Red-Black Trees

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Red-Black Trees

- AVL trees and (2,4) trees have very nice properties, but:
  - AVL trees might need **many rotations** after a removal
  - (2,4) trees might require **many split or fusion operations** after an update (insertion or deletion).
- **Red-black trees** are a data structure which requires only $\boldsymbol{O(1)}$ **structural changes** after an update in order to remain balanced.

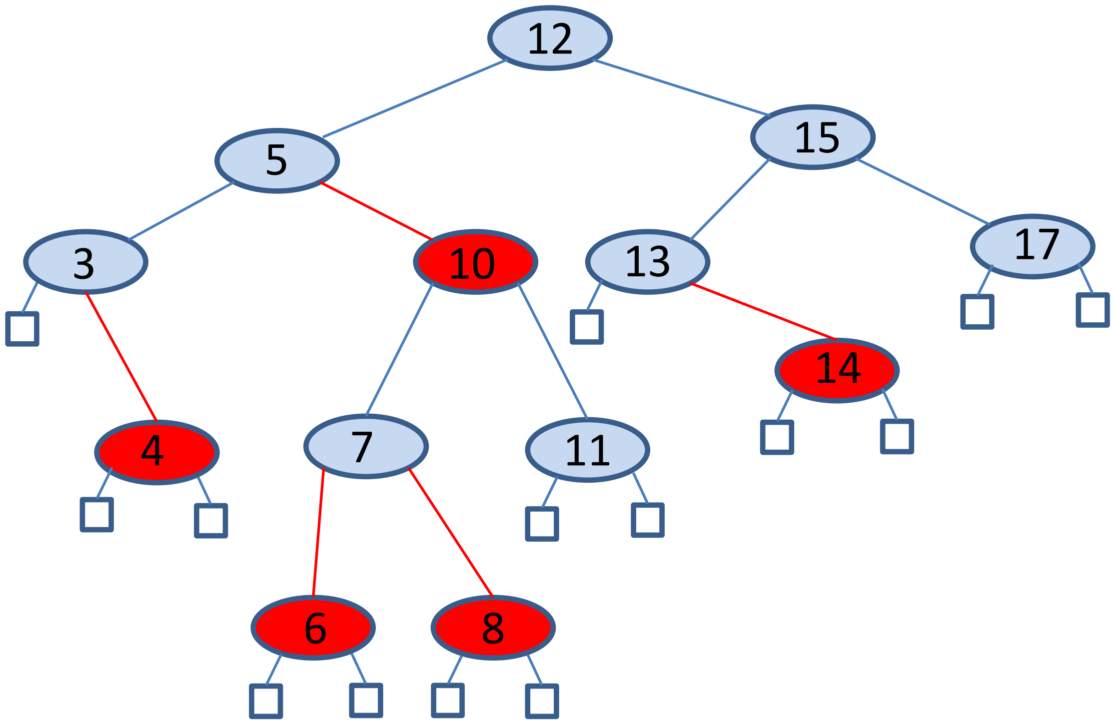
Definition

- A **red-black tree** is a binary search tree with nodes colored red and black in a way that satisfies the following properties:
  - **Root Property**: The root is black.
  - **External Property**: Every external node is black.
  - **Internal Property**: The children of a red node are black.
  - **Depth Property**: All the external nodes have the same **black depth**, defined as the number of black ancestors minus one (recall that a node is an ancestor of itself).

Definition (cont’d)

- Red-black trees will be used for implementing maps so **they will not be allowed to have duplicate keys**.

