On the Competitive Ratio of Online Sampling Auctions

Elias Koutsoupias[1⋆] and George Pierrakos[2]

[1] University of Athens, elias@di.uoa.gr,
[2] UC Berkeley, georgios@cs.berkeley.edu

Abstract. We study online profit-maximizing auctions for digital goods with adversarial bid selection and uniformly random arrivals. Our goal is to design auctions that are constant competitive with $\mathcal{F}^{(2)}$; in this sense our model lies at the intersection of prior-free mechanism design and secretary problems. We first give a generic reduction that transforms any offline auction to an online one, with only a loss of a factor of 2 in the competitive ratio; we then present some natural auctions, both randomized and deterministic, and study their competitive ratio; our analysis reveals some interesting connections of one of these auctions with RSOP, which we further investigate in our final section.

1 Introduction

The design of mechanisms that maximize the auctioneer's profit is a well-studied question in mechanism design. Most of the relevant literature assumes a prior on the distribution of bidders' values and aims at maximizing the expected profit [16]; the question of designing a profitable auction with no assumptions about the bids' distribution has only recently been addressed during the past decade. In *prior-free mechanism design* [10] we assume that bids are picked by an adversary and we want to design auctions that are profitable for any such input bid sequence. To analyze such auctions, prior-free mechanism design adopts the model of competitive analysis and compares the profit of every auction to some well-behaved benchmark.

Most of the work in prior-free mechanism design assumes that the bids are known in advance [10, 8, 13, 14]. Since almost all auctions today are happening online it makes sense to consider the online setting, where bidders arrive one at a time with a random order. In this setting, the design of a profitable, truthful auction reduces to making the "right" offer to every arriving bidder, using bids of previous bidders as the only information. We call such auctions *Online Sampling Auctions*.

This model bears a lot of similarities with the *secretary model*: the adversary picks the values of the elements, which are then presented in (uniformly) random order, and we are called to design an algorithm that

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maximizes the probability of picking the largest element. There is an extensive literature about online auctions and generalized secretary problems (for a survey see [3]). The online auctions studied there are social-welfare maximizing auctions, and the overall focus is on the competitive analysis. Given an online algorithm one can turn it into a *truthful* mechanism very easily (at least when people cannot misreport their arrival times) by simply charging every bidder its threshold value; this of course makes the profit of such auctions very hard to analyze. Our approach is the opposite one: in order to design online profit-maximizing auctions, we start with the truthful offline setting of prior-free mechanism design, and turn it into an online setting. This way we ensure our auctions are both truthful and constant-competitive with the profit-benchmark.

The work closer in spirit to ours is [11]. This paper studies limited-supply online auctions, where an auctioneer has k items to sell and bidders arrive and depart dynamically; the analysis assumes worst-case input bids and random arrivals and the main result is an online auction that is constant-competitive for both efficiency and revenue. The profit-benchmark considered for $k > 1$ items, is essentially the same as the one here, namely the optimal single price sale profit that sells at least two items, $\mathcal{F}^{(2)}$. The authors present an auction that acts in two phases, very much in the spirit of secretary algorithms, that is 6338-competitive with respect to this benchmark. Our auctions achieve much better competitive ratios (below 10), and are arguably simpler to analyze; however in our model we do not address the issue of possible arrival times misreports.

Online auctions for digital goods have also been studied before in [5, 7, 6, 4]. Their model is different from ours in that they do not assume random arrivals. Most of the algorithms presented in these papers are based on techniques from machine learning, and their performance depends on h, the ratio of the highest to the lowest bid. Our auctions are arguably more natural, and in most cases achieve better competitive ratios; however in our model auctions heavily rely on learning the actual values of past bids, and not just whether a bidder accepted or rejected the offer (as opposed to some of the auctions in [6]).

Finally, in an earlier work, Lavi and Nisan study worst case social-efficiency and profitability of online auctions for a different setting (not digital goods), taking the off-line Vickrey auction as a benchmark [15].

