On the performance of approximate equilibria in congestion games*

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Abstract

We study the performance of approximate Nash equilibria for congestion games with polynomial latency functions. We consider how much the price of anarchy worsens and how much the price of stability improves as a function of the approximation factor ϵ. We give almost tight upper and lower bounds for both the price of anarchy and the price of stability for atomic and non-atomic congestion games. Our results not only encompass and generalize the existing results of exact equilibria to ϵ-Nash equilibria, but they also provide a unified approach which reveals the common threads of the atomic and non-atomic price of anarchy results. By expanding the spectrum, we also cast the existing results in a new light.

1 Introduction

Algorithmic Game Theory has studied extensively and with remarkable success the computational issues of Nash equilibria. As a result, we understand almost completely the computational complexity of exact Nash equilibria (they are PPAD-complete for games described explicitly [16, 7] and PLS-complete for games described succinctly [19]). The results indicate a long-suspected drawback of Nash equilibria, namely that they cannot be computed effectively. Despite substantial recent progress [21, 19, 17, 33], it is still open whether approximate Nash equilibria share the same drawback. But in any case, they seem to provide a more reasonable equilibrium concept: It usually makes sense to assume that an agent is willing to accept a situation that is almost optimal to him.

In another direction, a large body of research in Algorithmic Game Theory concerns the degree of performance degradation of systems due to the selfish behavior of the users. Central to this area is the notion of price of anarchy (PoA) [20, 25] and the price of stability (PoS) [3]. The first notion compares the social cost of the worst-case equilibrium to the social optimum, which could be obtained if every agent followed obediently a central authority. The second notion is very similar but it considers the best Nash equilibrium instead of the worst one.

A natural question then is how the performance of a system is affected when its users are approximately selfish: What is the *approximate price of anarchy* and the *approximate price of stability*? Clearly, by allowing the players to be almost rational (within an ϵ factor), we expand the equilibrium concept and we expect the PoA to get worse. On the other hand, the PoS should improve. The question is how they change as functions of the parameter ϵ. This is exactly the question that we address in this work.

We study two fundamental classes of games: the class of atomic congestion games [26, 23] and the class of non-atomic congestion (or selfish routing) games [5, 15, 22]. Both classes of games played central role in the development of the area of the PoA [20, 27, 28]. Although the PoA and PoS of these games for exact equilibria was established long ago [27, 11, 10, 4]—and actually some work [29, 32, 8] addressed partially the question for the PoA of approximate equilibria—our results add an unexpected understanding of the issues involved.

While the classes of atomic and non-atomic games are conceptually very similar, dissimilar techniques were employed in the past to answer the questions concerning the PoA and PoS. Moreover, the qualitative aspects of the answers were quite different. For instance, the maximum PoA for the non-atomic case is attained by the Pigou network (Figure 1), while for the atomic case it is attained [4, 11] by a completely different network (with the structure of Figure 2).

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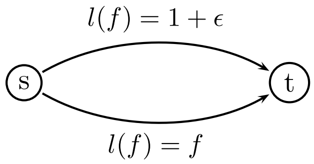

Figure 1: The Pigou network.

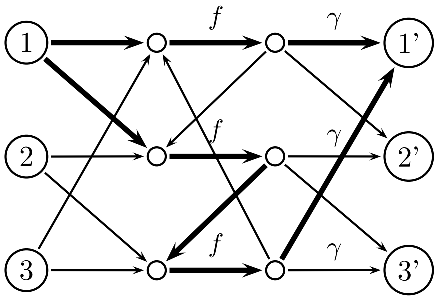

Figure 2: Lower bound for the PoA for selfish routing. There are n distinct edge latency functions: $l(f) = f$, $l(f) = \gamma$ (a constant which depends on ϵ), $l(f) = 0$ (omitted in the picture). There are n commodities of rate 1 with source i and destination i' and 2 paths that connect them. The two paths for the first commodity are shown in bold lines. A similar construction works for atomic games as well (Theorem 5).

There are two main differences between the atomic and non-atomic games. The first difference is the *"uniqueness"* of equilibria: Non-atomic congestion games have a unique exact Nash (or Wardrop as it is called in these games) equilibrium, and therefore the PoS is not different than the PoA. On the other hand, atomic congestion games may have multiple exact equilibria. Perhaps, because of this, the problem of determining the PoS proved more challenging [10, 6] for this case. New techniques needed for upper bounding the PoS for linear latencies which exploited the potential of these games; for polynomial latencies, the problem is still open. Furthermore, the lower bound for linear latencies is quite complicated and, unlike the selfish routing case, it has a dependency on the number of players (it attains the maximum value at the limit).

The second main difference between the two classes of games is *"integrality"*: In the case of atomic games, when a player considers switching to another strategy, he has to take into account the extra cost that he will add to the edges (or facilities) of the new strategy. The number of players on the new edges increases by one and this changes the cost. On the other hand, in the selfish routing games the change of strategies has no additional cost. A simple—although not entirely rigorous—way to think about it, is to consider the effects of a tiny amount of flow that ponders whether to change path: it will not really affect the flow on the new edges (at least for continuous cost functions).

When we consider approximate equilibria, the uniqueness difference dissolves and only the integrality difference remains. But still, determining the PoS seems to be a harder problem than determining the PoA.

1.1 Our contribution and related work

Our work encompasses and generalizes some fundamental results in the area of the PoA [27, 11, 10, 4] (see also the recently published book [24] for background information). Our techniques not only provide a unifying approach but they cast the existing results in a new light. For instance, the Pigou network (Figure 1) is still the tight example for the PoS, but not for the PoA. For the latter, the network of Figure 2 is tight for $\epsilon \leq 1$ and the network of Figure 4 is tight for larger ϵ.

We use the *multiplicative definition of approximate equilibria*: In atomic games, a player does not switch to a new strategy as long as his current cost is less than $1+\epsilon$ times the new cost. In the selfish routing games, we use exactly the same definition [27]: the flow is at an equilibrium when the cost on its paths is less than $1+\epsilon$ times the cost of every alternate path. There have been other definitions for approximate Nash equilibria in the literature. For example, for algorithmic issues of Nash equilibria, the most-studied one is the additive case [21, 16]. However, since the PoA is a ratio, the natural definition is the multiplicative one. A slightly different multiplicative approximate Nash equilibrium definition was used in [9], where they study convergence issues for congestion games.

There is a large body of work on the PoA in various models [24]. More relevant to our work are the following publications: For atomic congestion games, it was shown in [4, 11] that the pure PoA of linear latencies is 5/2. Later in [10], it is showed that the ratio 5/2 is tight even for correlated equilibria, and consequently for mixed equilibria. For weighted congestion games, the PoA is $1+\phi \approx 2.618$ [4].

For polynomial latencies of degree p, [11] showed that the PoA of pure equilibria is $p^{\Theta(p)}$, and [4] showed $p^{\Theta(p)}$ bounds for mixed equilibria and for weighted congestion games. Aland et al. [1] gave exact bounds for both weighted and unweighted congestion games.

