Triangulations of point sets, high dimensional Polytopes and Applications

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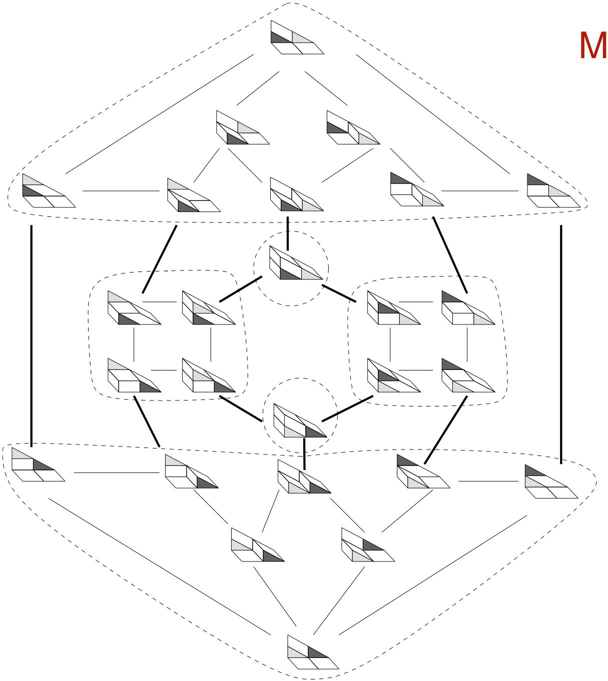
23 December 2009

Motivation



Outline

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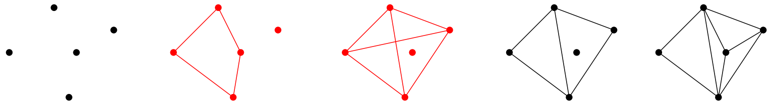
Polyhedral Subdivisions

Definition

A **polyhedral subdivision** of a point set A in $\mathbb{R}^d$ is a collection S of subsets of A called cells s.t.

- The cells cover $conv(A)$
- Every pair of cells intersect at a (possibly empty) common face

A polyhedral subdivision of A whose cells are all simplices, is called **triangulation**.

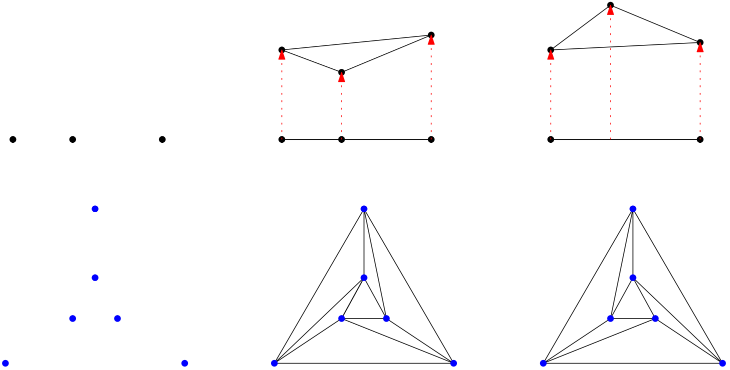


A point set, two invalid subdivisions, a polyhedral subdivision and a triangulation.

Regular Polyhedral Subdivisions

Definition

A polyhedral subdivision P of a point set A in $\mathbb{R}^d$ is **regular** if there exist a lifting function $w : A \to \mathbb{R}$ s.t. P is the projection to $\mathbb{R}^d$ of the lower hull of $\widetilde{A} = (a, w(a))$, $a \in A$.