2 Our model

We are going to study auctions of digital goods, where bidders arrive online. Formally we have n bidders with valuations $v_1, \ldots, v_n$ (where we assume $v_1 \geq \ldots \geq v_n$) and n identical items for sale. Bidders arrive with a random order, specified by the function $\pi : [n] \to [n]$, which is a permutation on $[n] = \{1, \ldots, n\}$; we assume uniform distribution over all different permutations of the n bids and adversarial (worst-case) choice of the values of the bids. In that sense our model is similar to the secretary model.

As each bidder arrives, we make her a take-it-or-leave-it offer for a copy of the item, for some price p. We want to make the offer before the bidder

declares her bid (or equivalently we do not want our offer to depend on her declared bid) so that our auction is *truthful* (i.e. it is in the bidder's best interest to bid her true value v_i); hence, from now on we shall use $b_1, \dots, b_n$ to refer to both bids and actual values of the players. Formally we want to make the j-th bidder b_{π_j}, an offer $p_j = p(b_{\pi_1}, \dots, b_{\pi_{j-1}})$; the bidder will accept the offer if $b_{\pi_j} \geq p_j$ and will pay p_j.
Our goal is to maximize the expected profit of our auction, defined as $\mathbb{E}\left[\sum_{i=1}^{n} p_j \cdot \mathbb{I}(b_{\pi_j} \geq p_j)\right]$. We are going to consider both deterministic and randomized pricing rules $p(b_{\pi_1}, \dots, b_{\pi_{j-1}})$; therefore the expectation is over all possible orderings of the input bids and –in the case of random pricings– over the randomization in our mechanism.
We are going to use the competitive framework proposed in [10] and compare the expected profit of our auctions to the profit of the best single price auction that sells at least two items, namely $\mathcal{F}^{(2)}(b_1, \dots, b_n) = \max_{i \geq 2} i \cdot b_i$.[3] We say that an online auction is ρ-competitive if its expected profit is at least $\mathcal{F}^{(2)}/\rho$. Our goal is to design constant-competitive auctions (i.e. auctions where ρ is a constant).

3 Online Sampling Auctions

3.1 Randomized Competitive Online Sampling Auctions

Our first result establishes the existence of constant-competitive online sampling auctions. In fact we show the stronger result that any truthful offline auction gives rise to a truthful online sampling auction, with competitive ratio at most twice as large.
We start by noticing that any truthful (offline) auction for digital goods has the following format: every bidder i is given a take-it-or-leave-it offer p_i which is a function of the bids of the other players $f(b_{-i})$; if the bidder accepts she pays p_i otherwise nothing (this follows from Myerson's theorem [16]). Then we notice that every such truthful offline auction gives rise to an online auction if we simply set the price offered to the j-th arriving bidder to be $p_j = f(b_{\pi_1}, \dots, b_{\pi_{j-1}})$, for the same function f; intuitively this means that we run the offline auction on the whole set of revealed bids, but actually charge only the bidder that has just arrived. Because we restrict our attention to truthful offline auctions, we know that the price offered to p_j will not depend on b_j and so we can offer the j-th bidder a price before she even reveals her bid. Our theorem now says that the resulting online auction has at most twice the competitive ratio of the offline auction.

Theorem 1. *If we turn an offline auction with competitive ratio ρ into an online auction, the competitive ratio of the online auction is at most 2ρ. More precisely, if b_k is the price of the optimal auction, then the competitive ratio of the online auction is at most $\rho \cdot k/(k-1)$.*[4]

[3] Notice however that in general our auctions will not be single-price auctions.
[4] We note that this looks very much like the $\frac{1}{4} \cdot \frac{k-1}{k}$ approximation ratio of [8], in a different model of course.

