IMPROVEMENTS ON KHRAPCHENKO'S THEOREM

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ABSTRACT: *We present an improvement of Khrapchenko's theorem which gives lower bounds for the size of Boolean* $\{\wedge,\vee,\neg\}$*-formulae. Our main theorem gives better lower bound than the original Khrapchenko's theorem or at least the same, although we know of no function where it gives an improvement factor larger that two. This lower bound is the largest eigenvalue of a certain matrix associated with the formula. Moreover, we give an approximation of this bound which is easier to compute and is never smaller than the bound given by Khrapchenko's theorem.*

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Let φ be a Boolean function on n variables and $L(\varphi)$ denote the number of leaves of the minimal size $\{\wedge,\vee,\neg\}$-formula that computes φ. For $A \subset \varphi^{-1}(0)$ and $B \subset \varphi^{-1}(1)$ we define the $|B| \times |A|$ matrix Q with $q_{ij} = 1$ if $i \in B$ and $j \in A$ differ in exactly one variable; otherwise $q_{ij} = 0$. Khrapchenko's theorem[3] can be restated as follows:

$$L(\varphi) \geq K_\varphi = \frac{1}{|A||B|}(\sum_{i,j} q_{ij})^2$$

Let $P_\varphi = QQ^T$ and $\overline{P}_\varphi = Q^TQ$. Obviously, for $i,j \in B$, $p_{\varphi,ij}$ is the number of their common neighbors in A, i.e. the number of elements in A that differ from both i and j in exactly one variable. Similarly for $\overline{p}_{\varphi,ij}$. Notice that P_φ and $\overline{P}_\varphi$ are symmetric matrices.

Let $\lambda(X)$ denote the largest eigenvalue of matrix X. By elementary properties of matrices[2], the nonzero eigenvalues of P_φ are the same as the nonzero eigenvalues of $\overline{P}_\varphi$.

Since trivially $\lambda(P_\varphi), \lambda(\overline{P}_\varphi) \geq 0$, we have that $\lambda(P_\varphi) = \lambda(\overline{P}_\varphi)$. Our main theorem relates the leaf size of φ to $\lambda(P_\varphi)$:

Theorem 1. For any Boolean function φ and any nonempty sets A,B defined as above:

$$L(\varphi) \geq \lambda(P_\varphi)$$

Proof. We prove the theorem by induction on the size of the minimal $\{\wedge,\vee,\neg\}$-formula that computes φ. For the basis case, $\varphi = x_i$, it is easy to see that $p_{\varphi,ii} = 0$ or 1 and $p_{\varphi,ij} = 0$ for $i \neq j$ and consequently, $\lambda(P_\varphi) \leq 1 = L(\varphi)$.

Suppose now that the theorem holds for ψ and θ. It suffices to show that it holds for *minimal* formulae of the forms $\varphi = \neg\psi$, $\varphi = \psi \wedge \theta$ and $\varphi = \psi \vee \theta$. The first case follows immediately from the fact that $\lambda(P_\varphi) = \lambda(\overline{P}_\varphi)$ and $L(\varphi) = L(\neg\varphi)$.

For the second case, $\varphi = \psi \wedge \theta$, let $B_\psi = B_\varphi = B$. Moreover, we can find $A_\psi \subset \psi^{-1}(0)$ and $A_\theta \subset \theta^{-1}(0)$ such that $A_\psi \cup A_\theta = A$ and $A_\psi \cap A_\theta = \emptyset$, e.g. $A_\psi = \psi^{-1}(0)$ and $A_\theta = A - A_\psi$. By the induction hypothesis, we have:

$$L(\varphi) = L(\psi) + L(\theta) \geq \lambda(P_\psi) + \lambda(P_\theta)$$

But $P_\varphi = P_\psi + P_\theta$ and because P_φ, P_ψ, P_θ are symmetric matrices we have that:

$$\lambda(P_\psi) + \lambda(P_\theta) \geq \lambda(P_\varphi)$$

Thus $L(\varphi) \geq \lambda(P_\varphi)$.

The case $\varphi = \psi \vee \theta$ is treated similarly and the theorem follows. □

The lower bound on the leaf size $L(\varphi)$ in this theorem is at least as good as the lower bound given by Khrapchenko's theorem. In other words, for any A, B as above: $K_\varphi \leq \lambda(P_\varphi)$. But in many cases it is not easy to apply Theorem 1, because of the difficulty in computing the largest eigenvalue of a matrix. However, it is easy to find lower bounds for the largest eigenvalue of a symmetric matrix. These are also lower bounds for the leaf size of the associated formula. We give here such a lower bound for the largest eigenvalue of the symmetric matrix P_φ. First let us define s_i to be the sum of the elements in column i of matrix Q, i.e. s_i is the number of neighbors in B of $i \in A$, and let

$$D_\varphi = \frac{1}{|B|}\sum_{i=1}^{|A|} s_i^2 = \frac{1}{|B|}\sum_{i,j} p_{\varphi,ij}$$

Similarly, for matrix $\overline{P}_\varphi$:

$$\overline{D}_\varphi = \frac{1}{|A|} \sum_{i,j} \overline{p}_{\varphi,ij}$$

D_φ and $\overline{D}_\varphi$ are easy to compute and it turns out that they both lie between the values given by Khrapchenko's theorem and our theorem.