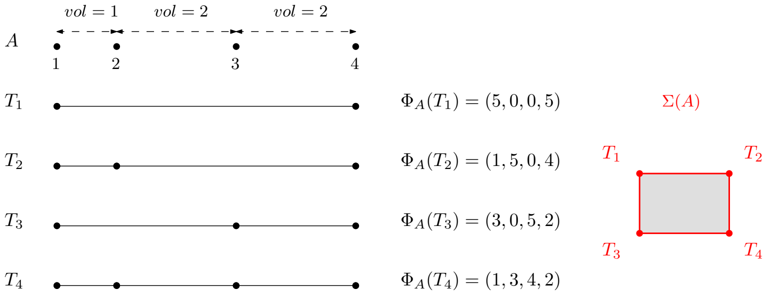


A point set with two regular triangulations and one with two non regular triangulations.

The Secondary Polytope

Definition

- Let T be a triangulation of A. Then the **GKZ-vector** of T is
$$\Phi_A(T) = \sum_{a_i \in A} \sum_{\sigma \in T : a_i \in \sigma} vol(\sigma) e_{a_i}$$
- The **Secondary polytope** $\Sigma(A)$ is the convex hull of GKZ-vectors for all triangulations of A.



A point set with 4 triangulations the GKZ vectors and the Secondary polytope.

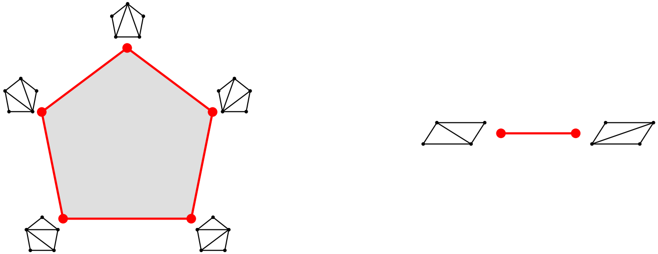
Secondary Polytopes

Definition

The operation of switching from one triangulation to another is called **bistellar flip**.

Theorem [Gelfand-Kapranov-Zelevinsky]

For every point set A of n points in $\mathbb{R}^d$ corresponds a Secondary polytope $\Sigma(A)$ with dimension $dim(\Sigma(A)) = n - d - 1$. The vertices correspond to the regular triangulations of A and the edges to bistellar flips.

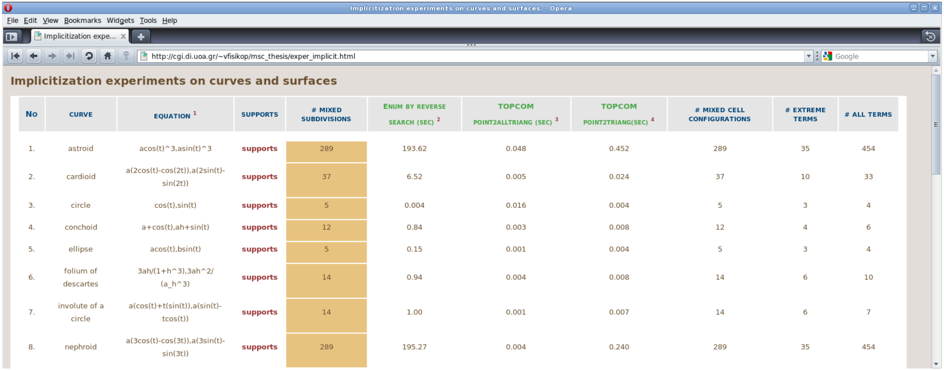


Secondary polytopes of a pentagon and a quadrilateral.

Enumeration of Triangulations

- Two approaches:
  1. **TOPCOM** [Rambau]: combinatorially characterize triangulations
  2. **Enumeration of Regular Triangulations by Reverse Search** [F. Takeuchi, Masada, H.Imai, K.Imai]
- Webpage with experiments:

  http://cgi.di.uoa.gr/∼vfisikop/msc_thesis/exper_implicit.html



Implicitization experiments on curves and surfaces. - Opera

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http://cgi.di.uoa.gr/~vfisikop/msc_thesis/exper_implicit.html