Example Red-Black Tree

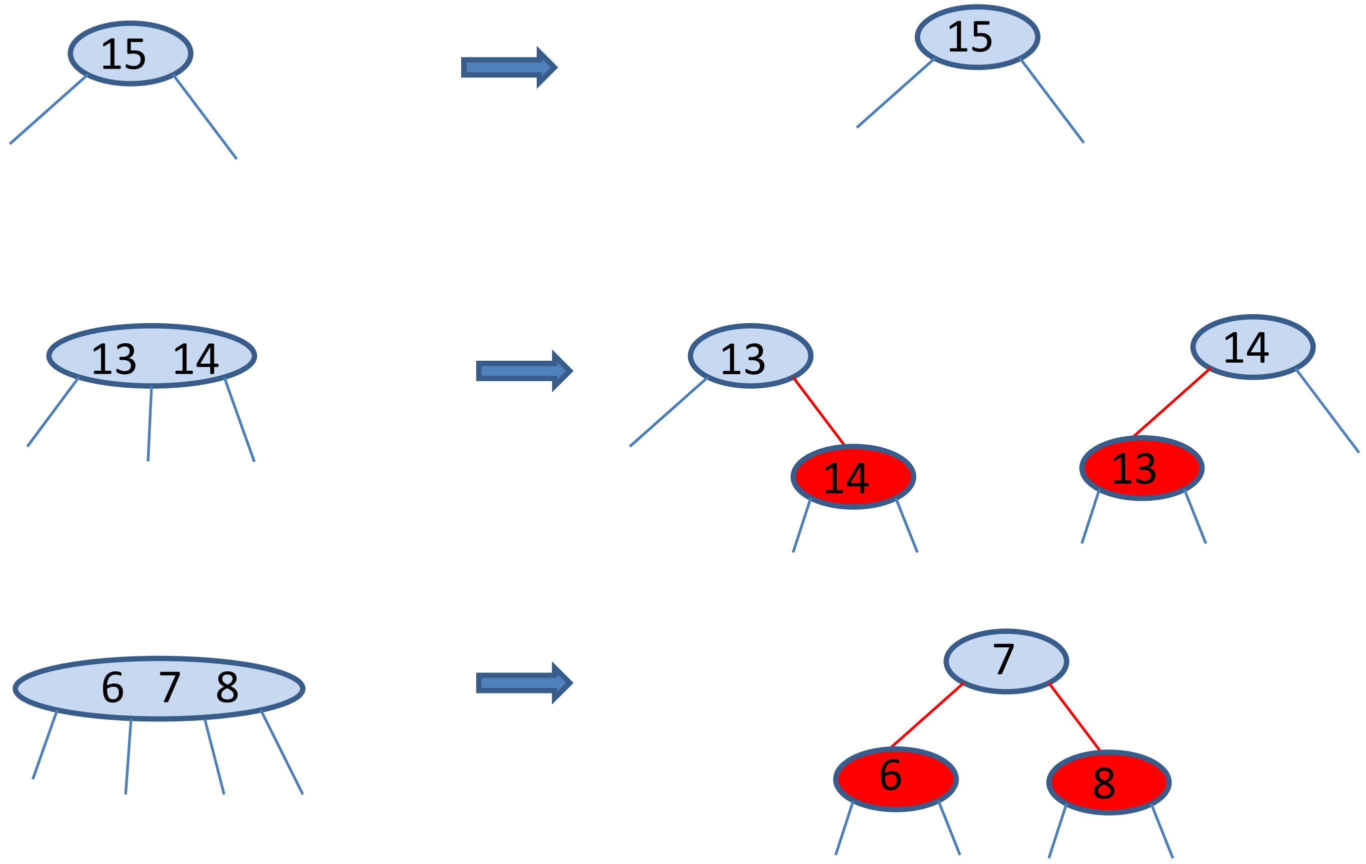


In our figures, we use **light blue color instead of black**.

