The subset-Sum Problem

$\{(s,t)\}$

$\text{Input}: s \subset \aleph, t \in \aleph$

$\text{Output}: s' \subseteq s \left| \sum_{s \in S'} s = t \right.$

$s = \{1,4,16,64,256,1040,1041,1093,1284,1344\}, t = 3754$

$\Downarrow$

$s' = \{1.16,64,256,1040,1093,1284\}$

$\bullet \text{NP-Complete } (\text{VC})$

$\rightarrow \text{optimization problem}$

$\text{output}: s' \subseteq s \left| \max_{S'} \sum_{s \in S'} s \leq t \right.$

MERGING two sorted Lists

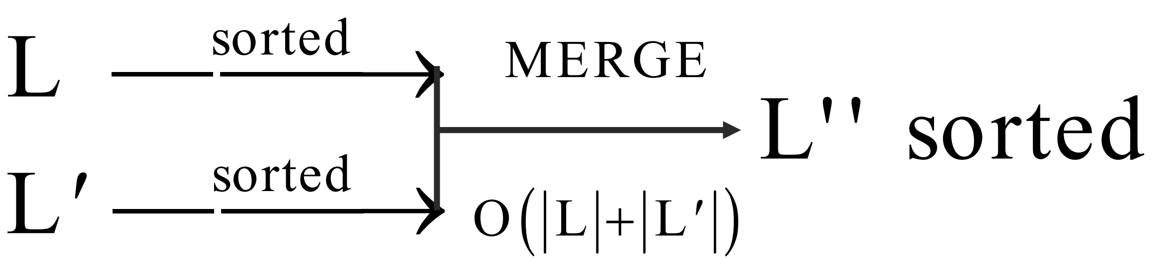
L: list of integers , x integer

$$L + x = \{ l_i + x \mid l_i \in L \}$$

$$L = \{1,2,3,5,9\}$$

$$\downarrow x = 2$$

$$L + x = \{3,4,5,7,11\}$$



Algorithms ~~complexity??~~

ESS(S,t)

$n = |s|$

$L_o = \{0\}$

for $i = 1$ to n do

$\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + X_i)$

$\quad$ remove from L_i $\quad X | X \rangle t$

return the largest element in L_n

$S_i = \{X_1, X_2, X_3, \ldots, X_i\}$

$S \subseteq S_i$

$$P_i = \left\{ X \middle| X = \sum_{s \in S} s \right\}$$

Example

$$\{1,4,5\}$$

$$S_1 = \{1\} \Rightarrow \Phi, \{1\} \Rightarrow P_1 \{0,1\}$$

$$S_2 = \{1,4\} \Rightarrow \Phi, \{1\}, \{4\}, \{1,4\} \Rightarrow P_2 \{0,1,4,5\}$$

$$S_3 = \{1,4,5\} \Rightarrow \Phi, \{1\}, \{4\}, \{5\}\{1,4\}, \{4,5\}, \{1,5\}, \{1,4,5\} \Rightarrow P_3 \{0,1,4,5,6,9,10\}$$

$$P_i = P_{i-1} \cup \left(P_{i-1} + X_i\right) \rightarrow ά\sigma\kappa\eta\sigma\eta\,!$$

$$\updownarrow$$

$$L_i \left(\text{ordered} \leq t\right)$$

Complexity

Πολ/τα ESS

$$\underbrace{|L_i| \to 2^i}_{\text{expo}}$$

Polynomial :

- t polynomial in $|S|$
- $\forall x \in S \Rightarrow x$ bounded by a polynomial in $|S|$

A fully Polynomial-time Approximation Scheme (FPTAS)