Implicitization experiments on curves and surfaces

No	Curve	Equation [1]	Supports	# Mixed subdivisions	Enum by reverse search (sec) [2]	TOPCOM point2alltriang (sec) [3]	TOPCOM point2triang(sec) [4]	# Mixed cell configurations	# Extreme terms	# All terms
1.	astroid	acos(t)^3,asin(t)^3	supports	289	193.62	0.048	0.452	289	35	454
2.	cardioid	a(2cos(t)-cos(2t)),a(2sin(t)-sin(2t))	supports	37	6.52	0.005	0.024	37	10	33
3.	circle	cos(t),sin(t)	supports	5	0.004	0.016	0.004	5	3	4
4.	conchoid	a+cos(t),ah+sin(t)	supports	12	0.84	0.003	0.008	12	4	6
5.	ellipse	acos(t),bsin(t)	supports	5	0.15	0.001	0.004	5	3	4
6.	folium of descartes	3ah/(1+h^3),3ah^2/(a_h^3)	supports	14	0.94	0.004	0.008	14	6	10
7.	involute of a circle	a(cos(t)+t(sin(t)),a(sin(t)-tcos(t))	supports	14	1.00	0.001	0.007	14	6	7
8.	nephroid	a(3cos(t)-cos(3t)),a(3sin(t)-sin(3t))	supports	289	195.27	0.004	0.240	289	35	454

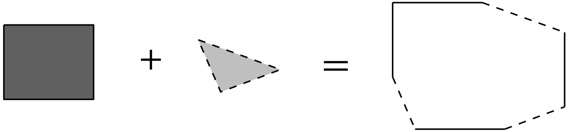
Outline

Minkowski Sum

Definition

The **Minkowski sum** of two convex polytopes P_1 and P_2 is the convex polytope:

$$P = P_1 + P_2 := \{p1 + p2 \mid p_1 \in P_1, p_2 \in P_2\}$$



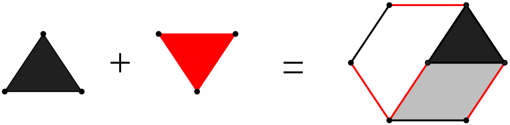
Minkowski sum of a square and a triangle.

Minkowski Cell

Let $A_1, \ldots, A_k$ points sets in $\mathbb{R}^d$ and $A = A_1 + \ldots + A_k$

Definition

- A subset of A is called **Minkowski cell** if it can be written as $F_1 + \cdots + F_k$ for certain subsets $F_1 \subseteq A_1, \ldots, F_k \subseteq A_k$.
- A Minkowski cell is fine if all F_i are affinely independent and $\sum_{i=1}^{k} dim(F_i) = d$.



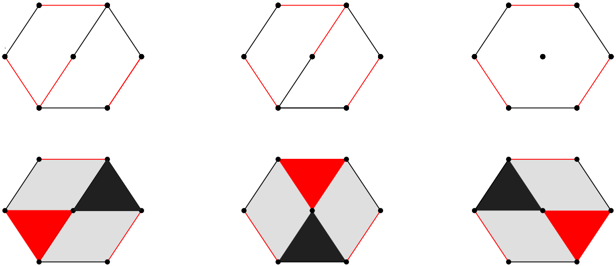
Two fine cells (black, gray) and a non-fine one (white).

Mixed Subdivisions

Definition

A polyhedral subdivision of A whose cells are all Minkowski cells, is called **mixed subdivision**.

- if all cells are fine is called fine mixed subdivision.
- if it is the projection of the lower hull for some lifting on A is called regular mixed subdivision.



3 regular mixed subdivisions and 3 regular fine mixed subdivisions.

The Cayley Trick

Definition

The **Cayley embedding** of $A_1, \ldots A_k$ is the point set

$$C(A_1, \ldots, A_k) = A_1 \times \{e_1\} \cup A_2 \times \{e_2\} \cup \ldots \cup A_k \times \{e_k\} \subseteq \mathbb{R}^d \times \mathbb{R}^{k-1}$$

where $e_1, e_2, \ldots, e_k$ are an affine basis of $\mathbb{R}^{k-1}$.

