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An approximation scheme for strip packing of rectangles with bounded dimensions

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Abstract

It is shown that for any positive ε the strip-packing problem, i.e. the problem of packing a given list of rectangles into a strip of width 1 and minimum height, can be solved within $1+\varepsilon$ times the optimal height, in linear time, if the heights and widths of these rectangles are all bounded below by an absolute constant $\delta > 0$. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let L be a list of n (not necessarily distinct) pairs of positive numbers (l_i, h_i), $1 \leqslant i \leqslant n$. Each of these pairs specifies the dimensions, say width and height, of a rectangle. In the strip-packing problem we want to find (or to approximate) the minimum height of a vertical strip of width 1 into which all these rectangles, which we will also call pieces, can be packed, i.e. from which these rectangles can be obtained using only horizontal cuts (for each "width" side) and vertical cuts (for each "height" side). It is assumed that all the widths are bounded from above by 1. This model applies to certain scheduling and stock-cutting problems [2]. Let us consider, for instance, the multiprocessor scheduling problem. Here, the pieces may represent jobs to be executed on an unlimited number of processors with a limited common memory. The width of each piece represents the amount of memory required by the corresponding job and the height represents the required processing time. The question addressed here is then equivalent to asking for a schedule for which the total execution time is minimum.

For any list L of rectangles let $\mathrm{OPT}(L)$ denote the minimum height needed to pack L and for any algorithm A let $A(L)$ denote the height used by A to pack L. If the inequality

$$A(L) \leqslant \alpha\, \mathrm{OPT}(L) + \beta \qquad (1)$$

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holds for some fixed constants α and β and for every list L of rectangles with heights at most a fixed constant h and $\mathrm{OPT}(L)$ sufficiently large, then α is called an *asymptotic worst-case performance bound* of A. The best asymptotic bound obtained so far for a polynomial time algorithm is $\frac{5}{4}$ and is due to Baker et al. [1] who improved previous results of Coffman et al. [4] and Golan [6].

Let $S(L)$ denote the total area of the pieces in L. Let the *performance ratio* of a packing algorithm A be the infimum of the α's for which (1) holds for every list L with sufficiently large $S(L)$. The purpose of this paper is to show that, for any given $\varepsilon>0$ and $\delta>0$, there exists a linear time algorithm which has performance ratio $\leqslant 1+\varepsilon$ *when restricted to lists of rectangles with both dimensions bounded below by* δ. We shall prove the following theorem.

Theorem. *For any given $\varepsilon>0$, $h>0$ and δ with $0<\delta\leqslant\min\{1,h\}$ there is an algorithm which packs any list L of rectangles with heights bounded above by h and heights and widths bounded below by δ, into a strip of width 1 and height at most*

$$(1+\varepsilon)\,\mathrm{OPT}(L)+\beta,$$

where β is an absolute constant. This algorithm has a time bound of the form C_1n+C_2 where n is the length of L and C_1 and C_2 are constants which depend only on ε, h and δ.

The hypothesis that the heights are bounded above is standard in the strip-packing problem. Kenyon and Rémila have shown recently that our additional hypothesis that the widths and the heights are bounded below is not necessary, i.e. the strip-packing problem is now known to have a polynomial time approximation scheme [9].

A similar approximation scheme was obtained by Fernandez de la Vega and Lueker [5] for the one-dimensional bin-packing problem, without any restriction on the sizes of the pieces. In [5] and in the present paper, the execution time grows worse than exponentially as a function of $1/\varepsilon$. Johnson [7] observed that, by letting ε depend on the instance L, one could use any such scheme to construct a polynomial time algorithm A such that $A(L)\leqslant\mathrm{OPT}(L)+o(\mathrm{OPT}(L))$. Johnson's observation applies, in particular, to the scheme defined in this paper. Karmarkar and Karp [8] gave several algorithms with execution time growing only as a polynomial function of $1/\varepsilon$ for the one-dimensional bin-packing problem.

