The class PLS

A local search problem P is in PLS if there are 3 polynomial-time algorithms Ap, Bp, Cp with :

1. Given a string $x \in \{0,1\}^*$, Ap determines if $x \in$ Dp and produces s_0
2. x , string s $\overset{Bp}{\Longrightarrow}$ if s is sol. and Computes the cost
3. x , sol. s $\overset{Cp}{\Longrightarrow}$ if s local opt and if it is not Cp outputs a neighbor s′ with (strictly) better cost

Theoretical results (for LS)

- Find a local optimum is – it EASY ?

Example Linear Programming

SIMPLEX . . .

Class PLS : polynomial time local search

(Johnson , Papadimitriou & Yannakakis ¨ 88¨)

-find initial solution in polynomial time

-cost in polynomial time

-find better solution in the neighborhood or not existence

in polynomial Time

Examples (even for simple neighborhoods !)

(Schäffer & Yannakakis " 91 ")

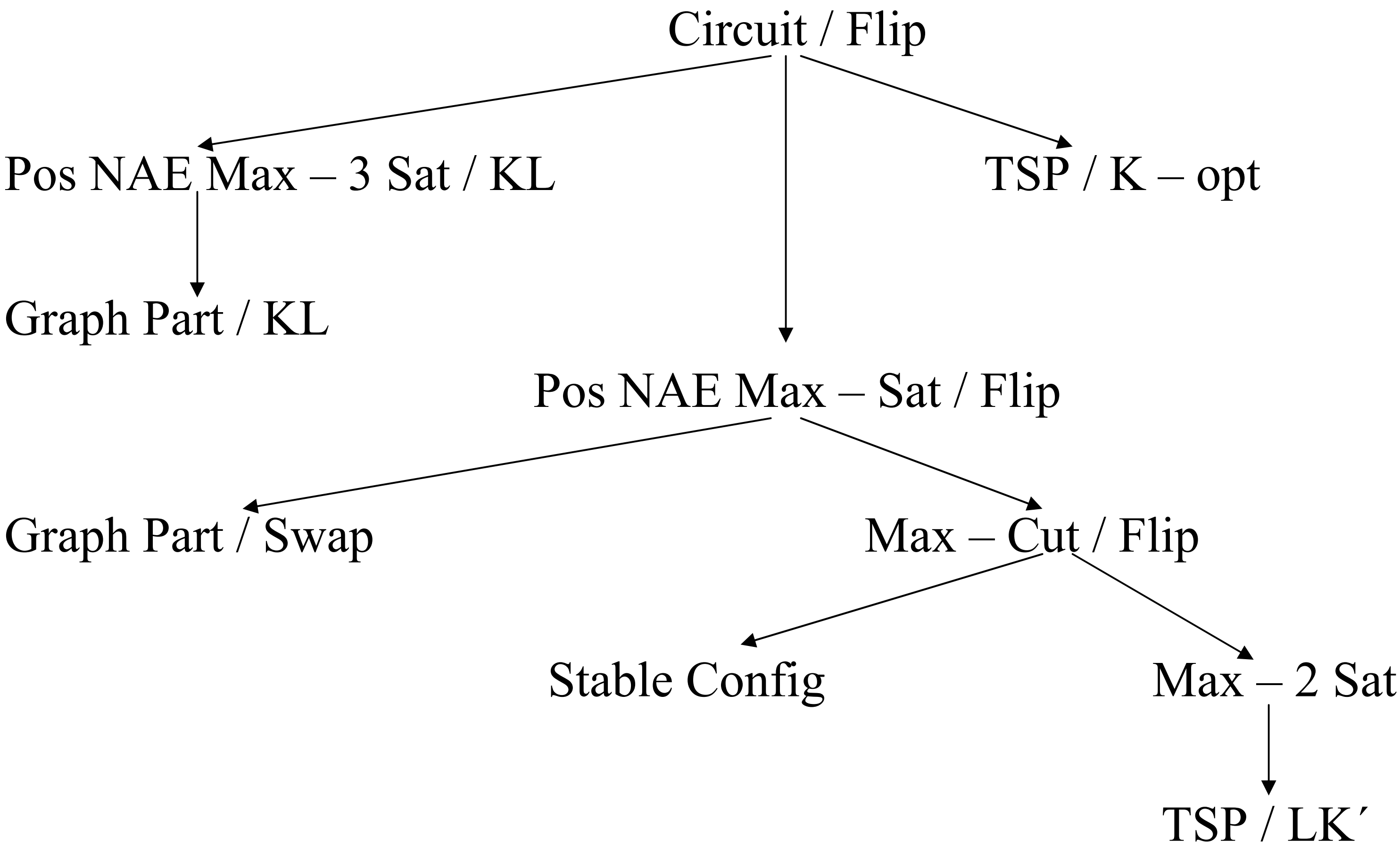
- 2 – SAT weighted
- TSP
- Bipartitioning of weighted graphs
- Max – Cut
- Stable configurations in Hopfield neural network model

