can be found in $O(r\,2^r\,n^{\frac{r}{2}})$ time.*

If $r \geq \delta$, then we proceed in a **greedy** manner

- $S$ is obtained by the following algorithm:
  - $t$ is the largest integer number such that $k \geq \sum_{i=1}^{t} |C_i| = k'$, where $C_1 \geq C_2 \geq \ldots \geq C_{m-1}$
  - return all the vertices of the cliques $C_1 \geq C_2 \geq \ldots \geq C_t$ and $k - k'$ vertices of clique $C_{t+1}$
- all cliques in $S$ are of size at least $|C_0|$ and we need at least $r$ cliques of size $|C_0|$ in order to fill $k$
  thus, $E(S) \geq rE(C_0)$
- $S^*$ involves exterior cliques of size at least $|C_0|$ and these cliques are selected by $S^*$ due to the edges between their overlaps in $C_0$
  thus, $E(S^*) \leq E(S) + E(C_0)$

## Lemma

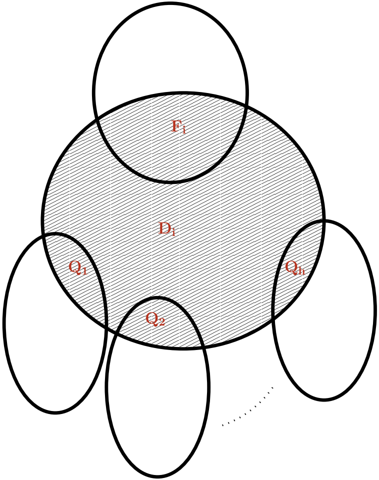
*For the case $r \geq \delta$, where $\delta = \frac{1-\epsilon}{\epsilon}$, $0 < \epsilon < 1$, there is an $(1-\epsilon)$-approximation algorithm for the DkS problem in a star of cliques.*

The complexity of the exhaustive optimal algorithm is exponential in $r \leq \delta = \frac{1-\epsilon}{\epsilon}$, that is exponential in $\frac{1}{\epsilon}$.

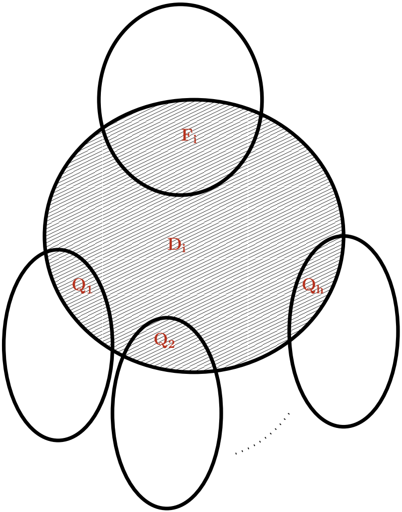
The complexity of the greedy approximation algorithm is $O(n\,logn)$.

**Theorem**

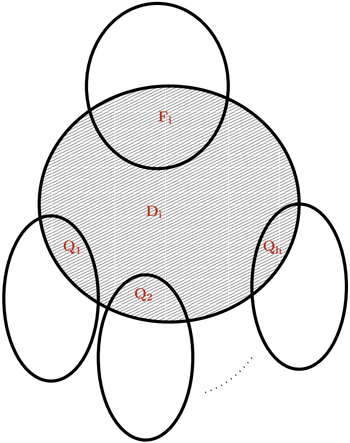
*There is a polynomial time approximation scheme for the DkS problem in stars of cliques.*




- $\mathbf{C_i}$ a non-leaf clique with $m_i \geq 1$ children
- $\mathbf{Q_h}$ the intersection of $C_i$ with its $h^{\text{th}}$ child clique
- $\mathbf{F_i}$ the intersection of the clique $C_i$ with its father clique
- $\mathbf{D_i}$ the vertices of a clique $C_i$ not belonging to any intersection




$\mathbf{f_i(j, a)}$ optimal solution to the D$j$S problem
on the subtree rooted at clique $C_i$
including *exactly a* vertices from the clique $F_i$.




$$f_i(j,a) = \begin{cases} \binom{j}{2}, & \text{if } j \leq \sum_{h=i_1}^{i_{m_i}} |Q_h| + |D_i| + a \\ \max\limits_{a+b+\sum_{h=i_1}^{i_{m_i}} j_h = j} \{ \sum\limits_{h=i_1}^{i_{m_i}} f_h(j_h, a_h) + \binom{a+b}{2} + (a+b)\sum\limits_{h=i_1}^{i_{m_i}} a_h + \\ \qquad \sum\limits_{\substack{i,j=i_1 \\ i \neq j}}^{i_{m_i}} \frac{a_i \cdot a_j}{2} \}, & \text{otherwise.} \end{cases}$$

- $a \in F_i$, $b \in D_i$, $a_h \geq 1$ vertices of each $Q_h$

### Theorem

*There is an $O(nk^{m+1})$ algorithm for the DkS problem on a tree of cliques of maximum degree $m$.*

### Corollary

*There is an $O(nk^3)$ optimal algorithm for the DkS problem on a path of cliques.*

- What is the complexity of the DkS problem on:
  - (Proper) Interval graphs
  - Permutation graphs
- Approximation algorithms for special graph classes:
  - Bipartite graphs
  - Comparability graphs
  - Chordal graphs
  - Planar graphs
  - Bounded degree graphs (even bipartite)
  - Regular graphs
- Better approximation algorithms for arbitrary graphs