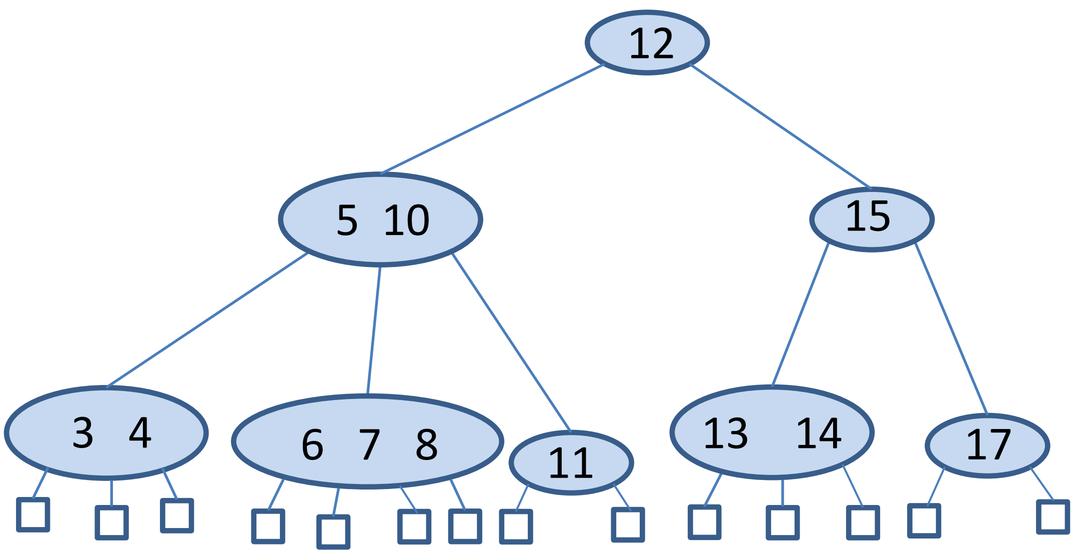
From (2,4) Trees to Red-Black Trees

- Given a (2,4) tree, we can **transform it into a red-black tree** by performing the following transformations for each internal node v:
  - If v is a 2-node, then keep the (black) children of v as is.
  - If v is a 3-node, then create a new red node w, give v's first two (black) children to w, and make w and v's third child be the two children of v (the symmetric operation is also possible; see next slide).
  - If v is a 4-node, then create two new red nodes w and z, give v's first two (black) children to w, give v's last two (black) children to z, and make w and z be the two children of v.

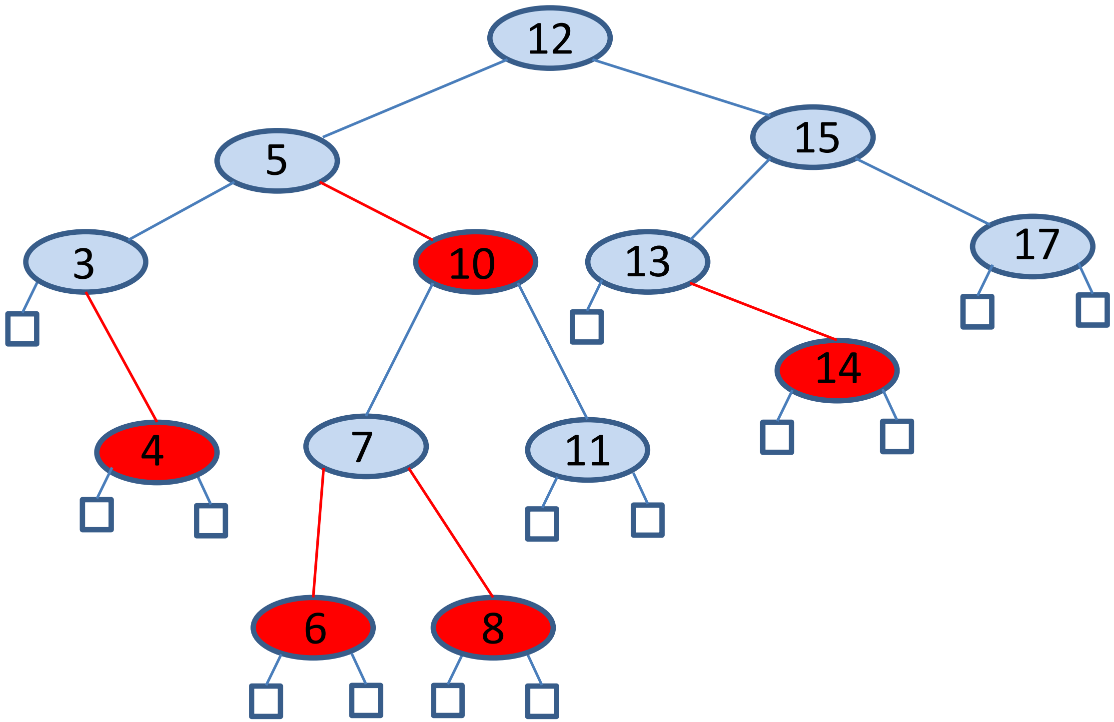
From (2,4) Trees to Red-Black Trees (cont'd)