PLS – complete problems

- If there exists a LS (in polynomial time) for a problem in the class then it will be the case for any other problem.

- ´´ For any problem in the class PLS – complete, the number of iterations with the standard local search is exponential unless P = NP ´´

Reductions


Circuit / Flip
Pos NAE Max – 3 Sat / KL
TSP / K – opt
Graph Part / KL
Pos NAE Max – Sat / Flip
Graph Part / Swap
Max – Cut / Flip
Stable Config
Max – 2 Sat
TSP / LK′

Which is the solutions quality of a LS ?

-for a given neighborhood

-even in exponential time !

Grover ´´ 92 ´´

- TSP with 2 – exchange
- Bipartitioning of a weighted graph with 2 – exchange

$$C_{\overline{loc}} \leq C_{AV}$$

local minimum

average cost of all solutions

The Quadratic Assignment Problem (QAP)

- n locations : distance
- n facilities: flow f_{ij}
- $\pi(i)=k$: facility i → location k

minimize the total cost

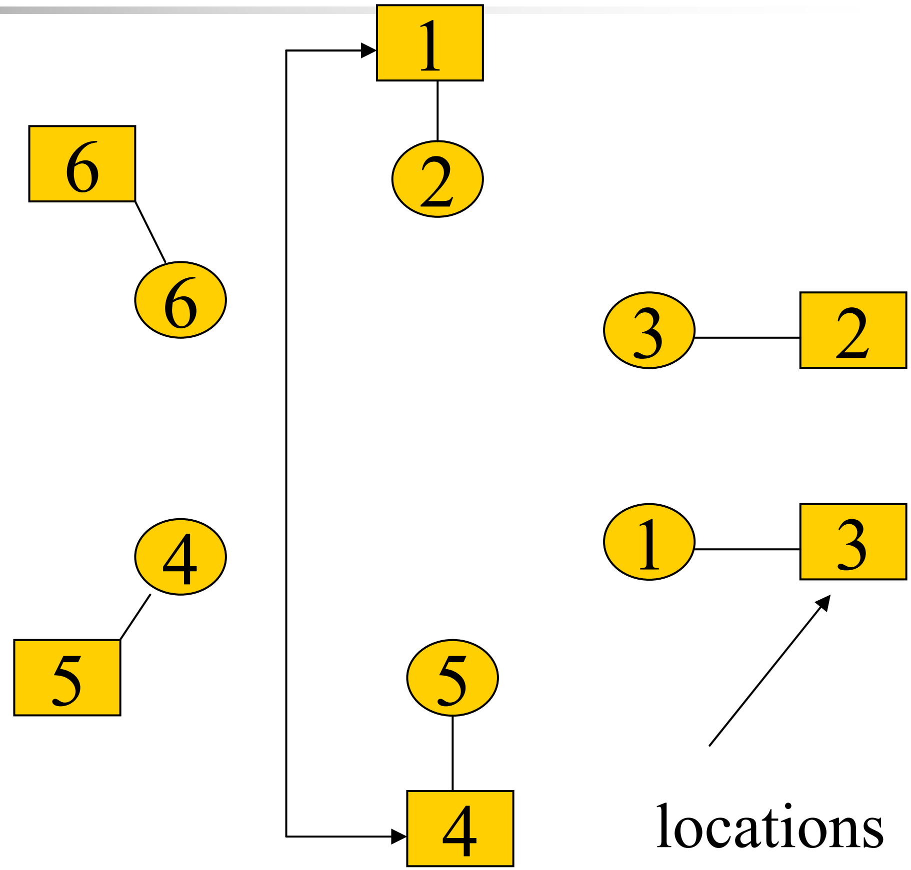
$$Min_{\pi \varepsilon \Pi} \sum_{i,k=1}^{n} f_{ik} d_{\pi(i)\pi(k)}$$

The QAP: an example

$\pi = 3\ 1\ 2\ 5\ 4\ 6$

2-exch.