Proof. Consider the first t bids of the online auction. The online auction runs the offline auction on them. The expected profit of the offline auction from the whole set of bids would be at least $\frac{1}{\rho}\mathcal{F}^{(2)}(b_{\pi_1}, \ldots, b_{\pi_t})$; by the random-order assumption about the input, the expected profit from every bid is equal and, in particular, the expected gain from b_{π_t} is at least:

$$\frac{1}{t}\frac{1}{\rho}\mathcal{F}^{(2)}(b_{\pi_1}, \ldots, b_{\pi_t})$$

With probability $\binom{t}{m}\binom{n-t}{k-m}/\binom{n}{k}$ the first t bids have **exactly** m of the highest k bids which contribute to the optimum. Also, for $m \geq 2$, $\mathcal{F}^{(2)}(b_{\pi_1}, \ldots, b_{\pi_t}) \geq m b_k$.[5] So, it follows that when $m \geq 2$, with the above probability the expected gain of the online auction from b_{π_t} is at least:

$$\frac{1}{t}\frac{1}{\rho} m b_k$$

So, the expected profit of the online auction is at least:

$$\begin{aligned}
&\sum_{t=2}^{n} \sum_{m=2}^{\min\{t,k\}} \frac{\binom{t}{m}\binom{n-t}{k-m}}{\binom{n}{k}} \frac{1}{t}\frac{1}{\rho} m b_k \\
&= \frac{1}{\rho} b_k \binom{n}{k}^{-1} \sum_{t=2}^{n}\sum_{m=2}^{k} \binom{t-1}{m-1}\binom{n-t}{k-m} \\
&= \frac{1}{\rho} b_k \binom{n}{k}^{-1} \sum_{t=1}^{n-1}\sum_{m=1}^{k-1} \binom{t}{m}\binom{n-1-t}{k-1-m} \\
&= \frac{1}{\rho} b_k \binom{n}{k}^{-1} \sum_{t=1}^{n-1} \left(\binom{n-1}{k-1} - \binom{n-1-t}{k-1}\right) \\
&= \frac{1}{\rho} b_k \binom{n}{k}^{-1} \left((n-1)\binom{n-1}{k-1} - \sum_{j=k-1}^{n-2} \binom{j}{k-1}\right) \\
&= \frac{1}{\rho} b_k \binom{n}{k}^{-1} \left((n-1)\binom{n-1}{k-1} - \binom{n-1}{k}\right) \\
&= \frac{k-1}{\rho} b_k,
\end{aligned}$$

where in the third equality we used the Chu-Vandermonde identity and in the second-to-last equality we used the identity $\sum_{j=k}^{n}\binom{j}{k} = \binom{n+1}{k+1}$; the Theorem now follows. □

We can now state our main Theorem:

Theorem 2. *The competitive ratio of Online Sampling Auctions is between 4 and 6.48.*

[5] Notice that when $m = 1$, there is no decent lower bound for $\mathcal{F}^{(2)}$; this is the reason that the online auction has larger competitive ratio than the offline auction.

Proof. The upper bound is given by the online version of the (offline) auction presented in [12] which achieves a competitive ratio of 3.24.
For the lower bound, consider the case where the input chosen by the adversary consists of two bids: if the price offered to the second bidder is strictly greater than the first bid the adversary will pick two identical bids and the online auction will have zero profit. If the auction's offer to the second bidder is less than or equal to the first bid the adversary will pick as input the bids $h+\epsilon, h$ (for sufficiently large ϵ), in which case the optimal profit is $2h$ but the auction has expected profit at most $h/2$. $\square$

At this point we note that the above Theorem greatly improves over the previously known bounds due to [11]. A natural question to ask now is whether we can bridge the gap between the lower and the upper bound. To this end we first study the competitive ratio that can be achieved by the online version of the *Sampling Cost Sharing* auction (SCS); this auction partitions bidders uniformly into two parts and extracts the optimal single price sale profit of each side from the other (if possible, otherwise it extracts no profit) [10]. We have the following:

Corollary 1. *The competitive ratio of the online version of SCS is at most 8. In the special case in which the optimal single-price auction for the whole set of bids sells the item to at least 5 buyers, the competitive ratio is at most 4.*

Proof. Using Theorem 1 and the bound on the competitive ratio of SCS proved in [10] we get that the online version of SCS will have competitive ratio at most $\frac{k}{k-1}\left(\frac{1}{2}-\binom{k-1}{\lfloor k/2\rfloor}2^{-k}\right)^{-1}$, which is less than 4 for $k \geq 5$. $\square$

Notice that the worst-case inputs for this auction are when the optimal single price b_k is large, i.e. k is small. In the following section we show that this is not the case for all auctions.