Proposition (the Cayley trick)

$$\left\{ \begin{array}{c} \text{polyhedral} \\ \text{subdivisions} \\ \text{of} \\ C(A_1, \ldots, A_k) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{mixed} \\ \text{subdivisions} \\ \text{of} \\ A_1 + \ldots + A_k \end{array} \right\}$$

$$\begin{array}{rcl} \text{regular polyhedral subdivisions} & \longleftrightarrow & \text{regular mixed subdivisions} \\ \text{regular } \textbf{triangulations} & \longleftrightarrow & \text{regular } \textbf{fine} \text{ mixed subdivisions} \end{array}$$

Outline

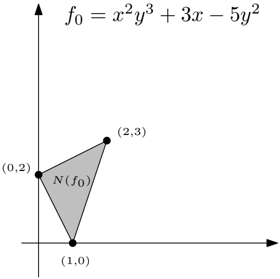
Newton polytopes

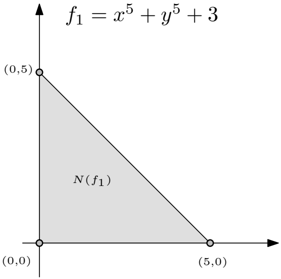
Definition

- The **support** of a polynomial $f_i = \sum c_{ij} \vec{x}^{\overrightarrow{a_{ij}}}$ is the set:

$$sup(f_i) = \{a_{ij} \in \mathbb{N}^{n_i} : c_{ij} \neq 0\}$$

- Given a polynomial f_i its **Newton polytope** $N(f_i)$ is the convex hull of its support.





Resultant

Let $f = f_0, f_1, \ldots, f_k$ a polynomial system on k variables.

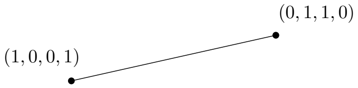
Definition

- The (sparse) **Resultant** of f is a polynomial $R \in K[c_{ij}]$ s.t. $R = 0$ iff f has a common root.
- The Newton polytope of the Resultant polynomial $N(R)$ is called the **Resultant polytope**.
- We call an **extreme term** of R a monomial which correspond to a vertex of $N(R)$.

Example

$$f_0 = c_{00} + c_{01}x$$
$$f_1 = c_{10} + c_{11}x$$

$$\mathbf{R = c_{00}c_{11} - c_{01}c_{10}}$$



Resultant Extreme Terms

Let $A_0, A_1, \dots, A_k$ s.t. $A_j = N(sup(f_j))$ in $\mathbb{Z}^k$, $A = A_0 + A_1 + \cdots + A_k$

Definition

A Minkowski cell σ is called **i-mixed** if for all j exists $F_j \subseteq A_j$ s.t.

$$\sigma = F_0 + \cdots + F_{i-1} + F_i + F_{i+1} + \cdots + F_k$$

where $|F_j| = 2$ (edges) for all $j \neq i$ and $|F_i| = 1$ (vertex).

Theorem [Sturmfels]

Given a polynomial system f and a regular fine mixed subdivision of the Minkowski sum of the Newton polytopes of its supports we get an extreme term of the resultant R which is equal

$$const \cdot \prod_{i=0}^{k} \prod_{\sigma} c_{iF_i}^{vol(\sigma)}$$

where $\sigma = F_0 + F_1 + \cdots + F_k$ is an i-mixed cell and $const \in \{-1, +1\}$.