Example (2,4) Tree



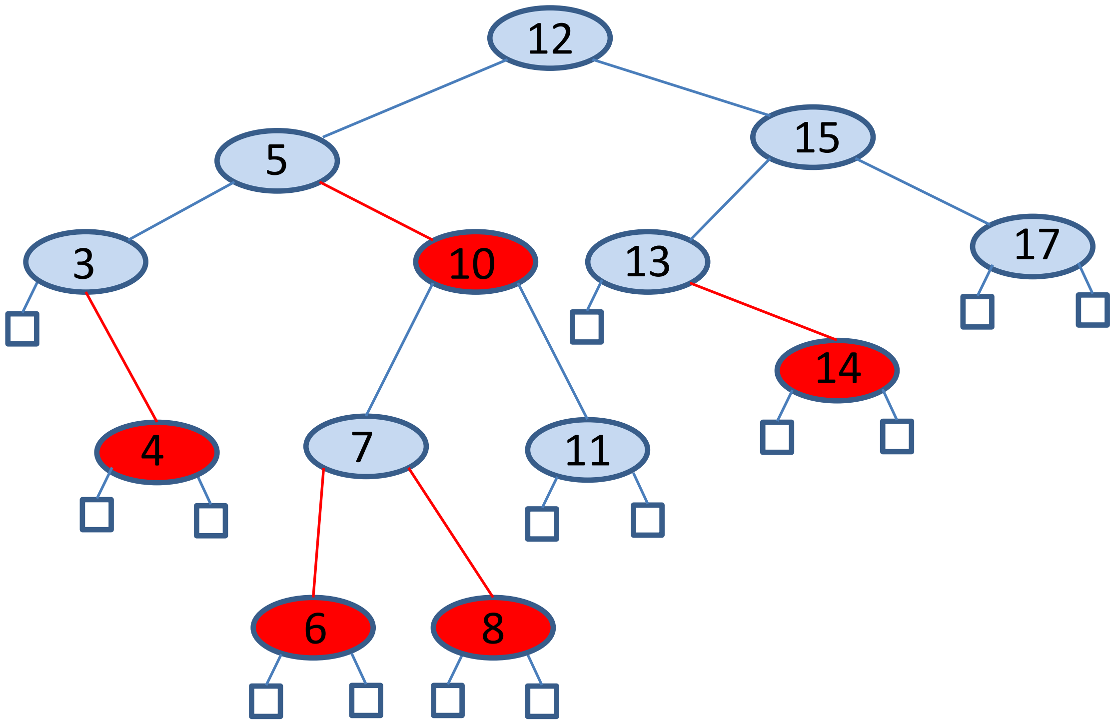
Corresponding Red-Black Tree



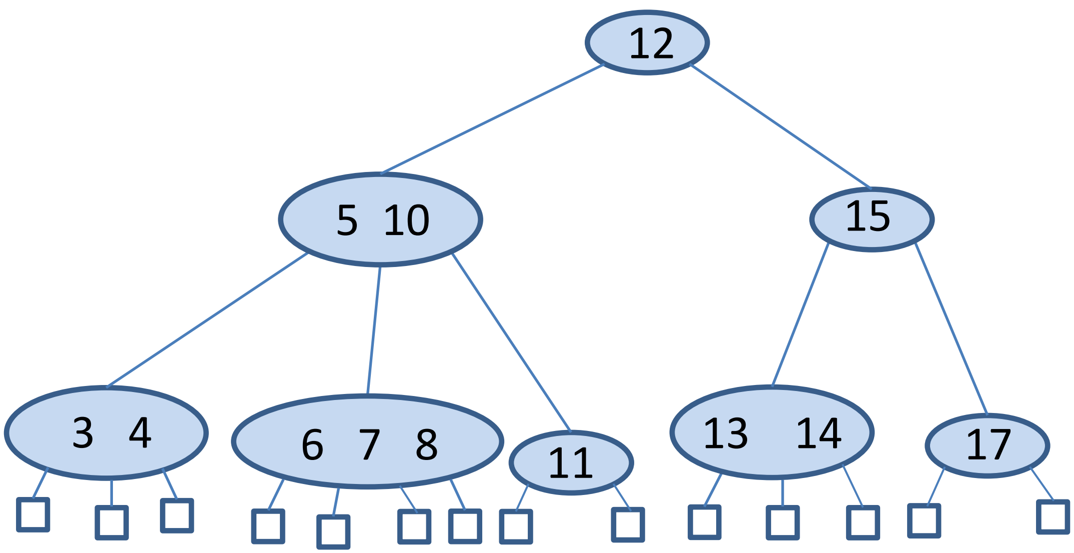
From Red-Black Trees to (2,4) Trees

- Given a red-black tree, we can **construct a corresponding (2,4) tree** by merging every red node v into its parent and storing the entry from v at its parent.
- The two children of v become left and right child of v in the new 3-node or 4-node.

Example Red-Black Tree



Corresponding (2,4) Tree



Proposition

- The height of a red-black tree storing n entries is $O(\log n)$.
- Proof?

Proof

- Let T be a red-black tree storing n entries, and let h be the height of T. We will prove the following:

$$\log(n+1) \leq h \leq 2\log(n+1)$$

- Let d be the common black depth of all the external nodes of T. Let T' be the (2,4) tree associated with T, and let h' be the height of T'.
- Because of the correspondence between red-black trees and (2,4) trees, we know that $\boldsymbol{h' = d}$.
- Hence, $d = h' \leq \log(n+1)$ by the proposition for the height of (2,4) trees. By the internal node property of red-black trees, we have $\boldsymbol{h \leq 2d}$ (the upper bound for the height is reached when every black node has only red children). Therefore, $h \leq 2\log(n+1)$.

Proof (cont'd)

- The other inequality, $\log(n+1) \leq h$ follows from the properties of proper binary trees and the fact that T has n internal nodes.

Search in a Red-Black Tree

- The algorithm for searching for the entry with key k in a red-black tree is exactly the same as the algorithm we presented for searching in a binary search tree.
- The worst-case complexity of this algorithm is $\boldsymbol{O(\log n)}$ where n is the number of entries in the tree.

Updates

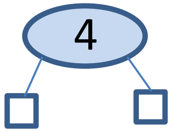