For the PoS of the atomic case, it was shown in [10, 6] that for linear congestion games it is $1+\sqrt{3}/3$. Also for the linear selfish routing games, [29, 32] give tight bounds for the PoA of ϵ-Nash equilibria when $\epsilon \leq 1$.

For the selfish routing paradigm and exact Nash equilibria, [27] gave the PoA (and consequently the PoS) for polynomial latencies of degree p with nonnegative coefficients and more general functions (see [13] or [24] for a simplified version of the proof); the results are extended to non-atomic games in [28].

In this work, we give almost tight upper and lower bounds for the PoA and PoS of atomic and non-atomic congestion games with polynomial latencies of degree p. We assume throughout that the coefficients of the polynomials are nonnegative. We give tight upper and lower bounds for the PoS and the PoA (as functions of the degree p and the approximation factor ϵ) for both atomic and non-atomic games. The only exception is the PoS for atomic games (Section 7), where we have weaker bounds; Figure 3 shows the upper and lower bounds for this case and linear latencies.

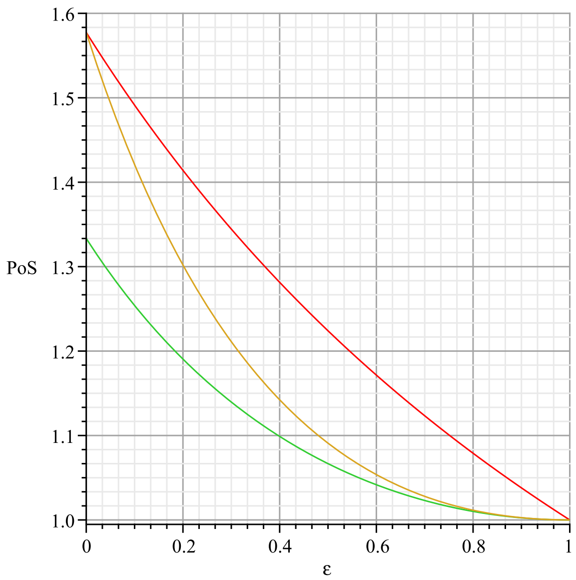

Figure 3: The upper and lower bound of the price of stability (Theorems 7 and 9) for atomic games and linear latencies. The lower line is the Pigou bound $\frac{4}{(1+\epsilon)(3-\epsilon)}$.

Our results reveal qualitative differences between exact and approximate Nash equilibria for non-atomic congestion games. These games have unique exact Nash equilibria, which makes the

PoS to be equal to the PoA. When we consider the larger class of approximate equilibria the uniqueness of equilibria vanishes and the PoS and PoA diverge. Another interesting finding of our work is that for $\epsilon = p$, the PoS drops to 1 in both the atomic and non-atomic case (this was known in the non-atomic case [30, 14]).

For all cases we give appropriate *characterizations* of the ϵ-Nash equilibria, which are of independent interest and generalize known characterizations of exact equilibria. These characterizations apply to every nondecreasing and continuous latency functions, not just polynomials. Especially for the PoS, our characterization involves a notion which generalizes the potential function of Rosenthal [26, 24].

All upper bounds proofs have the following outline: Using an appropriate characterization for the ϵ-Nash equilibria and suitable algebraic inequalities—which for non-atomic games involve real numbers and for atomic games involve nonnegative integers—we bound the cost of the ϵ-Nash equilibria. The cases of PoS however are more complicated. In these cases, the characterization of ϵ-Nash equilibria involves potential-like quantities. For exact Nash equilibria there is essentially a unique potential; but for approximate equilibria, there are many choices for the potential and we have to figure out which potential gives the best results. This together with the technical challenges of the algebraic inequalities makes the proofs for the PoS harder.

All the games that we consider in this work possess pure equilibria and our upper bound proofs are tuned to these, for simplicity. However, all our proofs can be directly extended to mixed and correlated equilibria—the difference will be an extra outer sum which corresponds to the expectation. The upper bounds of our theorems apply unmodified to the more general classes of *mixed* and *correlated* equilibria.

An unpublished preliminary version of this work [12], studied the same questions for linear latencies. Here we generalize the results to polynomial latencies which are technically much more challenging.

Due to space limitations, we move to the appendix the sections about atomic games.

2 Definitions

A congestion game [26], also called an exact potential game [23], is a tuple $(N, E, (\mathcal{S}_i)_{i\in N}, (f_e)_{e\in E})$, where $N = \{1, \ldots, n\}$ is a set of n players, E is a set of facilities, $\mathcal{S}_i \subseteq 2^E$ is a set of pure strategies for player i: a pure strategy $A_i \in \mathcal{S}_i$ is simply a subset of facilities and l_e is a cost (or latency) function, one for each facility $e \in E$. The cost of player i for the pure strategy profile $A = (A_1, \ldots, A_n)$ is $c_i(A) = \sum_{e\in A_i} l_e(n_e(A))$, where $n_e(A)$ is the number of players who use facility e in the strategy profile A.

Definition 1. A pure strategy profile A is an ϵ equilibrium iff for every player $i \in N$

$$c_i(A) \le (1+\epsilon)c_i(A_i, A_{-i}), \qquad \forall A_i \in \mathcal{S}_i \tag{1}$$

We believe that the multiplicative definition of approximate equilibria makes more sense in the framework that we consider: Given that the PoA is a ratio, we need a definition that is insensitive to scaling.

The social cost of a pure strategy profile A is the sum of the players cost

$$\mathrm{C}(A) = \sum_{i\in N} c_i(A).$$

The approximate PoA and PoS, is the social cost of the worst-case and best-case ϵ-equilibrium over the optimal social cost

$$PoA = \max_{A \text{ is an } \epsilon\text{-Nash}} \frac{\mathrm{C}(A)}{\mathrm{OPT}}, \quad PoS = \min_{A \text{ is an } \epsilon\text{-Nash}} \frac{\mathrm{C}(A)}{\mathrm{OPT}}.$$

Instead of defining formally the class of non-atomic congestion games, we prefer to focus on the more illustrative–more restrictive though–class of selfish routing games. The difference in the two models is that in a non-atomic game, there does not exist any network and the strategies of the players are just subsets of facilities (as in the case of atomic congestion games) and they do not necessarily form a path in a network.

Let $G = (V, E)$ be a directed graph, where V is a set of vertices and E is a set of edges. In this network we consider k commodities: source-node pairs (s_i, t_i) with $i = 1 \dots k$, that define the sources and destinations. The set of simple paths in every pair (s_i, t_i) is denoted by $\mathcal{P}_i$, while with $\mathcal{P} = \cup_{i=1}^{k} \mathcal{P}_i$ we denote their union. A flow f, is a mapping from the set of paths to the set of nonnegative reals $f : \mathcal{P} \to \mathbb{R}^+$. For a given flow f, the flow on an edge is defined as the sum of the flows of all the paths that use this edge $f_e = \sum_{P \in \mathcal{P}, e \in P} f_P$. We relate with every commodity (s_i, t_i) a traffic rate r_i, as the total traffic that needs to move from s_i to t_i. A flow f is feasible, if for every commodity $\{s_i, t_i\}$, the traffic rate equals the flow of every path in $\mathcal{P}_i$, $r_i = \sum_{P \in \mathcal{P}_i} f_P$. Every edge introduces a delay in the network. This delay depends on the load of the edge and is determined by a delay function, $l_e(\cdot)$. An instance of a routing game is denoted by the triple (G, r, l). The latency of a path P, for a given flow f, is defined as the sum of all the latencies of the edges that belong to P, $l_P(f) = \sum_{e \in P} l_e(f_e)$. The social cost that evaluates a given flow f, is the total delay due to f

$$C(f) = \sum_{P \in \mathcal{P}} l_P(f) f_P.$$

The total delay can also be expressed via edge flows $C(f) = \sum_{e \in E} l_e(f_e) f_e$.