An Algorithm

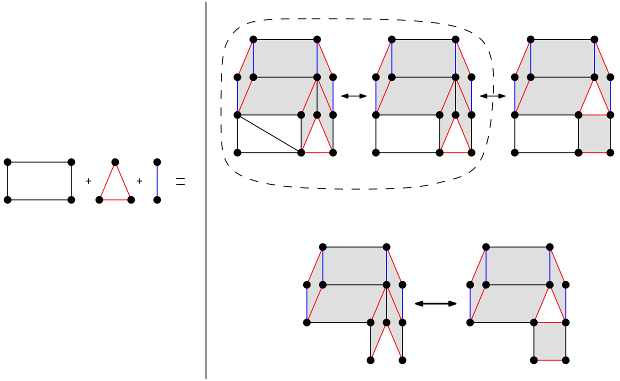
Input: A polynomial system $f = f_0, f_1, \dots, f_k$
Output: The Resultant polytope of f: $N(R)$

1. Compute the Newton polytopes of the supports of f: $A_0, \dots, A_k$
2. Compute the Cayley embedding $C(A_0, \dots, A_k)$
3. Enumerate all regular triangulations of $C(A_0, \dots, A_k)$.
   i.e. the regular fine mixed subdivisions of $A_0 + \cdots + A_k$
4. For each regular fine mixed subdivision compute the corresponding extreme term of R

Mixed Cells Configurations

Definition [Michiels, Verschelde]

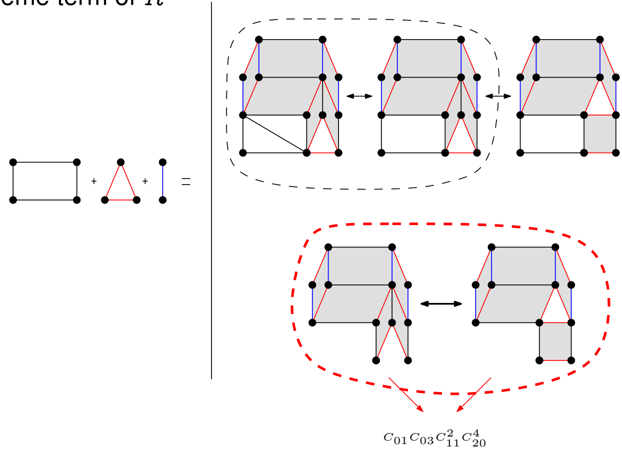
Mixed cells configurations are the equivalence classes of mixed subdivisions with the same i-mixed cells.



Ξ **Polytope** [Michiels, Cools]: Mixed cell configurations as vertices

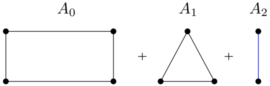
Resultant Extreme Terms Classification

- We care only about mixed cell configurations.
- Two regular fine mixed subdivisions of A may produce the same extreme term of R