Approximation scheme

$\Rightarrow$ "trimming" each L_i

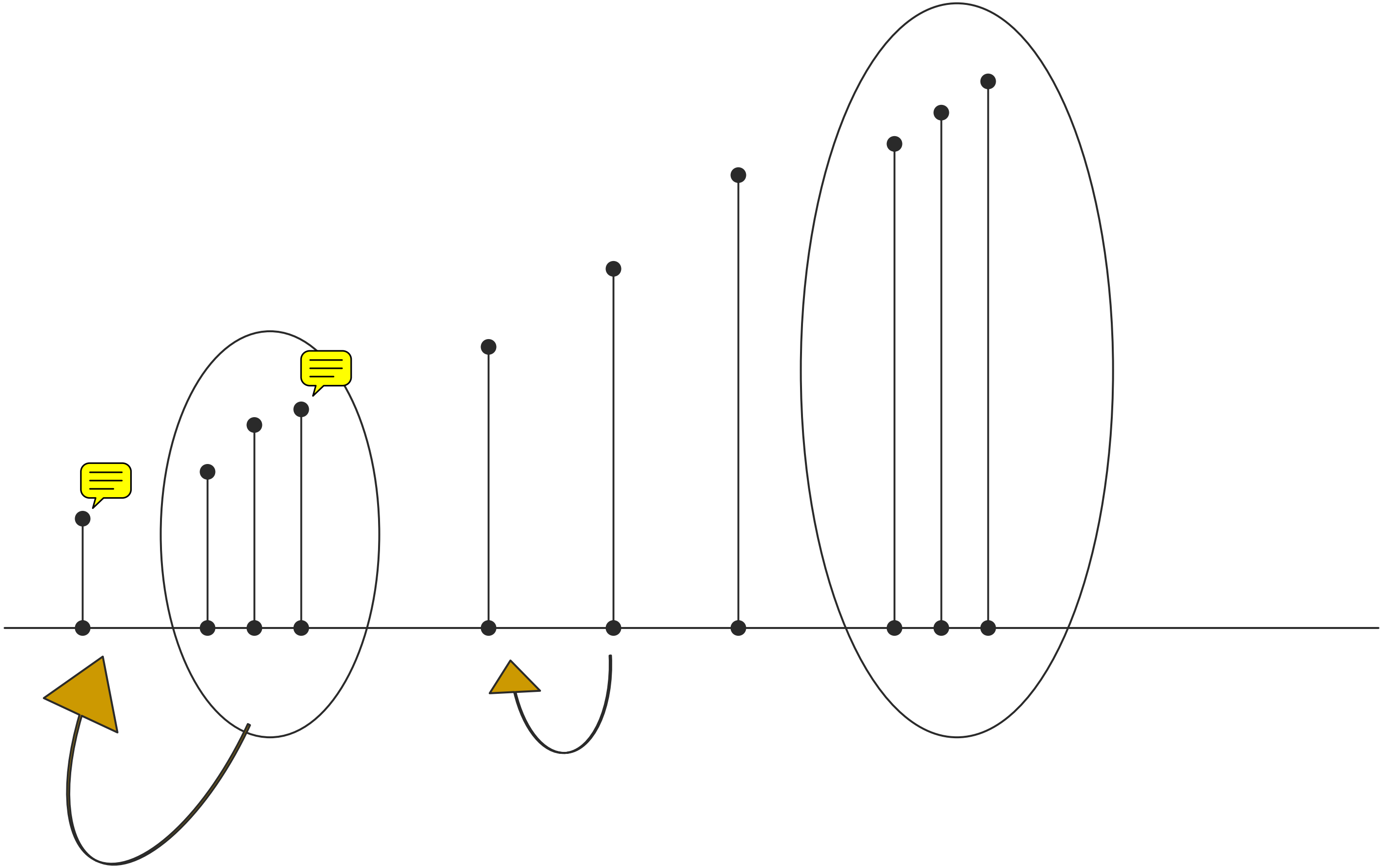
- $0 \langle \delta \langle 1$

$$L \xrightarrow{\text{trimming}} L' = L - \{y\}$$

$$\forall y \Rightarrow \exists z \in L' \left| \frac{y - z}{y} \right| \leq \delta$$

$$((1 - \delta) y \leq z \leq y)$$

Trimming



Representative

ex

$$\delta = 0.1$$

$$L = \{10,11,12,15,20,21,22,23,24,29\}$$

$$L' = \{10,12,15,20,23,29\}$$

$$(L' \subseteq L)$$

L' in polynmial time

The Trim algorithm

$L = \{y_1, y_2, \ldots, y_m\}$ ↗ sorted

$\mathrm{TRIM}(L, \delta)$

$m \leftarrow |L|$

$L' \leftarrow \langle y_1 \rangle$

$\mathrm{last} \leftarrow y_1$

for $i \leftarrow 2$ to m do if $\mathrm{last} \langle (1 - \delta) y_i$

then append y_i onto the end of L'

return L' $\qquad$ $\mathrm{last} \leftarrow y_i$

$L \xrightarrow{\Theta(m)} L'$ (sorted)

$S=\{x_1,x_2,...,x_n\}$ (in increasing order)

$t \in \aleph, 0 \langle \varepsilon \langle 1$

APPROX-SUBJECT-SUM (s,t,ε)

$n \leftarrow |S|$

$L_0 \leftarrow <0>$

P_i Trimmed $\left\{ \begin{array}{l} \text{for } i \leftarrow 1 \text{ to } n \\ \quad \text{do } L_i \leftarrow \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + X_i) \\ \quad\quad L_i \leftarrow \text{TRIM}(L_i, \varepsilon / n) \end{array} \right.$

let z be the largest value in L_n

return z

Example

ex:

$$L = \{104, 102, 201, 101\}, t = 308$$

$$\varepsilon = 0.20$$

$$\Downarrow$$

$$\delta = \frac{\varepsilon}{4} = 0.05$$

$$L_0 = \langle 0 \rangle,$$

$$\left\{ \begin{array}{l} L_1 = \langle 0, 104 \rangle, \\ L_1 = \langle 0, 104 \rangle, \\ L_1 = \langle 0, 104 \rangle, \end{array} \right\}$$