From now on, when we refer to flows, we mean feasible flows. We define (as in [27]) the ϵ-Nash or ϵ-Wardrop equilibrium flows:

Definition 2. A feasible flow f, is an ϵ-Nash (or ϵ-Wardrop) equilibrium, if and only if for every commodity $i \in \{1, \dots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \leq (1 + \epsilon) l_{P_2}(f)$.

3 Non-atomic PoA

In this section we give tight bounds for the approximate PoA for non-atomic congestion games. We present them for the special case of selfish routing (or network non-atomic congestion games), but they apply unchanged to the more general class of non-atomic congestion games. One of the technical difficulties here is that a different approach is required for small and large values of ϵ.

We start with a condition which relates the cost of ϵ-Nash equilibria to any other flow (and in particular the optimal flow). This is the generalization to approximate equilibria of the inequality established by Beckmann, McGuire, and Winston [5] for exact Wardrop equilibria.

Theorem 1. *If f is an ϵ-Nash flow and f^* is any feasible flow of a non-atomic congestion game:*

$$\sum_{e \in E} l_e(f_e) f_e \leq (1 + \epsilon) \sum_{e \in E} l_e(f_e) f_e^*.$$

Proof. Let f be an ϵ-approximate Nash flow, and f^* be the optimum flow (or any other feasible flow). From the definition of approximate Nash equilibria (Inequality (1)), we get that for every path p with non-zero flow in f and any other path p':

$$\sum_{e \in p} l_e(f_e) \leq (1 + \epsilon) \sum_{e \in p'} l_e(f_e^*).$$

We sum these inequalities for all pairs of paths p and p' weighted with the amount of flow of f and f^* on these paths.

$$\sum_{p,p'} f_p f_{p'}^* \sum_{e \in p} l_e(f_e) \leq (1 + \epsilon) \sum_{p,p'} f_p f_{p'}^* \sum_{e \in p'} l_e(f_e^*)$$

$$\sum_{p'} f_{p'}^* \sum_{e \in E} l_e(f_e) f_e \leq (1 + \epsilon) \sum_{p} f_p \sum_{e \in E} l_e(f_e^*) f_e^*$$

$$(\sum_{p'} f_{p'}^*) \sum_{e \in E} l_e(f_e) f_e \leq (1 + \epsilon) (\sum_{p} f_p) \sum_{e \in E} l_e(f_e^*) f_e^*$$

But $\sum_p f_p = \sum_{p'} f^*_{p'}$ is equal to the total rate for the feasible flows f and f^*. Simplifying, we get

$$\sum_{e\in E} l_e(f_e) f_e \leq (1+\epsilon) \sum_{e\in E} l_e(f_e) f^*_e.$$

$\square$

To bound the PoA, we will need the following lemma:

Lemma 1. *Let $g(x)$ be a polynomial with nonnegative coefficients of degree p. The inequality*

$$g(x)\, y \leq \alpha\, g(x)\, x + \beta\, g(y)\, y$$

holds for the following set of parameters:

$\alpha = \dfrac{p}{(p+1)(1+\epsilon)}$	$\beta = \dfrac{(1+\epsilon)^p}{p+1}$	*for every* $\epsilon \geq (1+p)^{1/p} - 1$
$\alpha = p/(p+1)^{1+1/p}$	$\beta = 1$	*for every* $\epsilon \geq 0$

Proof. We first consider the case of $\epsilon \geq (1+p)^{1/p} - 1$ and then we show how to adapt the proof for small values of ϵ.

To establish the lemma for every polynomial with nonnegative coefficients of degree p, it is sufficient to prove it for every one of its monomials. For $y = 0$ the lemma holds trivially. For $y > 0$, we divide by y^{k+1} and by letting $z = x/y$, it is sufficient to establish that $\alpha\, z^{k+1} - z^k + \beta \geq 0$ for every positive z and every $k \in [0, p]$.

The minimum value of $\alpha\, z^{k+1} - z^k + \beta$ is achieved when $z = k/((k+1)\alpha)$. This minimum value is

$$\beta - \frac{k^k}{(k+1)^{k+1}\, \alpha^k}.$$

Now, it is not hard to establish that, for every $\gamma \in (0,1)$ the expression $\frac{k^k}{(k+1)^{k+1}\gamma^k}$ as a function of k is convex. Since by its definition $\alpha < 1$, we get that the above expression achieves its maximum at the extreme points ($k = 0$ or $k = p$).

We now establish that the value of the expression is indeed nonnegative for the extreme values of k. For $k = p$, the minimum is achieved when $z = p/((p+1)\alpha) = 1+\epsilon$ and this minimum is 0. We also establish it for the other extreme value, $k = 0$: it is clearly true for every $\epsilon \geq (1+p)^{1/p} - 1$ (this is where we need to bound ϵ from below).

For small values of ϵ and $k = 0$, the above value of β does not satisfy the inequality. It is however sufficient to have $\beta = 1$. From the other extreme $k = p$, we need to have $\beta - \frac{p^p}{(p+1)^{p+1}\alpha^p} \geq 0$, which is satisfied for $\alpha = p/(p+1)^{1+1/p}$. $\square$

We have now what we need to establish the upper bound of the main theorem of this section. The matching lower bound are shown in the following two lemmas. Since the lower bound is based on network non-atomic games, we get that the theorem applies also to the special class of network non-atomic congestion games.

Theorem 2. *The PoA of non-atomic congestion games with latency functions polynomials of degree p with nonnegative coefficients is $(1+\epsilon)^{p+1}$, for every $\epsilon \geq (p+1)^{1/p} - 1$, and $(1/(1+\epsilon) - p/(p+1)^{1+1/p})^{-1}$, for every $\epsilon \leq (p+1)^{1/p} - 1$.*

Proof. The proof is based entirely on the inequality of Theorem 1:

$$\sum_{e\in E} l_e(f_e) f_e \leq (1+\epsilon) \sum_{e\in E} l_e(f_e) f^*_e.$$

We can bound the right hand side using Lemma 1, as follows

$$\sum_{e \in E} l_e(f_e) f_e^* \leq \sum_{e \in E} \alpha l_e(f_e) f_e + \beta l_e(f_e^*) f_e^*.$$

It follows that for large ϵ (i.e. $\epsilon \geq (p+1)^{1/p} - 1$), the PoA is at most

$$\frac{(1+\epsilon)\beta}{1-(1+\epsilon)\alpha} = (1+\epsilon)^{p+1},$$

and for small ϵ (i.e. $\epsilon \leq (p+1)^{1/p} - 1$), it is at most

$$\frac{(1+\epsilon)\beta}{1-(1+\epsilon)\alpha} = (1/(1+\epsilon) - p/(p+1)^{1+1/p})^{-1}.$$

The following two lemmas establish that these upper bounds are tight. □

Lemma 2 (Non-Atomic-PoA-Lower-Bound for $\epsilon \leq (p+1)^{1/p} - 1$)**.** *There are instances of non-atomic congestion games with polynomial latencies of degree p, with approximate PoA of at least* $(1/(1+\epsilon) - p/(p+1)^{1+1/p})^{-1}$*, for every* $\epsilon \leq (p+1)^{1/p} - 1$.