2 mixed cell configurations that yield the same extreme term of R

Secondary - Ξ - Resultant



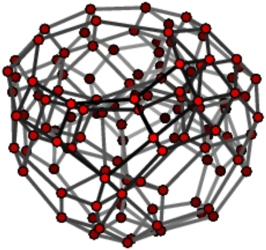
\# mixed subdivisions = 122

\# mixed cell configurations = 98

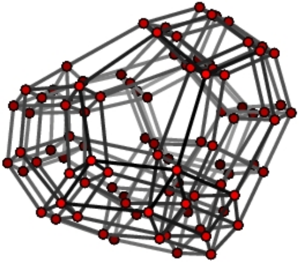
\# Resultant extreme terms = 8

mixed cell configurations

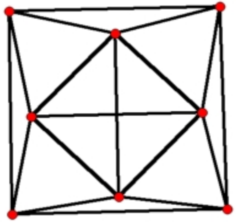
Resultant extreme terms



Secondary Polytope

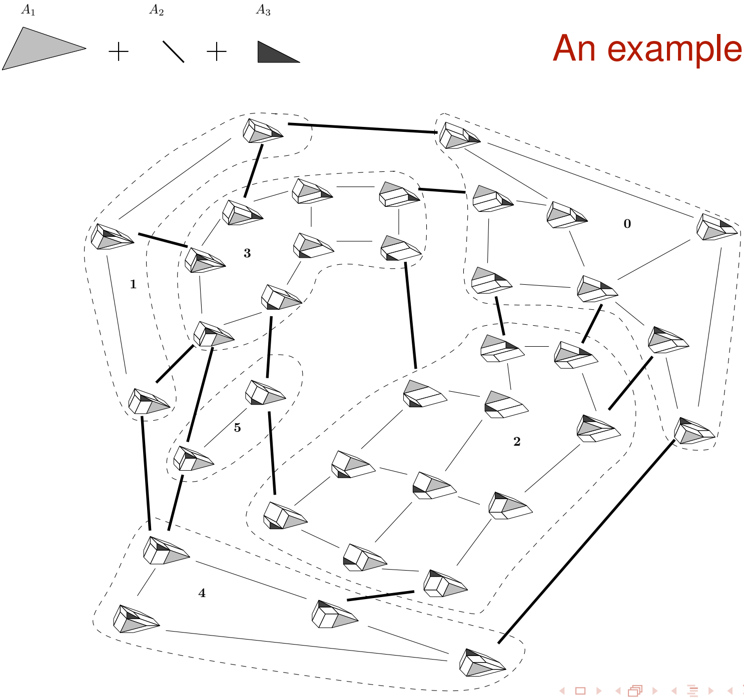


Ξ Polytope



Resultant Polytope

An example



Cubical Flips

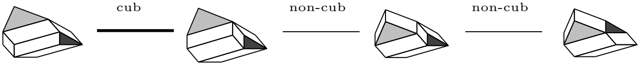
General position assumption: every two faces with the same dimension from two different A_i are not parallel.

Definition

- An **affine cube** is the Minkowski sum of nonparallel edges and vertices.
- Consider the set of cells that are changed by the flip. If the lifted points of this set form an affine cube then the flip is called **cubical.**

Theorem [Sturmfels]

Two regular fine mixed subdivisions produce the same extreme term of R iff they are connected by a sequence of non-cubical *flips.*



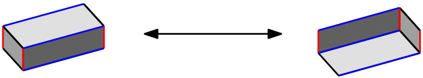
A Combinatorial Test for Cubical Flips

- Let S a regular fine mixed subdivision then S has a cubical flip iff exists a set $C \subseteq S$ of i-mixed cells s.t. $C = F_1 + F_2 + \cdots + F_k$ and

$$C = \begin{cases} C_1 = a_1 + F_2 + \cdots + F_i + \cdots + F_{k-1} + F_k \\ \vdots \\ C_i = F_1 + F_2 + \ldots \ + a_i + \cdots + F_{k-1} + F_k \\ \vdots \\ C_k = F_1 + F_2 + \cdots + F_i + \cdots + F_{k-1} + a_k \end{cases}$$

where $a_i \in F_i, |a_i| = 1, |F_i| = 2$.

- The cubical flip supported on C of a subdivision S to a subdivision S' consist of changing every a_i with $F_i - \{a_i\}$.



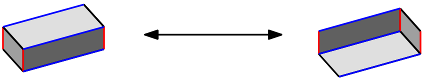
A Combinatorial Test for Cubical Flips

- Let S a regular fine mixed subdivision then S has a cubical flip iff exists a set $C \subseteq S$ of i-mixed cells s.t. $C = F_1 + F_2 + \cdots + F_k$ and

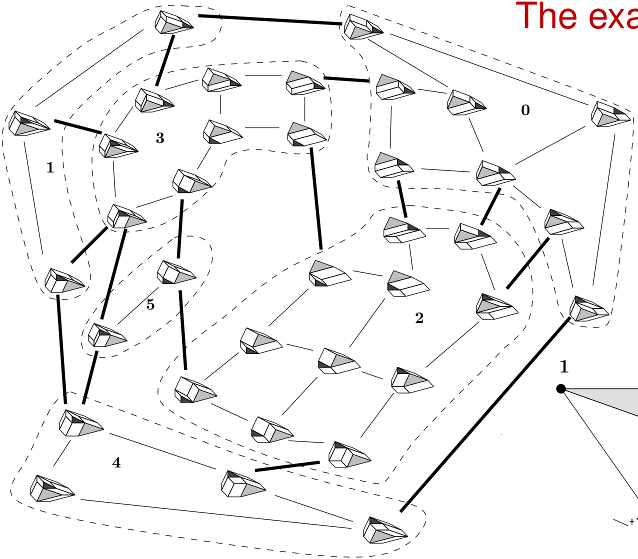
$$C' = \begin{cases} C'_1 = F_1 - \{a_1\} + F_2 + \cdots + F_i + \cdots + F_{k-1} + F_k \\ \vdots \\ C'_i = F_1 + F_2 + \ldots + F_i - \{a_i\} + \cdots + F_{k-1} + F_k \\ \vdots \\ C'_k = F_1 + F_2 + \cdots + F_i + \cdots + F_{k-1} + F_k - \{a_k\} \end{cases}$$