- Performing update operations (insertions or deletions) in a red-black tree is similar to the operations of binary search trees, but **we must additionally take care not to destroy the color properties.**
- For an update operation in a red-black tree T, **it is important to keep in mind the correspondence with a (2,4) tree T' and the relevant update algorithms for (2,4) trees.**

Insertion

- Let us consider the insertion of a new entry with key k into a red-black tree T.
- We will start with a few examples of insertions into an initially empty tree.

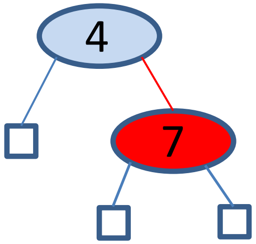
Initial Empty Tree

Insert 4



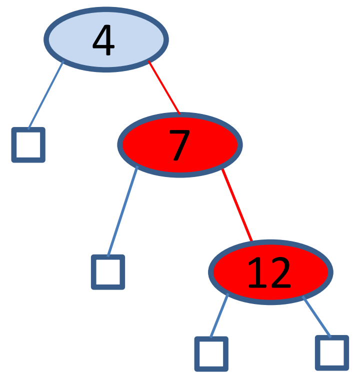
- Easy.

Insert 7



- Easy.

Insert 12



- In this case, the resulting tree **violates the internal property of red-black trees.** This problem needs to be fixed and we will see the details below.

Insertion (cont'd)

- Let us present the details of the algorithm for inserting a new entry with key k into a red-black tree T.
- We search for k in T until we reach an external node of T, and we replace this node with an internal node z, storing (k, i) and having two external-node children.
- If z is the root of T, we color z black, else we color z red. We also color the children of z black.
- This operation corresponds to inserting (k, i) into a node of the (2,4) tree T' with external-node children.
- This operation preserves the root, external, and depth properties of T, but **it might violate the internal property.**

Insertion (cont'd)

- Indeed, if z is not the root of T and the parent v of z is red, then we have **a parent and a child that are both red.**
- In this case, by the root property, v cannot be the root of T.
- By the internal property (which was previously satisfied), the parent u of v must be black.
- Since z and its parent are red, but z's grandparent u is black, we call this violation of the internal property a **double red** at node z.

