Dıscrete Knapsack

input $\quad |x| = n \quad \mathrm{c_i}, a_\mathrm{i}, b$ integers

output $\quad y \subseteq x \;$ s.t. $\sum_{\mathrm{x_i} \in \mathrm{y}} a_i \le b$

$$\max \sum_{\mathrm{x_i} \in \mathrm{y}} c_i$$

Greedy Knapsack

$$
\begin{array}{l}
\textit{begin} \\
\qquad p_i / a_i \quad \text{in non-increasing order} \\
\qquad (* x_1, x_2, \ldots, x_n *) \\
\qquad y := 0 \\
\qquad \textit{for } i := 1 \textit{ to } n \textit{ do} \\
\qquad\qquad \textit{if } b \geq a_i \textit{ then} \\
\qquad\qquad \textit{begin} \\
\qquad\qquad\qquad y := y \cup \{x_i\} \\
\qquad\qquad\qquad b := b - a_i \\
\qquad\qquad \textit{end} \\
\qquad \textit{return } y \\
\textit{end}
\end{array}
$$

Knapsack

$$x_i \in \{1,0\}$$

$$Max \sum_{i=1}^{n} c_i x \qquad \sum_{i=1}^{n} a_i x_i \leq b$$

$$\frac{c_{j1}}{a_{j1}} \geq \frac{c_{j2}}{a_{j2}} \geq \ldots \geq \frac{c_{jn}}{a_{jn}}$$

Knapsack

Παράδειγμα

$$\text{Max} \quad (k+1)x_1 + kx_2 + kx_3$$

$$(k+1)x_1 + kx_2 + kx_3 \le 2k$$

$$k = 1,2,\ldots$$

Heuristic

$$\frac{k+1}{k+1} \ge \frac{k}{k} \ge \frac{k}{k}$$

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 0 \\ x_3 = 0 \end{array}\right\} \qquad \rho = \frac{H}{Opt} = \frac{k+1}{2k} = \frac{1}{2}$$

$$H = k+1$$

Knapsack

$$\text{Max} \quad 2x_1 + kx_2 + kx_3$$

$$x_1 + kx_2 + kx_3 \leq k$$

$$H = \frac{2}{1} \geq \frac{k}{k}$$

$$x_1 = 1 \qquad \frac{\mathrm{H}}{\mathrm{Opt}} = \frac{2}{k} \xrightarrow[k \to \infty]{} 0$$

$$x_2 = 0$$

Greedy Knapsack

$$p_i = a_i = 1, \quad i = 1, \ldots, n-1$$

$$p_n = b - 1$$

$$a_n = b = kn$$

$$\left. \begin{array}{l} m^*(x) = b - 1 \\ m_G(x) = n - 1 \end{array} \right\} \Rightarrow \frac{m^*(x)}{m_G(x)} > k$$

k arbitrary large number

Knapsack

Algorithm

$$m_A(x) = \max\left(P_{\max}, m_G(x)\right)$$

$$P_{\max} = \max_{1 \le i \le n} \left\{P_i\right\}$$

$$\Downarrow$$

$$\frac{m^*(x)}{m_A(x)} < 2$$

Knapsack

Algorithm

$$\overline{P_j} = \sum_{i=1}^{j-1} P_i \le m_G(x)$$

$$\overline{a_j} = \sum_{i=1}^{j-1} a_i \le b$$

$$m^*(x) < \overline{P_j} + P_j \qquad \left(\overline{a_j} + a_j > b\right)$$

$$m^*(x) \le \overline{P_j} + \left(b - \overline{a_j}\right)\frac{P_j}{a_j} < \overline{P_j} + P_j$$

If $P_j \le \overline{P_j}$

$$m^*(x) < 2\overline{P_j} \le 2m_G(x) \le 2m_A(x)$$

If $P_j \ge \overline{P_j} \quad \left(P_{\max} \ge \overline{P_j}\right)$

$$m^*(x) < \overline{P_j} + P_j \le \overline{P_j} + P_{\max} < 2P_{\max} \le 2m_A(x)$$