$\pi_{ij} = 3\ 4\ 2\ 5\ 1\ 6$



The QAP

→PLS-complete(neighb. 2-exchange)

(schaffer & Yannakakis '91)

2-exchange:

$$\pi=(\pi(1),\dots,\pi(i),\dots,\pi(j),\dots\pi(n))$$

$$\pi_{ij}=(\pi(1),\dots,\pi(j),\dots,\pi(i),\dots\pi(n))$$

$$N(\pi)=(n*(n-1))/2$$

Notations

- $S(F) = \sum_{i,j} f_{ij}$
- $x = (x_1, x_2, ..., x_n)$

  $y = (y_1, y_2, ..., y_n)$

  $\pi y = (y_{\pi(1)}, y_{\pi(2)}, ..., y_{\pi(n)})$

  $\langle x, y \rangle_+ = \max_{\pi} \langle x, \pi y \rangle$

  $\langle x, y \rangle_- = \min_{\pi} \langle x, \pi y \rangle$

  $F_k = \sum_i f_{ki} = \sum_i f_{ik}$

  $\langle F, D \rangle_+ = \langle (F_1, ..., F_n), (D_1, ..., D_n) \rangle_+$
- $C_{AV} = \frac{1}{n!} \sum C(\Pi)$
- $C_{\overline{log}}$

Solution quality for the QAP ? E.Angel & V.Z.´´ 96 ´´ (DAM 98)

$\underline{Th_1 :}$

$$C_{\overline{loc}} \leq \frac{<F,D>_+}{S(F)S(D)} nC_{AV} = \frac{<F,D>_+}{2(n-1)} \leq \frac{n}{2} C_{AV}$$

$\underline{Th_2 :}$

$$\leq \frac{n}{2} C_{AV} \text{ in } O\left(n\log\frac{S(F)\times S(D)}{2} \right) \text{ iterat.}$$

Idea (deepest local search)

$$\delta_{ij}(\pi) = C(\pi_{ij}) - C(\pi)$$

$$\nabla^2 C(\pi) = \frac{1}{n(n-1)} \sum_{i,j} \delta_{ij}(\pi) \le -\frac{4}{n}\left(C(\pi) - C_{AV}\right) + a$$

$$\forall \pi \ \exists \ i, j \ t.q.$$

$$\delta_{ij}(\pi) \le \nabla^2 C(\pi)$$

$$C(\pi) \to C(\pi') \le C(\pi) - \frac{4}{n}\left(C(\pi) - C_{AV}\right) + a$$

$$1 = 1 - \frac{4}{n}\left(1 - C_{AV}\right) + a$$

$$C_{AV} = \frac{S(F) \times S(D)}{2n(n-1)}$$

$$1)\ \frac{C_{max}}{\frac{n}{2}C_{AV}} \rightarrow \infty$$

<u>ex.</u>

$$F = D = \begin{pmatrix} 0 & 1 & 1 & . & . & . & 1 \\ 1 & 0 & 1 & . & . & . & 1 \\ . & 1 & 0 & 1 & . & . & . \\ . & . & 1 & 0 & 1 & . & . \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & n^2 \\ 1 & . & . & . & 1 & n^2 & 0 \end{pmatrix}$$

$$\frac{n}{2}C_{AV} = \frac{(n-1)(3n+2)^2}{4}$$

$$C_{max} = \frac{n(n-1)}{2} - 1 + n^4$$

2) Traveling Salesman Problem (2-exchange)

- $D = \left(d_{ij}\right)$ adj. matrix

- $$F = \left(f_{ij}\right) = \begin{pmatrix} 0 & 1 & 0 & . & . & . & . & . & 0 & 1 \\ 1 & 0 & 1 & 0 & . & . & . & . & . & 0 \\ 0 & 1 & 0 & 1 & 0 & . & . & . & . & . \\ 0 & 0 & 1 & 0 & 1 & 0 & . & . & . & . \\ . & . & 0 & 1 & 0 & 1 & 0 & . & . & . \\ . & . & . & 0 & 1 & 0 & 1 & 0 & . & . \\ . & . & . & . & 0 & 1 & 0 & 1 & 0 & 0 \\ . & . & . & . & . & 0 & 1 & 0 & 1 & 0 \\ 0 & . & . & . & . & . & 0 & 1 & 0 & 1 \\ 1 & 0 & . & . & . & . & . & 0 & 1 & 0 \end{pmatrix}$$

$$S(F) = 2n$$

$$\langle F, D \rangle_+ = 2S(D)$$

Particular cases (TSP)