Insertion (cont'd)

- To remedy a double red, we consider two cases.
- **Case 1: the sibling w of v is black**. In this case, the double red denotes the fact that we have created in our red-black tree T a **malformed** replacement for a corresponding 4-node of the (2,4) tree T', which has as its children the four black children of u, v and z.
- Our malformed replacement has one red node (v) that is the parent of another red node (z) while we want it to have **two red nodes as siblings** instead.
- To fix this problem, we perform a **trinode restructuring (αναδόμηση τριών κόμβων)** of T as follows.

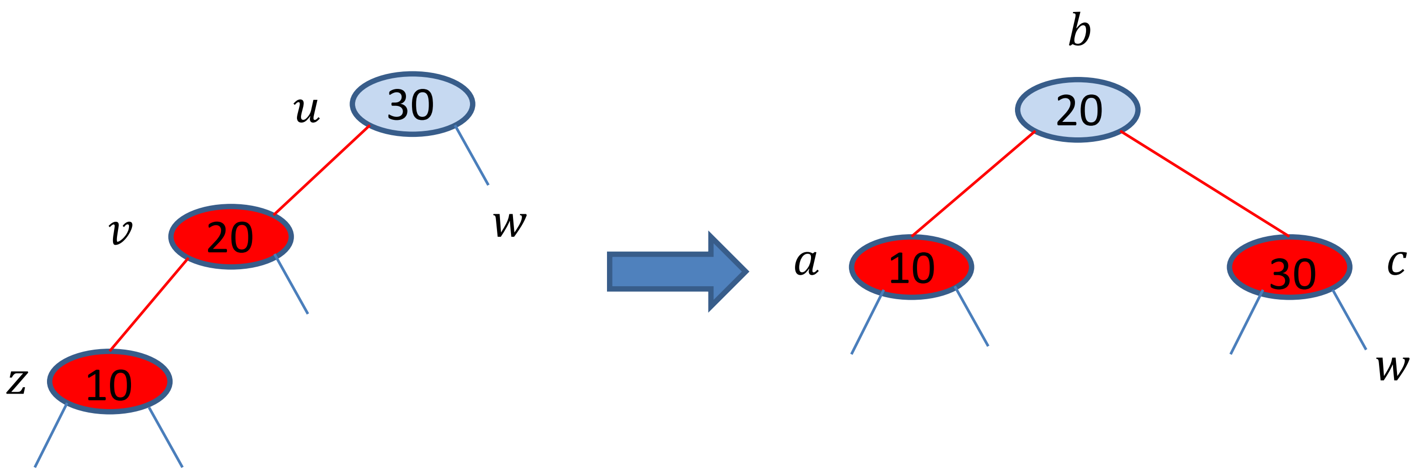
Trinode Restructuring

- Take node z, its parent v, and grandparent u, and temporarily relabel them as a, b and c, in left-to-right order, so that a, b and c will be visited in this order by an **inorder** tree traversal.
- Replace the grandparent u with the node labeled b, and make nodes a and c the children of b keeping inorder relationships unchanged.
- After restructuring, we color b black and we color a and c red. Thus, the restructuring **eliminates** the double red problem.