Proof. We construct an instance with $m+k$ commodities, each of them with unit flow, and $2(m+k)$ facilities. We choose m, k so that m/k gives an arbitrarily good approximation of $(p+1)^{1/p}$. There are two types of facilities (see Figure 2):

- $m+k$ facilities α_i, $i = 1, \ldots, m+k$, with latency $l_{\alpha_i}(x) = x^p$ and
- $m+k$ facilities β_i, $i = 1, \ldots, m+k$, with constant latency $l_{\beta_i}(x) = \gamma = m^{p+1}/(1+\epsilon) - km^p$.

For each commodity i there are two pure strategies, i.e.

$$A_i = \{\alpha_i, \ldots, \alpha_{i+m}\} \text{ and } P_i = \{\alpha_{i+m+1}, \ldots, \alpha_{i+m+k+1}\} \cup \{\beta_i\},$$

where the indices are taken cyclically (i.e., mod $m+k$).

First we prove that playing the first strategy $A = (A_1, \ldots, A_{m+k})$ is a ϵ-Nash equilibrium. The cost experienced by flow in commodity i is $c_i(A) = m^{p+1}$, as each commodity uses m facilities of type α with latency m^p each. The latency of strategy P_i is

$$c_i(P_i, A_{-i}) = km^p + \gamma = \frac{c_i(A)}{1+\epsilon}$$

and therefore A is an ϵ-Nash equilibrium.

In the optimal case, the flow uses strategy profile P, where commodity i has cost $c_i(P) = k^{p+1} + \gamma$ and so the PoA is

$$\frac{c_i(A)}{c_i(P)} = \frac{m^{p+1}}{k^{p+1} + \gamma} = (1/(1+\epsilon) - p/(p+1)^{1+1/p})^{-1}.$$

□

Lemma 3 (Non-Atomic-PoA-Lower-Bound for $\epsilon \geq (p+1)^{1/p} - 1$)**.** *There are instances of non-atomic congestion games with polynomial latencies of degree p, with approximate PoA of at least* $(1+\epsilon)^{p+1}$*, for every* $\epsilon \geq (p+1)^{1/p} - 1$.

Proof. Consider an undirected cycle of $m+k$ vertices $C = (v_1, \ldots, v_{m+k})$ with edge latency $l_e(f_e) = f_e^p$ on every edge e (Figure 4). We choose m, k so that m/k gives an arbitrarily good approximation of $1+\epsilon$. There are $m+k$ unit-demand commodities each one with source $s_i = v_i$ and destination $t_i = v_{i+m+1}$ (again indices are taken cyclically). Clearly, for each commodity i there are exactly two simple paths that connect s_i to t_i; i.e. the clockwise path using m edges and the counterclockwise path using k edges. If every commodity routes the traffic via the clockwise path, then the load on every edge is m. The latency experienced by players of each commodity is m^{p+1}–each commodity uses m edges each incurring cost m^p. The alternative path has length k and so the latency on that path is km^{p+1}, so this is an m/k-approximate flow. The optimal flow routes all the traffic counterclockwise having latency k^{p+1} per commodity; every edge has load k and every commodity uses k edges. This gives an approximate PoA of $(m/k)^{p+1} \approx (1+\epsilon)^{p+1}$. □

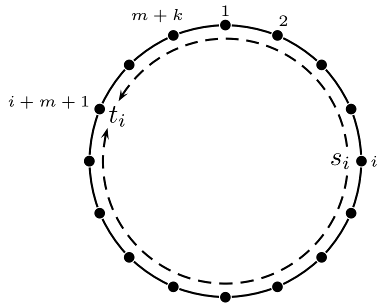

Figure 4: The lower bound for non-atomic linear latencies and large ϵ.

4 Non-atomic PoS

In this section we give tight upper and lower bounds for the non-atomic congestion games with polynomial latencies. We start with a theorem which characterizes the ϵ-Nash (or ϵ-Wardrop) equilibria for every non-atomic congestion game with arbitrary (not necessarily polynomial) latency function. It involves a potential-like function and generalizes a well-known characterization of exact Nash (Wardrop) equilibria [27].

Theorem 3. *In an non-atomic congestion game with latency functions $l_e(f_e)$, let $\phi_e(f_e)$ be any integrable functions which satisfy*

$$\frac{l_e(f_e)}{(1+\epsilon)} \leq \phi_e(f_e) \leq l_e(f_e), \tag{2}$$

for every $f_e \geq 0$ and define $\Phi_e(f_e) = \int_0^{f_e} \phi_e(t)\, dt$. For a flow f, define $\Phi(f) = \sum_{e \in E} \Phi_e(f_e)$. If a flow f minimizes the potential function $\Phi(f)$, it is an ϵ-Nash equilibrium.

Furthermore, when the latency functions are nondecreasing, for any other flow f':

$$\sum_{e \in E} \phi_e(f_e) f_e \leq \sum_{e \in E} \phi_e(f_e) f'_e$$

Proof. Consider a flow f and two paths p and p' with common endpoints. Suppose that the flow f on path p is positive. We want to compute how much $\Phi(f)$ changes when we shift a small amount $\delta > 0$ of flow from path p to path p'. More precisely, if f' denotes the new flow, we compute the following limit

$$\lim_{\delta \to 0} \frac{\Phi(f') - \Phi(f)}{\delta} = \sum_{e \in p'} \phi_e(f_e) - \sum_{e \in p} \phi_e(f_e) \tag{3}$$

When f minimizes Φ, then the above quantity is nonnegative. But then we can bound the cost of paths p and p' with the two terms of this quantity as follows:

$$l_p(f) = \sum_{e \in p} l_e(f_e) \leq (1+\epsilon) \sum_{e \in p} \phi_e(f_e) \leq (1+\epsilon) \sum_{e \in p'} \phi_e(f_e) \leq (1+\epsilon) \sum_{e \in p'} l_e(f_e) = (1+\epsilon) l_{p'}(f),$$

which implies that f is an ϵ-Nash equilibrium.

For the second part, we observe that the expression (3), which is nonnegative when f minimizes Φ, implies that for every path p on which f is positive and every other path p' we must have

$$\sum_{e \in p} \phi_e(f_e) \leq \sum_{e \in p'} \phi_e(f_e).$$

Consider now another flow f' which satisfies the rate constraints for the commodities and let us sum the above inequalities for all paths p and p' weighted with the amount of flow in f and f'.