$\text{Th}_1 \Rightarrow$ <u>conclusion :</u> $\boxed{C_{\overline{\text{loc}}} \leq C_{AV}}$

[Grover]

$$C_{\overline{\text{loc}}} \leq \frac{\langle F, D \rangle_+}{S(F)S(D)} n C_{AV}$$

3) Bipartitioning

- weighted graph G(V,E)
- 2-exchange

$$|A| = |B| = \frac{|V|}{2}(\pm 1)$$

$$- D = \left(d_{ij}\right) \text{ adj. matrix}$$

Particular cases (Bipartitioning)

$$F = (f_{ij}) = \begin{pmatrix} O & U \\ U & O \end{pmatrix} \begin{matrix} \updownarrow n/2 \\ \updownarrow n/2 \end{matrix}$$

$$U = (u_{ij}) = \begin{cases} 1, & i, j = 1, \ldots, \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{C_{loc} \leq C_{AV}}$$

[Grover]

$$C_{\overline{loc}} \leq \frac{\langle F, D \rangle_+}{S(F)S(D)} n C_{AV}$$

$\frac{n}{2} S(D)$ → $\langle F, D \rangle_+$

$n \frac{n}{2}$ → $S(F)$

m-densest (m-lightest)

- Graph $G(V, E)$, $|V| = n$

consider $m\ (\leq n)$, $m = 1, \ldots, n$

"Find a sub-graph $G'_G(V', E')$ with $|V'| = m$

and $|E'|$ maximum (minimum)

- NP-complete
  - maximum clique
  - maximum Independent set

Results (2-exchange)

$$F = \begin{pmatrix} \boxed{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \updownarrow m \\ \updownarrow n\text{-}m \end{matrix} \qquad D = (d_{ij})$$

$d_1 \geq d_2 \geq \ldots \geq d_n$ degrees of G

$$C^{+}_{loc} \geq \frac{(m-1)}{2(n-1)}(d_n + d_{n-1} + \ldots + d_{n-m+1})$$

$$C^{-}_{loc} \leq \frac{(m-1)}{2(n-1)}(d_1 + d_2 + \ldots + d_m)$$

$$\Uparrow$$

$$\underline{\underline{Th_1}} \begin{cases} S(F) = m(m-1) \\ S(D) = 2|E| \\ C_{AV} = \dfrac{m(m-1)}{n(n-1)}|E| \\ \langle F, D \rangle_{+} = (m-1)(d_1 + \ldots + d_m) \end{cases}$$

5) Stable Maximum

(Meyers ´´ 72 ´´)

" If $d_1 \geq d_2 \geq \ ... \ \geq d_n$ and $d_1 + d_2 + \ ... \ + d_k \leq n - k$, for $2 \leq k \leq n$ then <u>stable maximal $\geq k$ "</u>

(Angel & V.Z. ´´ 96 ´´)

" If $d_1 + d_2 + \ ... \ + d_k \leq \left\lfloor \frac{2a(n-1)}{k-1} \right\rfloor_*$ for a integer,

$2 \leq k \leq n$

$$\lfloor x \rfloor_* = \begin{cases} x - 1 & \text{if } x \text{ integer} \\ \lfloor x \rfloor & \text{otherwise} \end{cases}$$

<u>stable maximal</u> $\geq k - a + 1$ "