3.2 A Deterministic Online Sampling Auction: BPSF$_r$

The two online auctions considered in the previous section are randomized, like their offline counterparts. In this section we shift our focus on deterministic online sampling auctions. For the offline setting the following theorem from [10] wipes out all hope for such an auction.

Theorem 3 ([10]). *We say an auction is* symmetric *if its outcome is independent of the order of the bids. It then holds that no symmetric, deterministic, truthful auction is constant-competitive against $\mathcal{F}^{(2)}$.*

There exist asymmetric, deterministic auctions with constant competitive ratio, but most of them result from derandomization of randomized ones and are unnatural [1]. In the online setting where order matters anyway, we can hope to design a constant competitive and deterministic (truthful) auction, that is also natural.
To this end we define the Best-Price-So-Far auction: BPSF$_r$ is the (family of) auction(s) which offer as price the bid among the highest r

of the previous bids which maximizes the single price sale profit of past requests. We are going to focus our attention on two representatives of this family, BPSF_1 and BPSF_∞, henceforth denoted by BPSF. BPSF_1 is an interesting auction which offers as price the maximum revealed bid. BPSF is an auction that offers the j-th bidder the price $p_j = p(b_{\pi_1}, \ldots, b_{\pi_{j-1}}) = \arg\max_{i \leq j-1} i \cdot b_{\pi_i}$.

Theorem 4. *The expected profit of $BPSF_1$ is exactly $\sum_{i=2}^{n} \frac{1}{i} b_i$. Furthermore, if b_k is the price of the optimal auction, then the competitive ratio of $BPSF_1$ is $\frac{k}{H_k - 1}$ where $H_k = 1 + 1/2 + \cdots + 1/k$ is the k-th harmonic number, and this is tight.*

Proof. Notice that b_j is going to be offered as price exactly when b_j appears before $b_1, \ldots, b_{j-1}$. Every such bid is accepted if there is a higher bid after b_j appears. Thus b_j is going to be accepted at some point when $j \geq 2$. The probability that b_j appears before $b_1, \ldots, b_{j-1}$ is exactly $1/j$. It follows that the expected profit of BPSF_1 is $\sum_{i=2}^{n} \frac{1}{i} b_i$.
For the second fact, simply observe that when b_k is the price of the optimal auction, the online profit is at least $\sum_{i=2}^{k} \frac{1}{i} b_i \geq \sum_{i=2}^{k} \frac{1}{i} b_k = (H_k - 1) b_k$. Since the optimal profit is $k b_k$, the claim follows.
Finally, it is easy to verify that the above bound is tight for any set of n identical bids. □

Corollary 2. *Let b_k, be the optimal single price for the whole set of bids. If $k \leq 5$ then the competitive ratio of $BPSF_1$ is at most 4.*

Corollaries 1 and 2 show that if we knew in advance the number of buyers of the optimal single-price auction, we could achieve competitive ratio 4 against $\mathcal{F}^{(2)}$, thus matching the corresponding lower bound.
We saw that BPSF_1 is not constant competitive; it is also easy to see that the competitive ratio of BPSF_r can only decrease for larger r; the natural question to ask is if it will ever be constant. To this end we examine BPSF, which is arguably a very natural online auction: BPSF is the online version of the *Deterministic Optimal Price* (DOP) auction that offers bidder j the optimal single price of the other bidders, namely $p_j = p(b_{-j}) = \arg\max_{i \neq j} i \cdot b_i$. DOP is known not to be competitive [10]; we conjecture that BPSF on the contrary is constant-competitive:

Conjecture 1. The competitive ratio of BPSF is 4.

The competitive ratio of 4 is the same as the conjectured competitive ratio of RSOP. This is not a coincidence; in the next section we take a closer look into the similarities of RSOP and BPSF.