2. Proof of the theorem

As in [5], the main idea of the proof consists of reducing the problem to a restricted case where the number of distinct rectangles is bounded (by a function of ε, δ and h) and which can be approximately solved in constant time. However, an important difference is that in the case of one-dimensional bin packing it is almost trivial to deal with the small pieces whereas in the present case there does not seem to be a simple way to do so. This is why we need the condition that the heights and widths be bounded below. We will need the following propositions.

Proposition 2.1. *Any list L of rectangles satisfies*

$$\mathrm{OPT}(\mathrm{L}) \leqslant 2S(L) + h,$$

where h denotes the maximum height of the rectangles in L.

This proposition is a partial result of Coffman et al. [4] obtained in the course of their proof that a certain algorithm A, when applied to a list L of maximum height 1, gives a packing of this list with height $A(L) \leqslant 2\mathrm{OPT}(L) + 1$. (See [4, Theorem 1]). However, this algorithm requires sorting the list in the order of non-increasing height, and is not fast enough for our purposes. The next proposition gives a performance bound for a linear time algorithm on lists with a given minimum height.

Proposition 2.2. *There is a linear time algorithm A which packs any list L of n rectangles with maximum height h and minimum height δ into a strip of height*

$$A(L) \leqslant h(2\delta^{-1}S(L) + 1). \tag{2}$$

Proof. Pack the rectangles in L in the order in which they appear, using successive layers each of height h. Any two successive layers contain rectangles whose total area exceeds δ. Therefore, $A(L)$ satisfies

$$A(L) \leqslant h\lceil 2S(L)\delta^{-1}\rceil \leqslant h(2\delta^{-1}S(L) + 1).$$

The outline of the rest of this section is as follows. The reduction of a general list L to the case where the number of distinct rectangles is bounded is presented in Section 2.1. In Section 2.2 we present another reduction (independent from the previous one) to a (non-standard) bin-packing problem. In Section 2.3 we show how this bin-packing problem can be solved in constant time on lists with a bounded number of distinct rectangles. Next, we conclude the proof by deriving from the previous results an algorithm with the properties stated in the theorem. Finally, we present some remarks and open problems.

2.1. Reduction to the case of a bounded number of distinct rectangles

In the sequel we occasionally treat lists as multi-sets. For instance, the intended meaning of the notation $K = \bigcup_{j \in J} K_j$ where the K_j's are lists is that K is a list containing each piece belonging to some K_j with a multiplicity equal to the sum of its multiplicities in each K_j. Assuming that the K_j's are sorted, it is not hard to see that the list K can be constructed from the K_j's in linear time. Similarly, $L \backslash K$ denotes a list containing each piece x in L with multiplicity equal to $\max\{0, \mu_L(x) - \mu_K(x)\}$, where $\mu_L(x)$ (resp. $\mu_K(x)$) denotes the multiplicity of x in L (resp. K).

Let us define the type of a rectangle as the ordered pair (l,h) where l is the width and h is the height of this rectangle. Let $\mathscr{L}(n,h,\delta)$ denote the set of lists containing n rectangles with heights bounded above by h and heights and widths bounded below by δ. The purpose of this section is to prove the following proposition.

Proposition 2.3. *Let $\varepsilon_1 > 0$ be given. There is a linear time algorithm which, when applied to a list $L \in \mathscr{L}(n,h,\delta)$ with sufficiently large $S(L)$, gives two lists L_1 and L_2 with the following properties*:

(i) *L_1 satisfies $S(L_1) \leqslant (\varepsilon_1/3)S(L)$*;

(ii) *L_2 contains no more than a constant number $C = C(\varepsilon_1, \delta, h)$ of distinct types of rectangles*;

(iii) *any packing P of $L_1 \cup L_2$ can be converted in linear time into a packing of L whose height is at most $(1+\varepsilon_1)$ times that of P;.*

(iv) $\mathrm{OPT}(L_2) \leqslant (1+\varepsilon_1/3)\mathrm{OPT}(L)$.