Example

n=10

3 3 3 2 2 2 2 1 1 1

6≤10-2 Meyer

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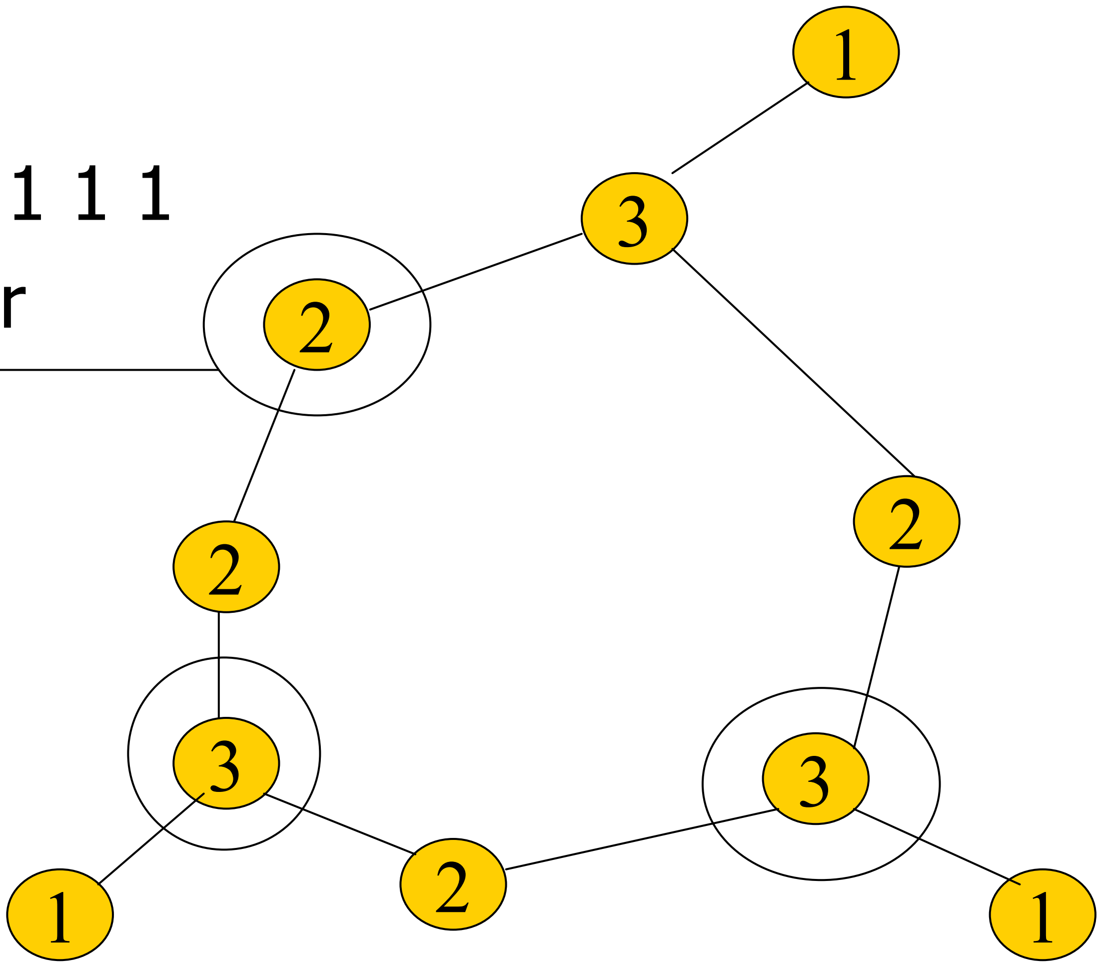
a=3

K=5

k-a+1=3



Comparison

- Meyer's result is deductive
- This result is constructive
- This condition is more flexible
- Result interesting in practice