More precisely:

$$\sum_{p,p'} f_p f'_{p'} \sum_{e\in p} \phi_e(f_e) \le \sum_{p,p'} f_p f'_{p'} \sum_{e\in p'} \phi_e(f_e)$$
$$\sum_{p'} f'_{p'} \sum_{e\in E} \phi_e(f_e) f_e \le \sum_{p} f_p \sum_{e\in E} \phi_e(f_e) f'_e.$$

But $\sum_{p'} f'_{p'} = \sum_p f_p$ is equal to the sum of the rates. If we remove from the expression this common factor, the second part of the theorem follows. □

We now study the PoS of polynomial latency functions of the form $\ell_e(f) = \sum_{k=0}^{p} a_{e,k} f_e^k$. For these latency functions we define the potential function which has derivatives $\phi_e(f_e) = \sum_{k=0}^{p} \zeta_k a_{e,k} f_e^k$, for some ζ_k to be determined later and which satisfy:

$$\frac{1}{1+\epsilon} \le \zeta_k \le 1 \tag{4}$$

Theorem 4. *The PoS for non-atomic congestion games with polynomial latencies of degree p is exactly*

$$\left((1+\epsilon)\left(1-\frac{p}{p+1}\left(\frac{1+\epsilon}{p+1}\right)^{1/p}\right)\right)^{-1},$$

for $\epsilon < p$, and it drops to 1 for $\epsilon \ge p$.

Proof. Our starting point is the second part of Theorem 3. Let f be a flow which minimizes the potential and let $\hat{f}$ be an optimal flow. We have that

$$\sum_{e\in E}\sum_{k=0}^{p} \zeta_k a_{e,k} f_e^{k+1} \le \sum_{e\in E}\sum_{k=0}^{p} \zeta_k a_{e,k} f_e^k \hat{f}_e.$$

We can bound from above the monomials $f_e^k \hat{f}_e$ as follows:

$$f_e^k \hat{f}_e \le \alpha_k f_e^{k+1} + \beta_k \hat{f}_e^{k+1},$$

where[1]

$$\alpha_k^k \beta_k = k^k/(k+1)^{k+1} \tag{5}$$

(and in particular $\alpha_0 = 0$ and $\beta_0 = 1$). We rearrange the terms to get:

$$\sum_{e\in E}\sum_{k=0}^{p} \zeta_k(1-\alpha_k) a_{e,k} f_e^{k+1} \le \sum_{e\in E}\sum_{k=0}^{p} \zeta_k \beta_k a_{e,k} \hat{f}_e^{k+1}.$$

The problem now is to select the parameters α_k, β_k, ζ_k to bound all the coefficients on the left side from below and all the coefficients on the right side from above:

$$\zeta_k\,(1-\alpha_k) \ge 1-\alpha_p \qquad\qquad \zeta_k\,\beta_k \le \beta_p \tag{6}$$

With these, the above inequality becomes

$$(1-\alpha_p)\sum_{e\in E}\sum_{k=0}^{p} a_{e,k} f_e^{k+1} \le \beta_p \sum_{e\in E}\sum_{k=0}^{p} a_{e,k} \hat{f}_e^{k+1},$$

[1] We use the following fact: for every nonnegative x, y, k, and for every positive α, β which satisfy $\alpha^k\beta = k^k/(k+1)^{k+1}$: $x^k y \le \alpha x^{k+1} + \beta y^{k+1}$. Proof: Let $z = x/y$. We need to have $\alpha z^{k+1} - z^k + \beta \ge 0$. Taking the derivative, we see that the expression is minimized when $z = k/((k+1)\alpha)$ and the minimum value is $\beta - k^k/((k+1)^{k+1}\alpha^k)$, which is 0 for our choice of α and β.

which will immediately bound the PoS from above by $\beta_p/(1-\alpha_p)$.

There is a complication in selecting the parameters α_k, β_k, ζ_k as functions of ϵ, in that we need to have different choices for small and large values of ϵ:

For $k \leq \epsilon$:	$\zeta_k = (k+1)/(1+\epsilon)$	$\beta_k = 1/(k+1)$
For $k > \epsilon$:	$\zeta_k = 1$	$\beta_k = 1/(1+\epsilon)$

The value of α_k is determined by Equation (5)[2]. We need to show that these values satisfy properties (4) and (6). We see immediately that property (4) is satisfied. For property (6), we need to do more work. We first observe that the second part of property (6) is satisfied always with equality. For the first part we get:

Case $\epsilon < k$: We first establish that $\alpha_k \leq k/(k+1)$. Indeed, from $\alpha_k^k \beta_k = k^k/(k+1)^{k+1}$, we get $\alpha_k^k = (1+\epsilon)\, k^k/(k+1)^{k+1} \leq (1+k)\, k^k/(k+1)^{k+1} = k^k/(k+1)^k$. We will also use the fact that the function $x^p/(x+1)^{p+1}$ is increasing in x when $x \leq p$.

We then have $\alpha_p^p = (1+\epsilon)\, p^p/(p+1)^{p+1} \geq (1+\epsilon) k^p/(k+1)^{p+1} = (1+\epsilon) k^k/(k+1)^{k+1} k^{p-k}/(k+1)^{p-k} \geq \alpha_k^k \alpha_k^{p-k} = \alpha_k^p$. Therefore, $\alpha_p \geq \alpha_k$, from which the first part of property (6) follows immediately.

Case $\epsilon \geq k$: In this case $\alpha_k = k/(k+1)$ and we need to show that $\zeta_k(1-\alpha_k) \geq 1-\alpha_p$ which is equivalent to $\alpha_p \geq \epsilon/(1+\epsilon)$. Since $\alpha_p^p = (1+\epsilon)\, p^p/(p+1)^{p+1}$, we use again the fact that the function $x^p/(x+1)^{p+1}$ is increasing in x when $x \leq p$ to get that $\alpha_p^p \geq (1+\epsilon)\, \epsilon^p/(1+\epsilon)^{1+p} \geq \epsilon^p/(1+\epsilon)^p$. The claim follows. Notice also that this analysis holds also for the special case $k=0$.

Therefore, the PoS is at most $\beta_p/(1-\alpha_p)$, where $\beta_p = 1/(1+\epsilon)$ and $\alpha_p^p \beta = p^p/(p+1)^{p+1}$. This establishes the upper bound of the theorem.

To establish the lower bound, we use the Pigou network with latencies $1+\epsilon$ and f_e^p and flow rate equal to 1. There is a unique ϵ-Nash equilibrium which uses the second edge with total latency 1. On the other hand, the optimal solution is to send flow x to the second edge and $1-x$ in the first edge and optimize for x. The social cost is $x^{p+1} + (1+\epsilon)(1-x)$. If we minimize this function, we match the upper bound. □

The fact that the PoS drops to 1 when $\epsilon = p$ was first shown in [30] and [14].

5 Conclusions

We considered the PoA and the PoS of approximate Nash equilibria for congestion games (atomic, selfish routing, and non-atomic) with polynomial latencies. We have used a unifying approach and obtained tight upper and lower bounds for all cases except for the PoS for atomic congestion games. This remains a challenging open problem even for the case of exact equilibria ($\epsilon = 0$).