Property (ii) is essential. It insures (see Section 2.3) that L_2 can be packed optimally in constant time depending only on δ, h and ε_1.

Proof. For any rectangle x we denote its width by $w(x)$. Let η satisfy $0 < \eta < 1$. Let m denote a positive integer to be fixed later. Define

$$l = \lceil \log_{\eta^{-1}}(h/\delta) \rceil.$$

For each j with $2 \leqslant j \leqslant l$, let M_j denote the list obtained from the sublist of L containing the pieces with heights in the interval $[h\eta^j, h\eta^{j-1})$ by setting the heights of all these pieces equal to $h\eta^j$. Let M_1 be defined similarly, but by choosing the heights in the *closed* interval $[h\eta, h]$.

We define now a grouping of the pieces, which are similar to the one used in Fernandez de la Vega and Lueker [5].

For $1 \leqslant j \leqslant l$, let us denote by $n_j = |M_j|$ the length of the list M_j, and let us define q_j by $n_j = mq_j + r_j$, $0 \leqslant r_j \leqslant m-1$.

For $1 \leqslant j \leqslant l$, let $K_j = y_{j,1}Q_{j,1}y_{j,2}Q_{j,2}\cdots y_{j,m}Q_{j,m}R_j$ be any list obtained by rearranging M_j in such a way that the following conditions (a)–(c) are satisfied.

(a) $|Q_{j,1}| = |Q_{j,2}| = \cdots = |Q_{j,m}| = q_j - 1$;

(b) for $1 \leqslant i \leqslant m-1$, each piece x in the list $Q_{j,i}$ satisfies $w(y_{j,i}) \leqslant w(x) \leqslant w(y_{j,i+1})$;

(c) each piece x in $Q_{j,m}R_j$ satisfies $w(x) \geqslant w(y_{j,m})$.

Let $J = \{j\colon S(M_j) \geqslant \varepsilon_1 S(L)/3l\}$. (Notice that, for each $j \in J$, q_j tends to infinity with $S(L)$).

Finally, for $j \in J$, set $K_{3,j} = y_{j,1}^{q_j} y_{j,2}^{q_j} \cdots y_{j,m}^{q_j}$ and $K_{2,j} = y_{j,2}^{q_j} y_{j,3}^{q_j} \cdots y_{j,m}^{q_j} y_j^{q_j+r_j}$, where y_j denotes the rectangle with width 1 and height $h\eta^j$.

Let

$$L_1 = \bigcup_{j \notin J} K_j, \qquad L_2 = \bigcup_{j \in J} K_{2,j}, \qquad L_3 = \bigcup_{j \in J} K_{3,j}, \quad K = \bigcup_{1 \leqslant j \leqslant l} K_j.$$

Assertion (i) is implied by the definition of L_1.

Notice that, in going from K_3j to K_2j, we add $q_j + r_j$ pieces of width 1 and height $h\eta^j$ (and suppress some other pieces). Observe that we have $K_j = M_j$. Again for $j \in J$, we have $q_j \leqslant m^{-1}|M_j| = m^{-1}|K_j|$, where the first inequality follows from the definition of q_j. We have also $S(K_{2,j}) \geqslant \delta h\eta^j mq_j$. This implies, with the previous inequality,

$$\frac{S(K_{2,j}\backslash K_{3,j})}{S(K_{2,j})} \leqslant \frac{(q_j + r_j)h\eta^j}{\delta h\eta^j mq_j} \leqslant \frac{q_j + m - 1}{\delta mq_j} \leqslant \frac{1}{\delta m} + \frac{1}{\delta q_j} \leqslant \frac{\varepsilon_1}{8} \tag{3}$$

for sufficiently large $S(L)$ if we choose $m = \lceil 9/(\varepsilon_1\delta)\rceil$ and use the fact that q_j tends to infinity with $S(L)$. This implies

$$S(L_2\backslash L_3) \leqslant (\varepsilon_1/8)S(L_2) \leqslant (\varepsilon_1/7)S(L_3).$$