where $a_i \in F_i, |a_i| = 1, |F_i| = 2$.

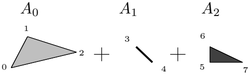
- The cubical flip supported on C of a subdivision S to a subdivision S' consist of changing every a_i with $F_i - \{a_i\}$.



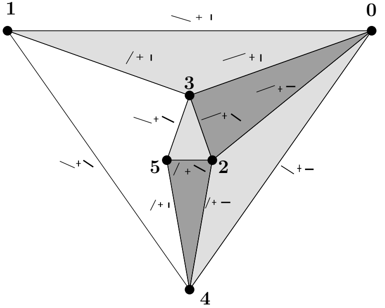
The example (continued)



Secondary Polytope



Resultant Polytope



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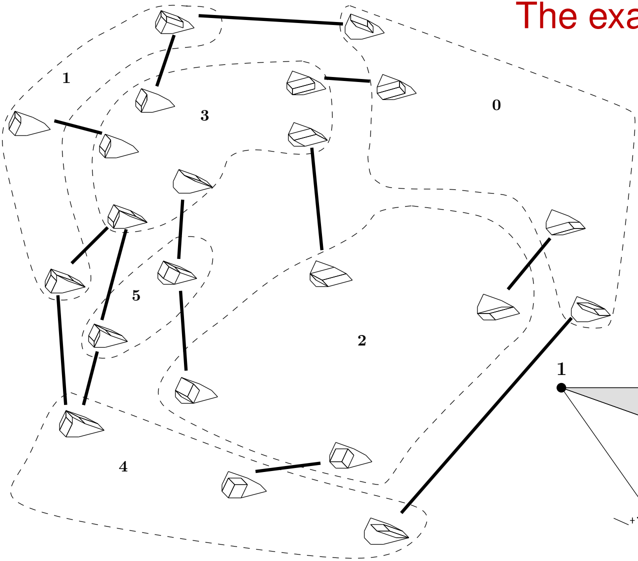
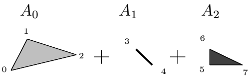
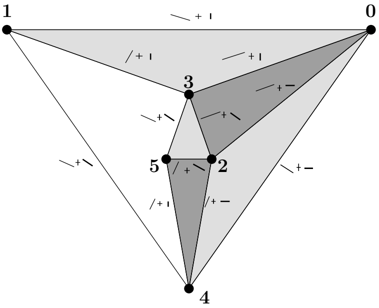