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$$\zeta_k = (q+1)/(1+\epsilon) \qquad\qquad \beta_k = 1/(q+1),$$

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Appendix

6 Atomic PoA

In this section we study the approximate PoA for the case of atomic congestion games. For large ϵ the PoA is roughly $(1+\epsilon)^{p+1}$.

The following lemma was implicitly used in Aland et al. [1]:

Lemma 4. *Let $g(x)$ be any polynomial of degree p with nonnegative coefficients. For every $\alpha, \beta, z \in \mathbb{N}$:*

$$\beta \cdot g(\alpha+1) \leq \kappa \cdot \alpha \cdot g(\alpha) + \lambda \cdot \beta \cdot g(\beta) \frac{z^2+3z+1}{2z+1} \beta^2,$$

where

$$\kappa = \frac{(z+2)^p - (z+1)^p}{(z+1)^{p+1} - z^{p+1}}, \qquad \lambda = \frac{(z+1)^{2\,p+1} - (z+2)^p\, z^{p+1}}{(z+1)^{p+1} - z^{p+1}}$$

Theorem 5 (Atomic-PoA)**.** *For any positive real ϵ, the approximate PoA of general congestion games with linear latencies is*

$$\frac{(1+\epsilon)\left((z+1)^{2p+1} - z^{p+1}(z+2)^p\right)}{(z+1)^{p+1} - (1+\epsilon)(z+2)^p + (1+\epsilon)(z+1)^p - z^{p+1}},$$

where $z \in \mathbb{N}$ is the maximum integer with $\frac{z^{p+1}}{(z+1)^p} \leq 1+\epsilon$.

Proof. Let $A = (A_1, \ldots, A_n)$ be an ϵ-Nash, and $P = (P_1, \ldots, P_n)$ be the optimum allocation. From the definition of ϵ-equilibria (Inequality (1)) we get

$$\sum_{e \in A_i} l_e\left(n_e(A)\right) \leq (1+\epsilon) \sum_{e \in P_i} l_e\left(n_e(A)+1\right).$$

If we sum up for every player i and use Lemma 4, we get

$$\begin{aligned}
\mathrm{C}(A) &= \sum_{i \in N} c_i(A) \\
&= \sum_{i \in N} \sum_{e \in A_i} l_e(n_e(A)) \\
&= \sum_{e \in E} n_e(A) l_e(n_e(A)) \\
&\leq (1+\epsilon) \sum_{e \in E} n_e(P) l_e\left(n_e(A)+1\right) \\
&\leq (1+\epsilon) \cdot \kappa \cdot \sum_{e \in E} n_e(A) l_e(n_e(A)) + (1+\epsilon) \cdot \lambda \cdot \sum_{e \in E} n_e(P) l_e(n_e(P)) \\
&= (1+\epsilon) \cdot \kappa \cdot \mathrm{C}(A) + (1+\epsilon) \cdot \lambda \cdot \textsc{opt},
\end{aligned}$$

where the last inequality follows from Lemma 4. The upper bound follows, by reordering and substituting κ and λ. We present the lower bound in the next lemma. $\square$

Lemma 5 (Atomic-PoA-Lower-Bound)**.** *For any real positive ϵ, there are instances of congestion games with polynomial latencies of degree p, for which the approximate PoA is at least equal to the expression of Theorem 5*

Proof. Let $z \in \mathbb{N}$ be the maximum integer with $\frac{z^{p+1}}{(z+1)^p} \leq 1+\epsilon$. We construct an instance with $z+2$ players and $2z+4$ facilities. There are two types of facilities:

- $z+2$ facilities of type α, with latency $l_e(x) = x^p$ and
- $z+2$ facilities of type β with latency $l_e(x) = \gamma x^p$, with

$$\gamma = \begin{cases} \frac{(z+1)^{p+1} - (1+\epsilon)(z+2)^p}{(1+\epsilon)(z+1)^p - z^{p+1}}, & \text{when } \frac{z^{p+1}}{(z+1)^p} < 1+\epsilon \\ 0, & \text{otherwise} \end{cases}$$

Player i has two alternative pure strategies, i.e.

$$A_i = \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{z+2}\} \cup \{\beta_{i+1}, \ldots, \beta_{i+1+z}\} \text{ and } P_i = \{\alpha_i, \beta_i\},$$

where the indices are taken cyclically (mod $z+2$).

First we prove that playing the first strategy $A = (A_1, \ldots, A_n)$ is a ϵ-Nash equilibrium. The cost of player i is

$$c_i(A) = (z+1)^{p+1} + \gamma z^{p+1},$$

as there are exactly $z+1$ players using facilities of type α and exactly z players using facilities of type β. If player i unilaterally switches to P_i he experiences cost

$$c_i(P_i, A_{-i}) = (z+2)^p + \gamma(z+1)^p = \frac{c_i(A)}{1+\epsilon},$$

which shows that A is an ϵ-Nash equilibrium.

The optimum allocation is the strategy profile P, where every player has cost $c_i(P) = 1+\gamma$ and so the PoA is

$$\frac{c_i(A)}{c_i(P)} = \frac{(z+1)^{p+1} + \gamma z^{p+1}}{1+\gamma} = \frac{(1+\epsilon)\left((z+1)^{2p+1} - z^{p+1}(z+2)^p\right)}{(z+1)^{p+1} - (1+\epsilon)(z+2)^p + (1+\epsilon)(z+1)^p - z^{p+1}}.$$

□

7 Atomic PoS

The PoS for atomic congestion games is by far the more difficult quantity that we treat in this work. Even the case of exact Nash equilibria is still open for polynomial latency functions. And the situation becomes even more complicated when we consider approximate equilibria. There are two sources of difficulty: one is to generalize appropriately the potential function which is essential for proving upper bounds on the PoS. The second difficulty, is that the technical depth of handling the situation increases considerably.

In general, the PoS is in many cases harder than the PoA. For example, there are still open problems from the first paper to study the PoS [3] concerning some important cases (for example undirected networks [18, 2]).

We have a characterization theorem which generalizes the notion of potential function and parallels Theorem 3 for non-atomic games.

Theorem 6. *In an atomic congestion game with latency functions $l_e(k)$, let $\phi_e(k)$ be any functions which satisfy*

$$\frac{l_e(k)}{(1+\epsilon)} \leq \phi_e(k) \leq l_e(k), \tag{7}$$

for every $k \geq 1$ and define $\Phi_e(k) = \sum_{t=1}^{k} \phi_e(t)$. For a strategy profile A, define $\Phi(A) = \sum_{e \in E} \Phi_e(n_e(A))$. If a strategy profile A is a local minimum of Φ, (i.e., $\Phi(A) \leq \Phi(P_i, A_{-i})$ for every i and P_i), then A is an ϵ-Nash equilibrium.