Example

$$\left\{ \begin{array}{c} L_2 = \langle 0, 102, 104, 206 \rangle, \\ L_2 = \langle 0, 102, 206 \rangle, \\ L_2 = \langle 0, 102, 206 \rangle, \end{array} \right\}$$

$$\left\{ \begin{array}{c} L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle, \\ L_3 = \langle 0, 102, 201, 303, 407 \rangle, \\ L_3 = \langle 0, 102, 201, 303 \rangle, \end{array} \right\}$$

$$\left\{ \begin{array}{c} L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle, \\ L_4 = \langle 0, 101, 201, 302, 404 \rangle, \\ L_4 = \langle 0, 101, 201, 302 \rangle, \end{array} \right\}$$

$$\text{optimal}: 104 + 102 + 101 = 307 \left(2 \% \right)$$

$$\varepsilon = 20 \%$$

Proof (approximation)

APPROX-SUBSET-SUM

↓

FPTAS

- $L_i \subseteq P_i$
- $z = \sum_{s \in S'} s, \quad S' \subseteq S \quad (output)$

$$\frac{|C - C^*|}{C^*} \leq \varepsilon(n)$$

$$\Updownarrow$$

$$\boxed{C^*(1 - \varepsilon) \langle C}$$

Proof (approximation)

$$\not{L}_i \xrightarrow{\text{Trim}} L_i'$$

$$y \xrightarrow{(\geq)} z$$

$$(1-\delta)y \leq z \leq y$$

$$\forall\ y \in P_i \,|\, y \leq t \Rightarrow \exists\ z \in L_i\ \underline{\underline{s.t}}$$

$$\left(1-\frac{\varepsilon}{n}\right)^i y \leq z \leq y\ (\acute{\alpha}\sigma\kappa\eta\sigma\eta)\ \text{by induction on } \underline{\underline{i}}$$

$$\text{If } y^* \in P_n \Rightarrow \exists\ z \in L_n\ \underline{\underline{s.t}}$$

$$\left(1-\frac{\varepsilon}{n}\right)^n y^* \leq z \leq y^*$$

↑ algorithm

$$\frac{d}{dn}\left(1-\frac{\varepsilon}{n}\right)^{n} > 0 \Rightarrow$$

$$\left(1-\frac{\varepsilon}{n}\right)^{n} \quad \overset{n>1}{\Rightarrow}$$

$$(1-\varepsilon) < \left(1-\frac{\varepsilon}{n}\right)^{n} \Rightarrow$$

$$(1-\varepsilon)\, y^{*} \leq z$$

The TIME :

After trimming

$\Downarrow$ successive elements

z, z' of L_i

$$\frac{z}{z'} > \frac{1}{1-\frac{\varepsilon}{n}}$$

$$|L_i| \le \log_{\frac{1}{1-\frac{\varepsilon}{n}}} t = \frac{\ln t}{-\ln\left(1-\frac{\varepsilon}{n}\right)} \le \frac{n \ln t}{\varepsilon}$$

$$\left[\frac{x}{1+x} \le \ln(1+x) \le x, \quad x > -1\right]$$

!!!!!

$$z>z' \ldots\ldots$$

$$\frac{z}{z'} \rangle \frac{1}{1-\delta} \Rightarrow z \rangle \left(\frac{1}{1-\delta}\right) z'$$

$$z_1 \le z_2 \langle \cdots \langle z_k \le t$$

$$t \ge z_k \rangle \left(\frac{1}{1-\delta}\right) z_{k-1} \rangle \left(\frac{1}{1-\delta}\right)^2 z_{k-2} \rangle \cdots \rangle \left(\frac{1}{1-\delta}\right)^k z_1 \ge \left(\frac{1}{1-\delta}\right)^k$$

$$\ln t \ge (L_k (\ln \frac{1}{1-\delta} \Rightarrow \frac{\ln t}{\ln \frac{1}{1-\delta}} \ge L_k \Rightarrow L_k \le \frac{\ln t}{-\ln(1-\delta)}$$

$$\boxed{\delta =} \frac{\varepsilon}{n} \qquad \delta \langle 1 \Rightarrow -\delta \rangle -1 \Rightarrow \ln(1-\delta) \le -\delta \Rightarrow -\ln(1-\delta) \ge \delta$$

$$L_k \le \frac{\ln t}{\delta} = \frac{n \ln t}{\varepsilon}$$