Secondary Polytope



Resultant Polytope



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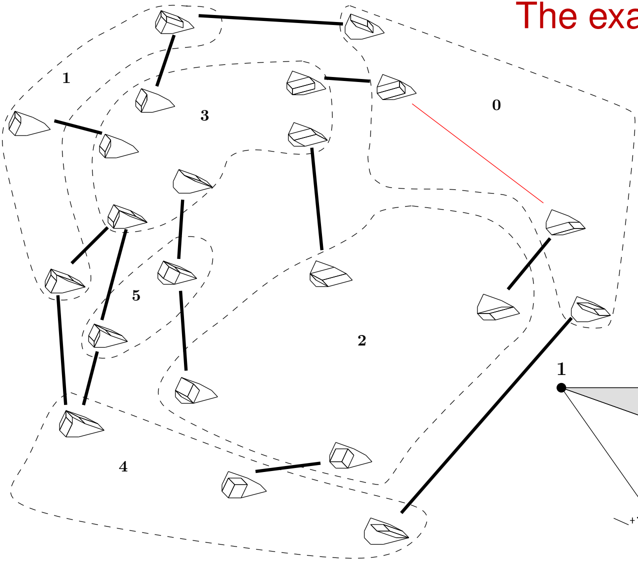
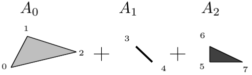
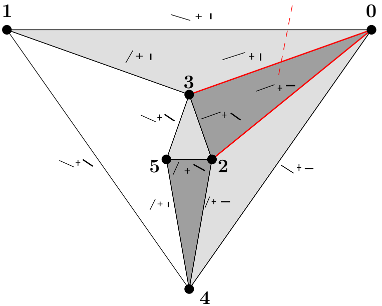


Secondary Polytope



Resultant Polytope



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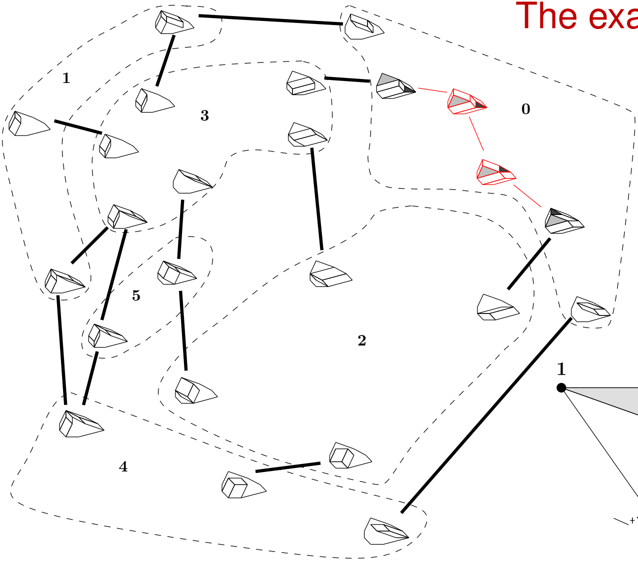
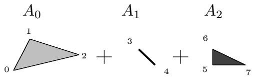
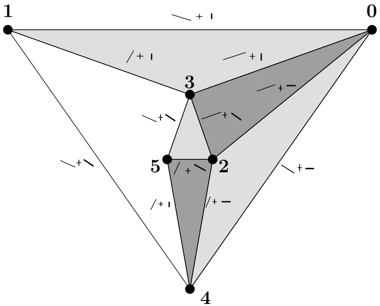


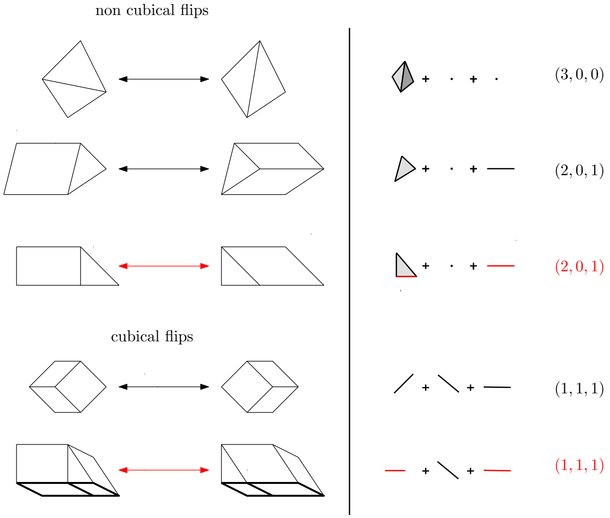
Secondary Polytope



Resultant Polytope



Degenerate Cases



Thank You!