4 On the competitive ratio of BPSF and RSOP

One of the simplest competitive auctions, and arguably the most studied [10, 9, 2] is the *Random Sampling Optimal Price* auction (RSOP). In RSOP the bidders are uniformly partitioned into two parts, and the optimal single price of each part (i.e. $\arg\max i \cdot b_i$) is offered to the bidders of the other part. RSOP is conjectured to be 4-competitive; to date the best upper bound is 4.68 [2].

In what follows we analyze the competitive ratio of BPSF and RSOP in more detail. We see that the analyses of the two auctions bear a lot of similarities and we suggest a possible approach for both auctions. We believe that our approach may be a promising direction for proving both Conjecture 1 and that RSOP is 4-competitive as well.[6]

We first introduce some notation. Let $B = \{b_1, \ldots, b_n\}$ be the set of all bids and $B_2 = \{b_2, \ldots, b_n\}$. Given a specific partition of bids $b_1, \ldots, b_n$ in two parts, we use $(b_{j_1}, \ldots, b_{j_k})$ to denote the side of the partition that does not contain the highest bid b_1, i.e. by writing $(b_{j_1}, \ldots, b_{j_k})$ we assume implicitly that $j_1 \geq 2$ and also $b_{j_1} \geq \ldots \geq b_{j_k}$. Finally let

$$y(b_{j_1}, \ldots, b_{j_k}) = \max\{b_{j_1}, 2b_{j_2}, \ldots, kb_{j_k}\},$$

the optimal single price sale profit from $(b_{j_1}, \ldots, b_{j_k})$ and let $z(b_{j_1}, \ldots, b_{j_k})$ be the profit from offering the optimal price of $(b_{j_1}, \ldots, b_{j_k})$ to the other side.

We next show how to write the expected profits of RSOP and BPSF in terms of z and y.

For RSOP it is straightforward; just notice that the adversary can always pick a large enough b_1 so that the profit from the side of the partition not containing b_1 will always be 0 [9]. We then have:

$$RSOP = \sum_{S \subseteq B_2} z(S) 2^{-n+1}$$

For BPSF the expression is less straightforward. We have:

Lemma 1. *The expected profit of BPSF is* $\sum_{S \subseteq B_2} z(S) \binom{n-1}{|S|}^{-1} n^{-1}$.

Proof. Let $Profit(S, b_i)$ denote the profit we get if we offer the optimal single price for S to bid $b_i \notin S$. In what follows, the expectation operator is used to denote expectation over the non-uniform distribution on the collection of sets $S \subseteq B_2$ induced by the random arrival order of the bids.[7] We have:

$$\begin{aligned}
BPSF &= \sum_{b_i} \mathbb{E}_{S \subseteq B_2, b_i \notin S} \left[Profit(S, b_i)\right] \\
&= \frac{1}{n} \cdot \sum_{b_i} \sum_{k=0}^{n-1} \mathbb{E}_{S \subseteq B_2, |S|=k, b_i \notin S} \left[Profit(S, b_i)\right] \\
&= \frac{1}{n} \cdot \sum_{b_i} \sum_{k=0}^{n-1} \sum_{S \subseteq B_2, |S|=k, b_i \notin S} \frac{Profit(S, b_i)}{\binom{n-1}{k}} \\
&= \frac{1}{n} \cdot \sum_{b_i} \sum_{S \subseteq B_2, b_i \notin S} \frac{Profit(S, b_i)}{\binom{n-1}{|S|}} \\
&= \frac{1}{n} \cdot \sum_{S \subseteq B_2} \frac{1}{\binom{n-1}{|S|}} z(S)
\end{aligned}$$

[6] The very technical approach of [2], although coming very close, does not seem to be able to prove the Conjecture.

[7] As opposed to expectation taken over a uniformly random choice of a set $S \subseteq B_2$.

where in the second equality we used the fact that b_i will be in the $k+1$ position with probability $1/n$, in the third equality we used the fact that all orderings have the same probability and in the last equality we used the fact that $z(S) = \sum_{b_i \notin S} Profit(S, b_i)$. $\square$

The next lemma establishes an interesting relation between the y values and the optimal single price sale profit $\mathcal{F}^{(2)}$.