Furthermore, when the latency functions are nondecreasing, for any other strategy profile P:

$$\sum_{e \in E} \phi_e(n_e(A))\, n_e(A) \leq \sum_{e \in E} \phi_e(n_e(A)+1)\, n_e(P). \tag{8}$$

Proof. To show that A is an ϵ-Nash equilibrium, we need to show $c_i(A) \leq (1+\epsilon)\, c_i(P_i, A_{-i})$, for every i and P_i. We have $c_i(A) = \sum_{e \in A_i} l_e(n_e(A))$ and $c_i(P_i, A_{-i}) = \sum_{e \in P_i} l_e(n_e(P_i, A_{-i}))$. Since A is a local minimum, we have $\Phi(A) \leq \Phi(P_i, A_{-i})$ which is equivalent to

$$\sum_{e \in A_i} \phi_e(n_e(A)) \leq \sum_{e \in P_i} \phi_e(n_e(P_i, A_{-i})). \tag{9}$$

We use this to compare $c_i(A)$ and $c_i(P_i, A_{-i})$:

$$c_i(A) = \sum_{e \in A_i} l_e(n_e(A)) \le (1+\epsilon) \sum_{e \in A_i} \phi_e(n_e(A)) \le (1+\epsilon) \sum_{e \in P_i} \phi_e(n_e(P_i, A_{-i})) \le (1+\epsilon)\, c_i(P_i, A_{-i}).$$

This shows that A is ϵ-Nash equilibrium. To show the useful inequality (8) of the theorem when l_e is nondecreasing, we use inequality (9). This inequality implies

$$\sum_{e \in A_i} \phi_e(n_e(A)) \le \sum_{e \in P_i} \phi_e(n_e(A)+1),$$

because $n_e(P_i, A_{-i}) \le n_e(A) + 1$. If we sum these inequalities for every i, we get the desired inequality (8). □

The above theorem gives us a way to bound the PoS. We treat here only the case $l_e(k) = k^p$. and we leave the generalization to polynomials with nonnegative coefficients for the final version of the paper. We will use the following:

$$\phi_e(k) = \frac{\epsilon \cdot (k-1) \cdot (k^p - (k-1)^p) + p \cdot k^p}{p \cdot (1+\epsilon)}. \tag{10}$$

This ϕ_e satisfies

$$\frac{k^p}{(1+\epsilon)} \le \phi_e(k) \le k^p,$$

as one can easily verify using the fact $k^p - (k-1)^p \le p\, k^{p-1}$. In fact, for $k = 1$, the left bound is tight, and in the limit, as n tends to ∞, the right bound becomes tight: $\phi_e(k)/k^p \to 1$.

To bound the PoS we use two inequalities to compare the cost of an approximate Nash equilibrium A to the cost of some other strategy profile P: the first is Inequality (8) and the second is the inequality which expresses the fact that A is a *global* minimum: $\Phi(A) \le \Phi(P)$, that is,

$$\sum_{e \in E} \Phi_e(n_e(A)) \le \sum_{e \in E} \Phi_e(n_e(P)), \tag{11}$$

where

$$\Phi_e(k) = \sum_{t=1}^{k} \phi_e(t). \tag{12}$$

We multiply Inequality (8) by nonnegative δ and (11) by nonnegative γ and we rearrange the terms to get

$$0 \le \sum_{e \in E} \left(\delta\, (\phi_e(n_e(A)+1)\, n_e(P) - \phi_e(n_e(A))\, n_e(A)) + \gamma\, (\Phi_e(n_e(P)) - \Phi_e(n_e(A)))\right).$$

Our aim is to find appropriate γ and δ and as small as possible ρ that satisfy

$$\delta\, (\phi_e(n_e(A)+1)\, n_e(P) - \phi_e(n_e(A))\, n_e(A)) + \gamma\, (\Phi_e(n_e(P)) - \Phi_e(n_e(A))) \le \\ \rho\, l_e(n_e(P))\, n_e(P) - l_e(n_e(A))\, n_e(A). \tag{13}$$

This would show that the PoS is at most ρ, since $l_e(n_e(A))\, n_e(A)$ and $l_e(n_e(P))\, n_e(P)$ are the social cost of the A and P (for edge e).

We can select any nonnegative γ and δ, but we have strong evidence that a good choice is to select $\delta = (\rho - 1)/(\phi(2) - \phi(1))$ and $\gamma = \rho\,(1+\epsilon) - \delta$ (notice their dependency on ρ). Both these values are nonnegative and we obtain the following theorem:

Theorem 7. *Let ϕ_e and Φ_e be as in Equations (10) and (12). Then the PoS for atomic congestion games with polynomial latencies of degree p is at most*

$$\rho = \max_{a,b\in N} \frac{\Phi(b) - \Phi(a) - (\phi(a+1)\,b - \phi(a)\,a) + (\phi(2) - \phi(1))\,a^{p+1}}{\Phi(b) - \Phi(a) - (\phi(a+1)\,b - \phi(a)\,a) + (\phi(2) - \phi(1))\;(b^{p+1} - (1+\epsilon)\,(\Phi(b) - \Phi(a)))}. \tag{14}$$

In particular, for linear latencies ($p = 1$), the PoS is at most

$$\frac{1+\sqrt{3}}{\epsilon+\sqrt{3}}.$$

Proof. The general bound follows from the above reasoning by solving (13) for ρ and plugging in the values for γ and δ. An important technical point is that with these choices of γ and δ, the denominator of the above expression is non-negative for every a and b (otherwise, we could not solve Inequality (13) for ρ).

For the case of $p = 1$, we have $\phi(k) = k - \epsilon/(1+\epsilon)$, and $\Phi(k) = k(k+1)/2 - k\,\epsilon/(1+\epsilon)$. The bound becomes

$$\rho = \max_{a,b\in N} \frac{b^2 - 2ba + 3a^2 - b - a}{(2-\epsilon)b^2 - 2ba + (2+\epsilon)a^2 - (2-\epsilon)b - \epsilon a}.$$

We can rewrite this as $\rho = \max_{a,b\in N} \frac{A+B}{A\epsilon+B}$, where $A = a^2 - b^2 - a + b$ and $B = 2(a^2 - ab + b^2 - b)$. To show the bound, it suffices to show that $A/B \leq 1/\sqrt{3}$, for all nonnegative integers a and b; taking into account that B is nonnegative, this is equivalent to $B - A\sqrt{3} \geq 0$. Remarkably, this does not involve the parameter ϵ at all. To show the inequality, we find that

$$B - A\sqrt{3} = \frac{1}{2}\left(\sqrt{3}+2\right)\left(\left(b + \sqrt{3}a - 2\,a - \frac{1}{2}\right)^2 + (3\sqrt{3}-5)\,a - \frac{1}{4}\right).$$

This is clearly positive for $a \geq 2$ because $(3\sqrt{3}-5)\,2 \geq 1/4$. For $a = 0$, we have

$$B - A\sqrt{3} = \frac{1}{2}\left(\sqrt{3}+2\right) b\,(b-1)\,,$$

which is clearly nonnegative for integer b. Finally, for $a = 1$ we have

$$B - A\sqrt{3} = \frac{1}{2}\left(\sqrt{3}+2\right)(b-1)\left(b - 4 + 2\sqrt{3}\right),$$

which is also clearly nonnegative. □

We don't have a closed form of the general bound of the previous theorem. We can use it however to establish that the PoS drops to 1 exactly when ϵ becomes equal to the degree of the polynomial, i.e., $\epsilon = p$.