$$\text{" } C_{\overline{loc}} \text{ includes at most } a-1 \text{ edges "}$$

$$\text{we remove at most } a-1 \text{ vertices}$$

$$\text{for obtaining a maximal Independent set}$$

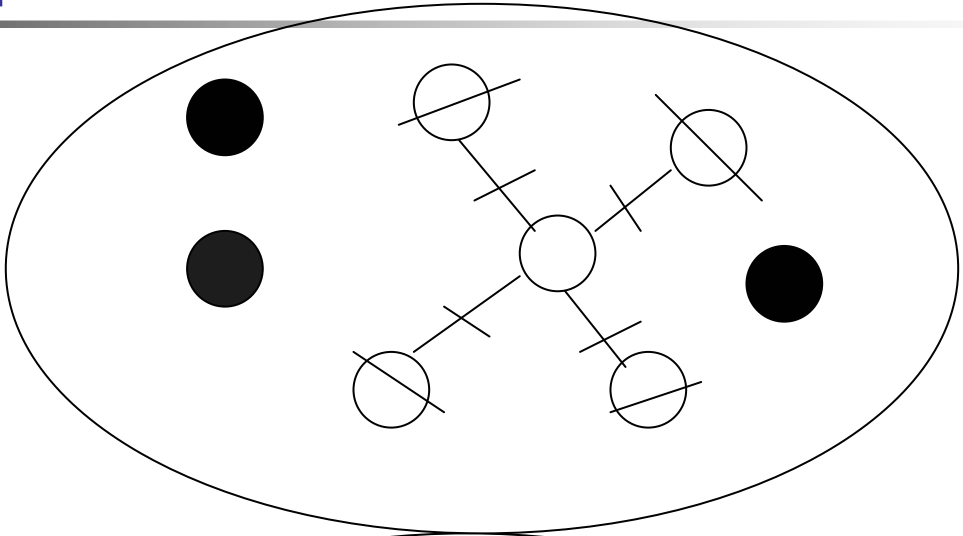
Example

ex

- a=5

  4 edges

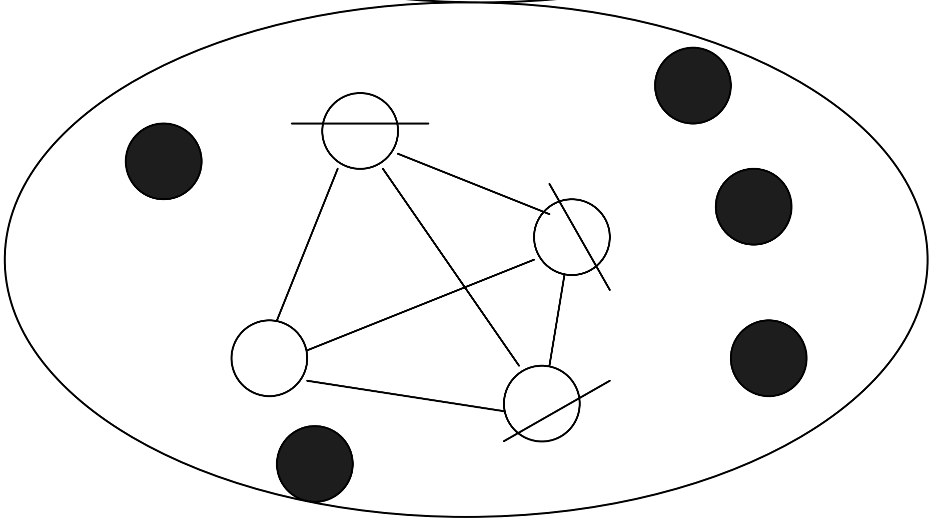
  4 vertices



- a=7

  6 edges

  3 vertices



LS Drawbacks

- Local optimum
- “good“ neighborhoods
- exploration strategies
- Performances guarantee ?
- Parallelization ?

Conclusions

- Some new results on the performance of Local Search and its variants
- The k-densest and the k-lightest problems
- A constructive result for the max. Ind. Set Pr.

(Th. MEYER)

Polynomial searchable Neighborhoods

- Local Search ($P \neq NP$)
  - **exact solution!**
  - **ε-approximate solution! (TSP Papad)**
  - **Local optimum**
    - Polynomial → **Unweighted cases**
      Approximation
    - Expo → **Weighted cases**
      Approximation? (PLS-complete)

Maximum Cut (Example with approximation 2 with N()=SWAP)

- Instance: G=(V,E)
- Solution: $V_1,V_2 \mid V_1 \cap V_2 = \emptyset, V_1 \cup V_2 = V$
- Measure: $\mathrm{Max}|\{[u,v] \in E \mid u \in V_1, v \in V_2\}|$
- Initial solution: $V_1 = \emptyset, V_2 = V$ feasible

SWAP Neighborhood

$N(V_1,V_2)=\{(V_{1\kappa},V_{2\kappa}),\kappa=1,\ldots,|V|\}$

$V_{1\kappa}=V_1-\{v_\kappa\}$, $V_{2\kappa}=V_2\cup\{v_\kappa\}$ if $v_\kappa\in V_1$