Lemma 2. *For any $i \in [n], i \geq 2$ it holds that:*

$$\sum_{S \subseteq B_2:\ b_2 \in S} y(S) \geq 2^{n-3}\ ib_i$$

Proof. Let b_i be the optimum single price for the whole set of bids, i.e. $\mathcal{F}^{(2)} = ib_i$ (although our result holds for any bid b_i).
We will introduce a mapping between the set of sequences $X = \{S \subseteq B_2 |\ b_2 \in S \ \&\ b_i \notin S\}$ and the set $Y = \{S \subseteq B_2 |\ b_2 \notin S \ \&\ b_i \in S\}$. Given a sequence of bids $S \in X$ let $t = \max\{j : j < i, b_j \in S\}$.[8] We then define the following mapping for each bid $b_j \in S$:

$$f(b_j) = \begin{cases} b_{j+i-t} & : \text{if } j < i \\ b_j & : \text{if } j > i \end{cases}$$

It is easy to see that the mapping $g : X \longrightarrow Y$ defined as $g(b_{j_1}, \ldots, b_{j_k}) = (f(b_{j_1}), \ldots, f(b_{j_k}))$ is in fact a bijection. Also note that $b_1 \geq \ldots \geq b_n$ implies that $y(S) \geq y(g(S))$. Hence we have:

$$\begin{aligned}
\sum_{S \subseteq B_2:\ b_2 \in S} y(S) &= \sum_{S \subseteq B_2:\ b_2 \in S, b_i \in S} y(S) + \sum_{S \subseteq B_2:\ b_2 \in S, b_i \notin S} y(S) \\
&\geq \sum_{S \subseteq B_2:\ b_2 \in S, b_i \in S} y(S) + \sum_{S \subseteq B_2:\ b_2 \in S, b_i \notin S} y(g(S)) \\
&= \sum_{S \subseteq B_2:\ b_2 \in S, b_i \in S} y(S) + \sum_{S \subseteq B_2:\ b_2 \notin S, b_i \in S} y(S) \\
&= \sum_{S \subseteq B_2:\ b_i \in S} y(S) \\
&= 2^{n-i} \sum_{j=0}^{i-2} \binom{i-2}{j} (j+1) \cdot b_i
\end{aligned}$$

For the last equality consider all possible positions of b_i in S. There can be j bids larger than b_i where j ranges from 0 to $i-2$; there are $\binom{i-2}{j}$ ways to pick these bids and 2^{n-i} ways to pick the bids that are smaller than b_i and for this specific position the coefficient of b_i is $(j+1)$.[9]
A straightforward calculation shows that $\sum_{j=0}^{i-2} \binom{i-2}{j}(j+1) = i2^{i-3}$, and the claim follows. $\square$

[8] Notice that this is a non-empty set, as it contains $j = 2$.

[9] If b_i is an arbitrary bid with $i \geq 2$, rather than the optimum single price as stated in the beginning of the proof, then the last equality should be replaced with an inequality, and the claim still goes through.

It is now easy to see that the following claim implies that RSOP is indeed 4-competitive.

Conjecture 2.

$$\sum_{S \subseteq B_2} z(S) \geq \sum_{S \subseteq B_2:\ b_2 \in S} y(S)$$

The corresponding claim for BPSF is:

Conjecture 3.

$$\sum_{S \subseteq B_2} z(S) \binom{n-1}{|S|}^{-1} n^{-1} \geq \sum_{S \subseteq B_2:\ b_2 \in S} y(S) 2^{-n+1}$$

We believe that Conjectures 2 and 3 both hold and that RSOP and BPSF are both 4-competitive. We attempted to prove the Conjectures using a number of relations between the z and y values; analytical and numerical simulations show that one can sum up individual relations between $z(S)$ and $y(S)$ for any set S of bids, like the ones presented in Appendix A, in order to get the result.