Clearly,

$$\mathrm{OPT}(L_2) \leqslant \mathrm{OPT}(L_3) + \mathrm{OPT}(L_2\backslash L_3).$$

This gives, using Proposition 2.1 applied to the list $L_2\backslash L_3$ and the obvious inequality $S(L_3) \leqslant \mathrm{OPT}(L_3)$

$$\mathrm{OPT}(L_2) \leqslant \mathrm{OPT}(L_3) + (2\varepsilon_1/7)\mathrm{OPT}(L_3) + h \leqslant (1 + \varepsilon_1/3)\mathrm{OPT}(L_3)$$

for sufficiently large $S(L)$ and assuming $\varepsilon_1 \leqslant 1$. This implies, of course,

$$\mathrm{OPT}(L_2) \leqslant (1 + \varepsilon_1/3)\mathrm{OPT}(L)$$

concluding the proof of assertion (iv).

Clearly, there exists a one-to-one mapping from the multiset $L_1 \cup L_2$ to the multiset K which is non-increasing on both coordinates. Thus, a packing containing the two lists L_1 and L_2 gives trivially a packing of K. We can then obtain in an obvious way, a packing of L from a packing of $L_1 \cup L_2$ by making a linear transformation in the vertical direction with parameter η^{-1}. Therefore, assertion (iii) will be true if we choose $\eta = (1 + \varepsilon_1)^{-1}$.

In order to conclude the proof of Proposition 2.3, it remains to verify that the involved computations can, indeed, be done in linear time. This is clear for the computations which follow the construction of the K_j's (since m and η are constants for given h, δ and ε_1). Concerning the construction of the K_j's, it suffices to observe as in [5], that finding for some fixed index j a list satisfying the conditions imposed on K_j, amounts essentially to find the elements $y_{j,1}, \ldots, y_{j,m}$ in the list M_j, i.e. to solve a fixed number of instances of the "selection problem" which is well known to be linear [3]. □

2.2. *Reduction of approximate strip packing to a two-dimensional bin-packing problem*

Proposition 2.4. *Let h and $\varepsilon > 0$ be given. Consider a list L of rectangles with heights bounded above by h and with $S(L)$ sufficiently large. There is a number H such that* *any packing of L into bins of height H using no more than* $(1+\varepsilon/2)\mathrm{OPTBIN}_H(L)$ *bins,* ($\mathrm{OPTBIN}_H(L)$ *denotes the minimum number of such bins into which L packs), can be converted into a strip-packing of L whose height is at most* $(1+\varepsilon)\,\mathrm{OPT}(L)$ *in linear time.*

Proof. We claim that

$$(H-h)(\mathrm{OPTBIN}_H(L)-1)\leqslant \mathrm{OPT}(L)\leqslant H\,\mathrm{OPTBIN}_H(L). \tag{4}$$

The right-hand side inequality is trivial (just pile the bins one above the other). For the left-hand side inequality, observe that from any strip packing of L of height l, say, we can deduce a packing into $k=\lceil l/(H-h)\rceil\leqslant(l/(H-h))+1$ bins of height H: just put in the $(j+1)$th bin all the pieces which are stictly contained between the levels $j(H-h)$ and $j(H-h)+H$ in the strip packing, $0\leqslant j\leqslant k-1$. This concludes the proof of the claim.