$V_{1\kappa}=V_1\cup\{v_\kappa\}$, $V_{2\kappa}=V_2-\{v_\kappa\}$ if $v\in V_2$

2-approximation

- (V_1, V_2) local optimum with N

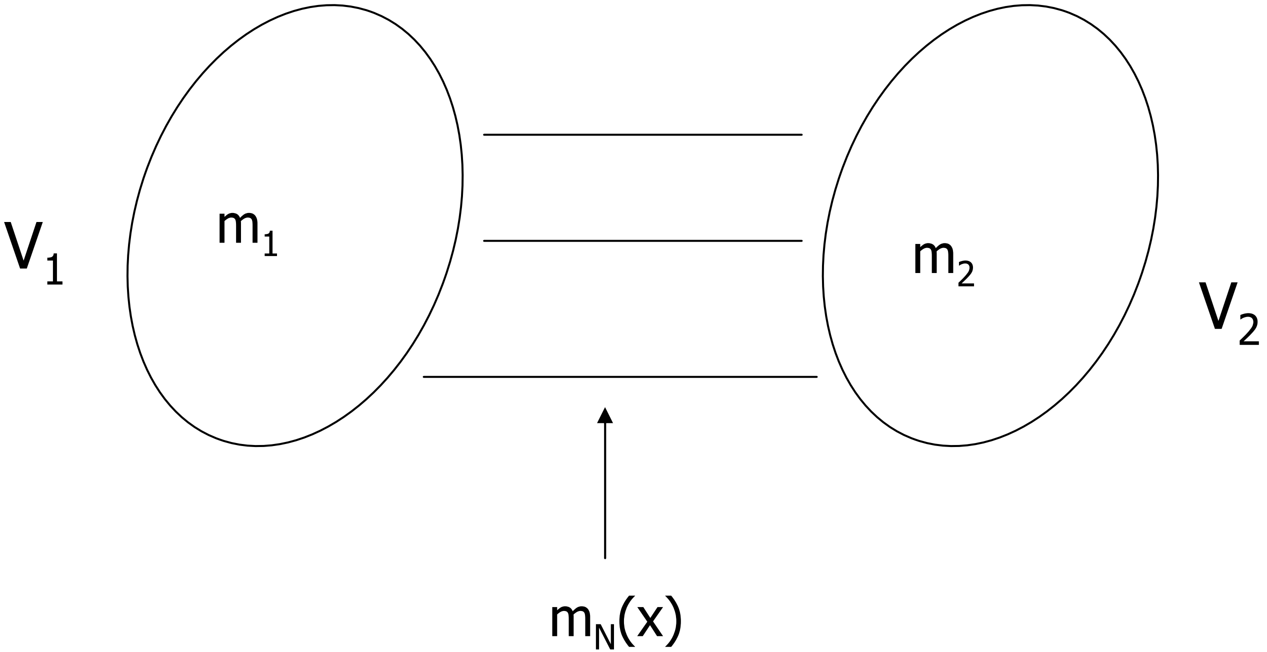
$$\Downarrow$$

$$\frac{m^*(x)}{m_N(x)} \leq 2$$

$\Rightarrow |E| = m \quad \Rightarrow m^*(x) \leq m$

if $m_N(x) \geq m/2 \Rightarrow$

Under a local optimum



$$m = m_1 + m_2 + m_N(x)$$

Under a local optimum

<u>def</u> $(\forall v_i)$

$m_{1i} = \{v | v \in V_1 \text{ and } (v, v_i) \in E\}$

$m_{2i} = \{v | v \in V_2 \text{ and } (v, v_i) \in E\}$

if (V_1, V_2) local Optimum

$\Downarrow \forall v_i \in V_1$

$|m_{1i}| - |m_{2i}| \leq 0$

$\Downarrow \forall v_i \in V_2$

$|m_{2j}| - |m_{1j}| \leq 0$

Under a local optimum

$$\sum_{\theta_i \in V_1} (|m_{1i}| - |m_{2i}|) = 2m_1 - m_N(x) \le 0$$

$$\sum (|m_{2j}| - |m_{1j}|) = 2m_2 - m_N(x) \le 0$$

$$\Downarrow$$

$$m_1 + m_2 - m_N(x) \le 0 \Rightarrow m - m_N(x) - m_N(x) \le 0 \Rightarrow$$

$$\boxed{m_N(x) \ge m/2}$$

FUTURE WORK

- ?!
- Proposed bound for QAP is it attained?
- TSP with 2-OPT is it in PLS - complete?
- Other problems with guaranteed LS
- Parallelization of PBs in PLS
- ´´ Good ´´ neighborhood structures
- Scheduling scheme in SA