5 Conclusion

There is a number of open questions from this work: the obvious ones are to prove that BPSF is indeed 4-competitive and see what this proof implies for the competitive ratio of RSOP. Proving that BPSF is constant competitive for some other constant is also interesting, and probably much easier. Finally, it would be interesting to see if there is a natural online sampling auction with competitive ratio at most 4, for all values k of the optimal single price b_k.

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A Relation of z and y values

In order to prove Conjectures 2 and 3 we need a strong lemma that captures the relation of the z and y values. In the appendix we list three such lemmata, of increasing strength. Numerical and analytical simulations in MAPLE12 have verified that Lemma 5 is enough to prove Conjecture 2 (by just summing up for all subsets $S \subseteq B_2$) for up to $n = 20$ bids.

Lemma 3.

$$z(b_{j_1}, \ldots, b_{j_k}) \geq \min_{t=1,\ldots,k} \left(\frac{j_t - t}{t} \right) \cdot y(b_{j_1}, \ldots, b_{j_k})$$

Proof. Let b_{j_t} be the optimal price for $(b_{j_1}, \ldots, b_{j_k})$, i.e.

$$t \cdot b_{j_t} = \max\{b_{j_1}, 2b_{j_2}, \ldots, kb_{j_k})$$

Then

$$\begin{aligned} z(b_{j_1},\ldots,b_{j_k}) &= (j_t - t)b_{j_t} \\ &= \frac{j_t - t}{t} \cdot t b_{j_t} \\ &= \frac{j_t - t}{t} \cdot y(b_{j_1},\ldots,b_{j_k}) \\ &\geq \min_{t=1,\ldots,k} \left(\frac{j_t - t}{t}\right) \cdot y(b_{j_1},\ldots,b_{j_k}) \end{aligned}$$

□

Notice that the term $\left(\frac{j_t-t}{t}\right)$ is the same quantity as the one minimized in the random walk of [9] and it also appears in the analysis of [2].
The following relation is stronger, in that by summing up all for all S we immediately get $\sum_{S\subseteq B_2:\ b_2\in S} y(S)$ and some more terms, whose sum we then need to show is positive.

Lemma 4.

$$z(b_{j_1},\ldots,b_{j_k}) \geq y(b_{j_1},\ldots,b_{j_k}) - \max\left(0, \max_{t=2,\ldots,k}\left\{\frac{2t - j_t}{t-1}\right\}\right) \cdot y(b_{j_2},\ldots,b_{j_k})$$

Proof. Let b_{j_t} be the optimal price for $(b_{j_1},\ldots,b_{j_k})$, i.e.

$$t \cdot b_{j_t} = \max\{b_{j_1}, 2b_{j_2},\ldots,kb_{j_k})$$

Then

$$\begin{aligned} z(b_{j_1},\ldots,b_{j_k}) &= (j_t - t)b_{j_t} \\ &= tb_{j_t} - (2t - j_t)b_{j_t} \\ &= tb_{j_t} - \frac{2t - j_t}{t-1}(t-1)b_{j_t} \\ &= y(b_{j_1},\ldots,b_{j_k}) - \frac{2t - j_t}{t-1}(t-1)b_{j_t} \\ &\geq y(b_{j_1},\ldots,b_{j_k}) - \max\left(0, \max_{t=2,\ldots,k}\left\{\frac{2t - j_t}{t-1}\right\}\right) \cdot y(b_{j_2},\ldots,b_{j_k}) \end{aligned}$$

where we need $\frac{2t-j_t}{t-1}$ to be positive for the inequalities to work correctly, which is why we take $\max\left(0, \max_{t=2,\ldots,k}\left\{\frac{2t-j_t}{t-1}\right\}\right)$. □

In order to optimally handle the negative terms showing up in the RHS of Lemma 4 we used the following, more elaborate bound.