Now, assume that a bin packing of L using no more than $(1+\varepsilon/2)\,\mathrm{OPTBIN}_H(L)$ bins of height H has been found. It gives a strip packing of height $(1+\varepsilon/2)H\,\mathrm{OPTBIN}_H(L)$ which, by the left-hand side of (4), will be optimal within $1+\varepsilon$ as desired if the inequality

$$(1+\varepsilon/2)H\,\mathrm{OPTBIN}_H(L)\leqslant(1+\varepsilon)(H-h)(\mathrm{OPTBIN}_H(L)-1)$$

holds. This will be true if

$$\frac{1+\varepsilon/2}{1+\varepsilon}\leqslant\frac{H-h}{H}\,\frac{\mathrm{OPTBIN}(L)-1}{\mathrm{OPTBIN}(L)},$$

which clearly holds for large enough H and $\mathrm{OPTBIN}(L)$. Since $\mathrm{OPTBIN}_H(L)\geqslant H^{-1}S(L)$, once H is selected we can make $\mathrm{OPTBIN}_H(L)$ large enough by picking $S(L)$ large enough. □

2.3. A constant time algorithm for bi-dimensional bin packing with a bounded number of distinct rectangles

Proposition 2.5. *Let* $\delta>0$, $h>0$, $H\geqslant h$ *and* $m\in\mathbb{N}$ *be given. There is an algorithm which, for any sequence of triples* $(l_1,h_1,n_1),\dots,(l_m,h_m,n_m)$ *with* $\delta\leqslant l_i\leqslant 1$, $\delta\leqslant h_i\leqslant h$, $n_i\in\mathbb{N}, 1\leqslant i\leqslant m$, *finds in constant time an optimal packing of the multiset of rectangles* $(l_1,h_1)^{n_1},\dots,(l_m,h_m)^{n_m}$ *into bins of height* H.

Proof. Given the set of types of rectangles $\{(l_1,h_1),\dots,(l_m,h_m)\}$ define a bin type as an m-tuple of non-negative integers $(k_1,k_2,\dots,k_m)$ with the property that the multi-set of rectangles $(l_1,h_1)^{k_1},\dots,(l_m,h_m)^{k_m}$ can be packed into a bin of height H.

Notice first that, because of the lower bounds imposed on both dimensions, at most $q=\lfloor H/\delta^2\rfloor$ rectangles can enter into the same bin and thus the number of bin types is bounded above by

$$\binom{m+q-1}{m-1},$$

which is the number of ways one can choose m non-negative integers which add to q.

Now, it remains to select within the multisets which are not discarded by this obvious area argument, those which are actually bin types. Let us show first that the problem of deciding whether or not some set K of rectangles whose cardinality is bounded by a fixed integer q, can be packed into a bin of width 1 and fixed height can be solved in constant time. Let us say that a packing into a bin is *left-bottom justified* if no rectangle in this packing can be moved downwards or to the left without overlapping other rectangles. Clearly, if K packs into a bin B, there exists a left-bottom justified packing of K in B and we can therefore consider only such packings. Let $|K| = n$ with $n \leqslant q$. We claim that if $P = P_n$ is a left-bottom justified packing of K, there exist left-bottom justified packings $P_o = \emptyset$, $P_1, \ldots, P_n = P$, where for each $0 \leqslant k \leqslant n-1$, P_{k+1} is obtained from P_k by adding a new piece (and leaving the positions of the pieces in P_k unchanged). Using induction, it suffices to prove that P contains a (left-bottom justified) piece p whose upper- and right-hand side are in contact with no other piece so that the packing obtained by removing p is also left-bottom justified. To this end, let us define the sequence of pieces $p_1, \ldots, p_k, \ldots$ where p_1 is the rightmost piece of P with highest upper side and, for each k, p_{k+1} is the highest piece with has a vertical contact with the right side of p_k. Thus, the upper side of p_{k+1} has no contact with any other piece. Clearly, this sequence is finite, ending with a piece $p = p_h$ say, which has neither upper nor right-hand side contact. We can thus choose $p = p_h$. and this concludes the proof of the claim.

For any left-bottom justified packing P let us call a corner defined by the right side of a piece (or the left side of the bin) and the upperside of another piece (or the bottom side of the bin) an "active corner". Such corners and only such corners may be occupied by an other piece to extend P. Note that

(a) the horizontal component of the left edge of the new piece must match the horizontal component of the left edge of the bin or the right edge of some piece already packed, and

(b) the vertical component of the bottom edge of the new piece must match the vertical component of the bottom edge of the bin or the top edge of some piece already packed.