Theorem 8. *The PoS for atomic congestion games with polynomial latencies of degree p is 1 when $\epsilon = p$. In other words, the strategy with optimal social cost is a p-Nash equilibrium. This is tight: for every $q < p$, there are congestion games for which the optimum is not a q-Nash equilibrium.*

We simply observe that for $\epsilon = p$, $\Phi(a) = a^{p+1}/(p+1)$ by which expression (14) turns out to be equal to 1 (for all a and b). The fact that the PoS for $\epsilon < p$ is higher than 1 follows from Theorem 5 and the following general proposition.

Proposition 1. *The approximate PoS for the class of atomic congestion games with polynomial latencies is greater or equal to the approximate PoS of non-atomic congestion games.*

To see that the proposition holds, consider a non-atomic game with latencies $l_e(f_e)$. We can turn it into an atomic game with n players, where n is very high, and latencies $l'_e(k) = l_e(k/n)$. Any ϵ-Nash equilibrium for the non-atomic game is an ϵ-Nash equilibrium for the atomic game. This in general is not sufficient to establish the proposition because the opposite may not hold. It suffices however to apply it to the Pigou construction at the end of the proof of Theorem 4, which gives the optimal value for the PoS of non-atomic games. The special structure of the Pigou construction guarantees that the resulting atomic game has trivially a unique ϵ-Nash equilibrium.

The discussion before Theorem 7 is the rough outline of all upper-bound proofs in this work. For the PoS of non-atomic congestion games, (Theorem 4), the situation is much simpler for two reasons: We don't need to use the global minimum inequality (i.e., we take $\gamma = 0$), and the parameters $n_e(A)$ and $n_e(P)$ are nonnegative reals, not integers; the hardest part was to find appropriate generalization of the potential. For the PoA of non-atomic (Theorem 2) and atomic (Theorem 5) games, the situation is substantially simpler, because we don't need a potential (or its generalization) and we have only one inequality. Among all 4 bounds, the atomic PoS is the only one which involves the combination of two types of inequalities (local and global constraints for the potential).

7.1 Lower bound for linear latencies

Theorem 4 and Proposition 1 show that the Pigou game gives a lower bound for non-atomic and atomic congestion games. While for the class of the non-atomic games this is tight, it is not tight for the class of atomic games. We construct here a game, which is much more complicated, and has PoS substantially higher than the Pigou game for linear latencies. The lower bound of this construction still does not match the upper bound of Theorem 7, except for the extreme values of $\epsilon = 0$ and $\epsilon = 1$. For $\epsilon = 0$, we get that the PoS is $1 + 1/\sqrt{3} \approx 1.57$, which matches the PoS of exact equilibria [10, 6]. It then decreases as a function of ϵ, and drops to 1 for $\epsilon = 1$.

Theorem 9 (Atomic-PoS-Lower-Bound-Linear)**.** *There are linear congestion games whose ϵ-Nash equilibria have PoS approaching*

$$2\,\frac{3+\epsilon+3\,\epsilon^3+2\,\epsilon^4+\left(1+\epsilon+\epsilon^2\right)\sqrt{2\,\epsilon^4+3\,\epsilon^3+\epsilon+3}}{6+2\,\epsilon+6\,\epsilon^3+4\,\epsilon^4+\epsilon\left(5+2\,\epsilon-\epsilon^2\right)\sqrt{2\,\epsilon^4+3\,\epsilon^3+\epsilon+3}}.$$

For small ϵ, this is approximately $1.5774 - 1.7955\epsilon$.

Proof. We construct a game, where the ratio of its unique *(and therefore best)* ϵ-Nash equilibrium, over the optimum profile, is high. There are $n = n_1 + n_2$ players divided into two sets G_1 and G_2, with $|G_1| = n_1$ and $|G_2| = n_2$. Each player i in G_1 has two strategies A_i and P_i. The players in G_2 have a unique strategy D. The strategy profile $A = (A_1, \ldots, A_{n_1}, D, \ldots, D)$ will be the unique ϵ-Nash equilibrium and $P = (P_1, \ldots, P_{n_1}, D, \ldots, D)$ will be the profile with optimal social cost.

There are 3 types of facilities:

- n_1 facilities α_i, $i = 1, \ldots, n_1$, each with cost function $l_{\alpha_i}(x) = \alpha x$. Facility α_i belongs only to strategy P_i.
- $n(n-1)$ facilities β_{ij}, $i, j = 1, \ldots, n_1$ with $i \neq j$, each with cost $l_{\beta_{ij}}(x) = \beta x$. Facility β_{ij} belongs only to strategies A_i and P_j.
- 1 facility γ with cost $l_\gamma(x) = x$. Facility γ belongs to A_i for $i = 1, \ldots, n_1$ and to D.

We first compute the cost of each player in G_1, for every possible strategy profile with k players playing strategies A_i and $n_1 - k$ players playing strategies P_i. By symmetry, we need only to consider the cost $cost_A(k)$ of player 1 and the cost $cost_P(k)$ of player n_1 of the strategy profile $S_k = (A_1, \ldots, A_k, P_{k+1}, \ldots, P_{n_1}, D, \ldots, D)$. Therefore,

$$cost_A(k) \quad = \quad (2n_1 - k - 1)\beta + (n_2 + k).$$

Similarly, we compute

$$cost_P(k) \quad = \quad \alpha + (n_1 + k - 1)\beta.$$

We now want to select the parameters α and β so that the strategy profile $A = S_{n_1}$ is the unique ϵ-Nash equilibrium of the game. In particular, we require that at every strategy profile S_k, player i, $i = 1, \ldots, k$, has no reason to switch to strategy P_i, because it is $(1+\epsilon)$ times more costly. This is expressed by the constraints

$$(1 + \epsilon) \cdot cost_A(k) \leq cost_P(k - 1), \quad \text{for every } k = 1, \ldots, n_1. \tag{15}$$

All these constraints are linear in k and they are satisfied by equality when

$$\alpha = (n_1 - 1)\frac{(1 + \epsilon)(1 + 2\epsilon)}{2 + \epsilon} + n_2 + 1 \qquad\qquad \beta = \frac{1 + \epsilon}{2 + \epsilon}$$

In summary, for the above values of the parameters α and β, we obtain the desired property that the strategy profile A is a ϵ-dominant equilibrium (in the strong sense where each player has a unique optimal strategy which is a factor ϵ factor better than any other). If we increase α by any small positive δ, inequality (15) becomes strict and the ϵ-dominant equilibrium is unique (and therefore unique ϵ-Nash equilibrium).

We now want to select the value of the parameter so that the ratio of the cost of A over the optimum is as high as possible. The PoS is

$$\frac{n_1 cost_A(n_1) + n_2(n_1 + n_2)}{n_1 cost_P(0) + n_2^2},$$

which gives the bound of the theorem when we take the limit for n_1 and n_2 and optimize their ratio. $\square$