Proposition 1. For any formula φ and any A, B defined as above we have:

$$K_\varphi \leq D_\varphi \leq \lambda(P_\varphi)$$

Proof. We have that

$$K_\varphi = \frac{1}{|A||B|} (\sum_{i=1}^{|A|} s_i)^2$$

and

$$D_\varphi = \frac{1}{|B|} \sum_{i=1}^{|A|} s_i^2$$

Using the inequality $(\sum_{i=1}^{n} x_i)^2 \leq n \sum_{i=1}^{n} x_i^2$, it is easy to see that $K_\varphi \leq D_\varphi$ and the equality holds iff all s_i's are equal. On the other hand, since $\lambda(P_\varphi)$ is the largest eigenvalue of the symmetric matrix P_φ we have:

$$\lambda(P_\varphi) = \max_{x \neq 0} \frac{x^T P_\varphi x}{x^T x}$$

Choosing $x = 1$, i.e. x is the vector with all entries equal to 1, we have that:

$$D_\varphi \leq \lambda(P_\varphi)$$

and equality holds iff $x = 1$ is the eigenvector of P_φ associated with $\lambda(P_\varphi)$. $\square$

Obviously, we can replace D_φ with $\overline{D}_\varphi$ in the above proposition.

Our theorem cannot help us find large lower bounds. The next proposition illustrates its limitations.

Proposition 2. The best lower bound that Theorem 1 can give is n^2:

$$\lambda(P_\varphi) \leq n^2$$

Proof. By Gerŝgorin's theorem[4] the largest eigenvalue of a matrix with nonnegative entries is at most the maximum of its row sums. So,

$$\lambda(P_\varphi) \leq \max_i \sum_j p_{\varphi,ij} \leq n^2$$

because each neighbor of $i \in B$ can contribute at most n to the sum and there are at most n neighbors of i. $\square$

Finally, we give an example where Theorem 1 gives almost twice the lower bound obtained by Khrapchenko's original theorem. Let $\varphi(x_1, x_2, \ldots, x_n) = 1$ iff exactly k of the x_i's are 1. Then let $B = \varphi^{-1}(1)$ and let $A \subset \varphi^{-1}(0)$ contain all the neighbors of B. Khrapchenko's theorem gives the following lower bound of the leaf size of φ:

$$K_\varphi = \frac{n^2(k+1)(n-k+1)}{n^2-(2k-1)n+2k^2} \approx (k+1)n$$

(assuming $k \ll n$). It is not difficult to see that:

$$\overline{D}_\varphi = K_\varphi, \quad D_\varphi = \lambda(P_\varphi) = (2k+1)n - 2k^2 \approx (2k+1)n$$

Notice that both Theorem 1 and Khrapchenko's theorem involve two sets $A \subset \varphi^{-1}(0)$ and $B \subset \varphi^{-1}(1)$. For the same sets A and B, Theorem 1 can give much better results. In the example above, if we pick $A = \varphi^{-1}(0)$ and $B = \varphi^{-1}(1)$ then Khrapchenko's theorem gives

$$K_\varphi = \frac{n^2\binom{n}{k}}{2^n - \binom{n}{k}} = o(1)$$

while Theorem 1 gives

$$\lambda(P_\varphi) = (2k+1)n - 2k^2 \approx (2k+1)n$$

The reason is that K_φ may decrease when we pick larger sets A and B, while $\lambda(P_\varphi)$ cannot. This last remark, based on elementary properties of symmetric matrices, suggests that Theorem 1 gives the best result when $A = \varphi^{-1}(0)$ and $B = \varphi^{-1}(1)$. On the other hand, in order to get the best results from Khrapchenko's theorem, one has to pick appropriate A and B. Intuitively, the best A and B for Khrapchenko's theorem are the subsets of 'large' values in the eigenvectors of $\overline{P}_\varphi$ and P_φ, associated with $\lambda(\overline{P}_\varphi)$ and$\lambda(P_\varphi)$, respectively.

It is probably worth mentioning here that we know of no Boolean function where our method improves upon Khrapchenko's theorem by a factor larger than two, when A, B are chosen appropriately.

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