It follows immediately that a packing with k pieces has at most $(k+1)^2$ active corners. This assertion together with the previous claim imply that we can decide in time $\leqslant (q!)^3$ whether or not some given multi-set of pieces can enter into a bin. (To see this, order the pieces in every possible way and observe that a left-bottom justified packing with k pieces can be extended to a left-bottom justified packing with $k+1$ pieces in at most $(k+1)^2$ ways.) The total time needed to find the distinct possible types of bins is thus bounded above by the constant

$$(q!)^3\binom{m+q-1}{m-1}.$$

It remains to show that, having found the (bounded) set of possible types of bins, we can find an optimal packing of our multi-set of rectangles in constant time. Exactly as in [5], this problem amounts to solving an integer linear program in which both the number of variables and the number of constraints are bounded above by constants. This can, indeed, be done in constant time (see [10]). □

2.4. End of the proof

In order to get a packing of a list L with approximation ratio $1+\varepsilon$ get first the lists L_1 and L_2 with the properties stated in Proposition 2.3 with $\varepsilon_1=\delta\varepsilon/10h$ (and $\eta=(1+\varepsilon_1)^{-1}$). By using the algorithm of Proposition 2.2, pack the list L_1 within height at most

$$2h\delta^{-1}S(L_1)+h\leqslant\frac{\varepsilon S(L)}{15}+h\leqslant(\varepsilon/6)\mathrm{OPT}(L).$$

Then, use the algorithm described in Section 2.3, with $H=7\varepsilon^{-1}$, to obtain in constant time (since the number of distinct types in L_2 is bounded) an optimal packing of L_2 into bins of height H. Convert this packing in the obvious way into a strip packing of L_2 with height bounded by $(1+\varepsilon/6)\mathrm{OPT}(L_2)\leqslant(1+\varepsilon/6)(1+\varepsilon_1/3)\mathrm{OPT}(L)$ where the first bound results from Proposition 2.4 with $\varepsilon/3$ in place of ε and the second bound is obtained by using assertion (iv) of Proposition 2.3. Using assertion (iii) in Proposition 2.3, we can deduce from these packings of L_1 and L_2 a packing of L whose height is bounded above by

$$(1+\varepsilon_1)(1+\varepsilon/6)(1+\varepsilon_1/3)\mathrm{OPT}(L)+(\varepsilon/6)\mathrm{OPT}(L)\leqslant(1+\varepsilon)\mathrm{OPT}(L)$$

for $\varepsilon\leqslant 1/2$ and $\varepsilon_1\leqslant\varepsilon/2$ (by a routine check). This concludes the description of the algorithm and the proof of the theorem. □

3. Summary and conclusions

We have shown that the strip-packing problem can be solved within $1+\varepsilon$ in linear time if the dimensions of the pieces to be packed are bounded below by a positive constant. Our work raises the following questions and remarks.

- The reduction to two-dimensional bin-packing problem with a bounded number of pairwise distinct pieces is possible here because we can stretch slightly the pieces in the vertical direction with only a small loss in the objective function. Apparently, a similar reduction is not possible in the standard two-dimensional bin-packing problem. As it was mentioned by Fernandez de la Vega and Lueker [5], a basic obstruction comes from the fact that there is no natural order in the set R^k for any $k \geqslant 2$.

The search for the allowable bin types is done in Section 2.3 by a brute force method which suffices for our needs. It would be interesting to find a more efficient algorithm. More specifically we ask if there exists an algorithm which, given a list of n rectangles, decides in exponential time, i.e. in time bounded by C^n where C is an absolute constant, whether or not this list can be packed into a rectangle with given dimensions.

– Consider the following "bi-dimensional knapsack" problem: given a list L of rectangles, what is the maximum of the total area of a sublist of rectangles in L which can be packed in the unit square. Can this problem be solved within $1+\varepsilon$ in polynomial time?

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