Lemma 5. *Let $(b_{j_1},\ldots,b_{j_k})$ be a set of at least 2 bids and λ a real in $[0, j_1 - 1]$. We can bound $z(b_{j_1},\ldots,b_{j_k})$ with*

$$z(b_{j_1},\ldots,b_{j_k}) \geq \lambda y(b_{j_1},\ldots,b_{j_k}) + \mu y(b_{j_2},\ldots,b_{j_k}), \tag{1}$$

where μ is defined by

$$\mu = \begin{cases} \frac{k}{k-1}\min_{t=1,\ldots,k}\{\frac{j_t - t - \lambda t}{t}\}\}, & \textit{when } \min_{t=2,\ldots,k}\{j_t - t - \lambda t\} \geq 0 \\ \min_{t=2,\ldots,k}\{\frac{j_t - t - \lambda t}{t-1}\}, & \textit{otherwise} \end{cases}$$

Proof. Assume that

$$
\begin{aligned}
y(b_{j_1},\dots,b_{j_k}) &= t\cdot b_{j_t}\\
y(b_{j_2},\dots,b_{j_k}) &= (s-1)\cdot b_{j_s}
\end{aligned}
$$

From these we get that $tb_{j_t} \geq sb_{j_s}$ and $(s-1)b_{j_s} \geq (t-1)b_{j_t}$. Notice that the latter holds even when $t = 1$.
Assume that $\min_{r=2,\dots,k}\{j_r - r - \lambda r\} \geq 0$. We will show that inequality (1) is satisfied for $\mu = \frac{k}{k-1}\min_{r=1,\dots,k}\{\frac{j_r - r-\lambda r}{r}\}\}$. We will use the fact that μ is nonnegative and the inequality $tb_{j_t} \geq sb_{j_s}$. Indeed we have,

$$
\begin{aligned}
\lambda y(b_{j_1},\dots,b_{j_k}) + \mu y(b_{j_2},\dots,b_{j_k}) &= \lambda t b_{j_t} + \mu(s-1)b_{j_s}\\
&\leq \lambda t b_{j_t} + \mu(s-1)\frac{t}{s}b_{j_t}\\
&\leq \lambda t b_{j_t} + \mu(k-1)\frac{t}{k}b_{j_t}\\
&\leq \lambda t b_{j_t} + \frac{k}{k-1}\frac{j_t - t - \lambda t}{t}(k-1)\frac{t}{k}b_{j_t}\\
&= (j_t - t)b_{j_t}\\
&= z(b_{j_1},\dots,b_{j_k})
\end{aligned}
$$

Now we consider the case of $\min_{r=2,\dots,k}\{j_r - r - \lambda r\} < 0$. Assume first that $t \geq 2$. We will now show that inequality (1) is satisfied for $\mu = \min_{r=2,\dots,k}\{\frac{j_r - r - \lambda r}{r-1}\}\}$. We will use the fact that μ is now negative and the inequality $(t-1)b_{j_t} \leq (s-1)b_{j_s}$. Indeed we have,

$$
\begin{aligned}
\lambda y(b_{j_1},\dots,b_{j_k}) + \mu y(b_{j_2},\dots,b_{j_k}) &= \lambda t b_{j_t} + \mu(s-1)b_{j_s}\\
&\leq \lambda t b_{j_t} + \mu(t-1)b_{j_t}\\
&\leq \lambda t b_{j_t} + \frac{j_t - t - \lambda t}{t-1}(t-1)b_{j_t}\\
&= (j_t - t)b_{j_t}\\
&= z(b_{j_1},\dots,b_{j_k})
\end{aligned}
$$

The case $t = 1$ must be handled separately because $t-1$ appears in the denominator in the above. When $t = 1$ we have that

$$
\begin{aligned}
z(b_{j_1},\dots,b_{j_k}) &= (j_1 - 1)b_{j_1}\\
&= \lambda b_{j_1} + (j_1 - 1 - \lambda)b_{j_1}\\
&\geq \lambda y(b_{j_1},\dots,b_{j_k})\\
&\geq \lambda y(b_{j_1},\dots,b_{j_k}) + \mu y(b_{j_2},\dots,b_{j_k}).
\end{aligned}
$$

Notice that we used the fact that $\lambda \leq j_1 - 1$ and that $\mu \leq 0$. $\square$