Trinode Restructuring vs. Rotations

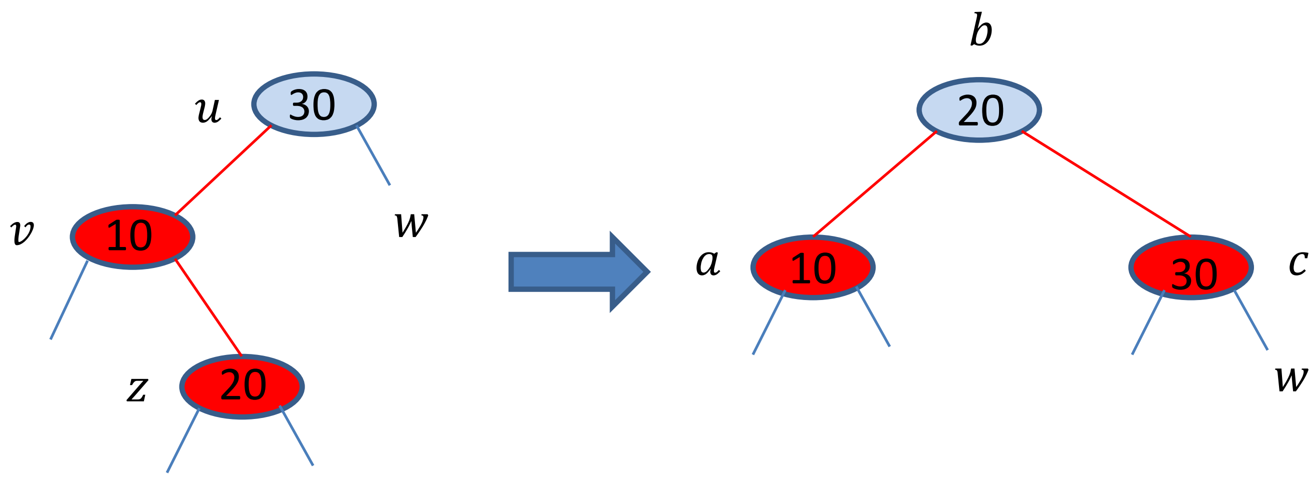
- The trinode restructuring operation we have just described corresponds exactly to the four kinds of **rotations** we discussed for AVL trees.
- Below we show graphically the four possible subcases of Case 1 for the nodes v, u, z and w and the rotations that will restore the internal property.

Trinode Restructuring Graphically



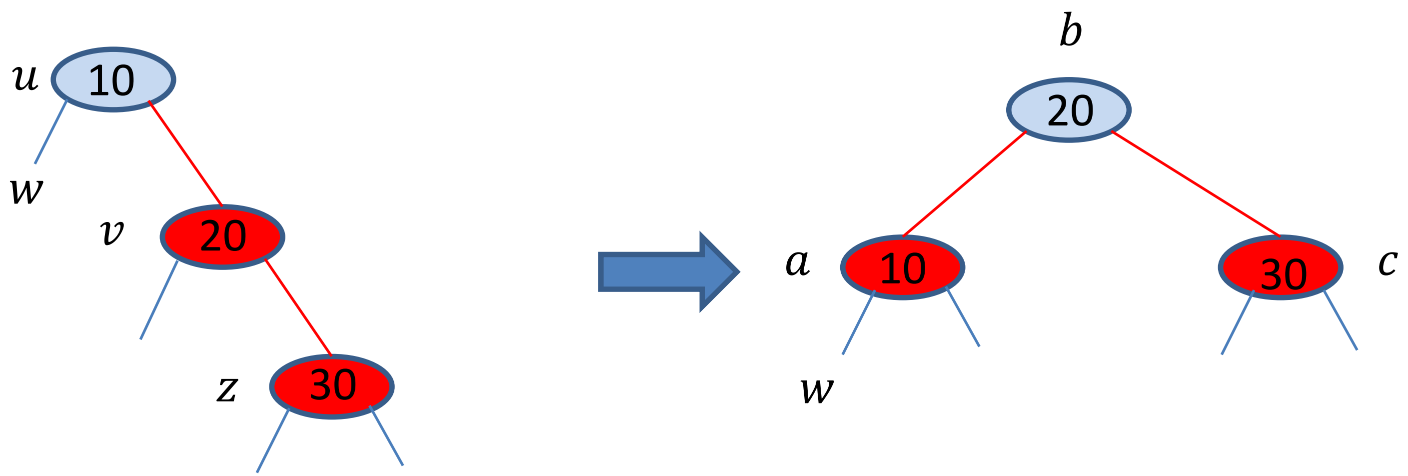
- **Right rotation** at u

Trinode Restructuring Graphically (cont'd)



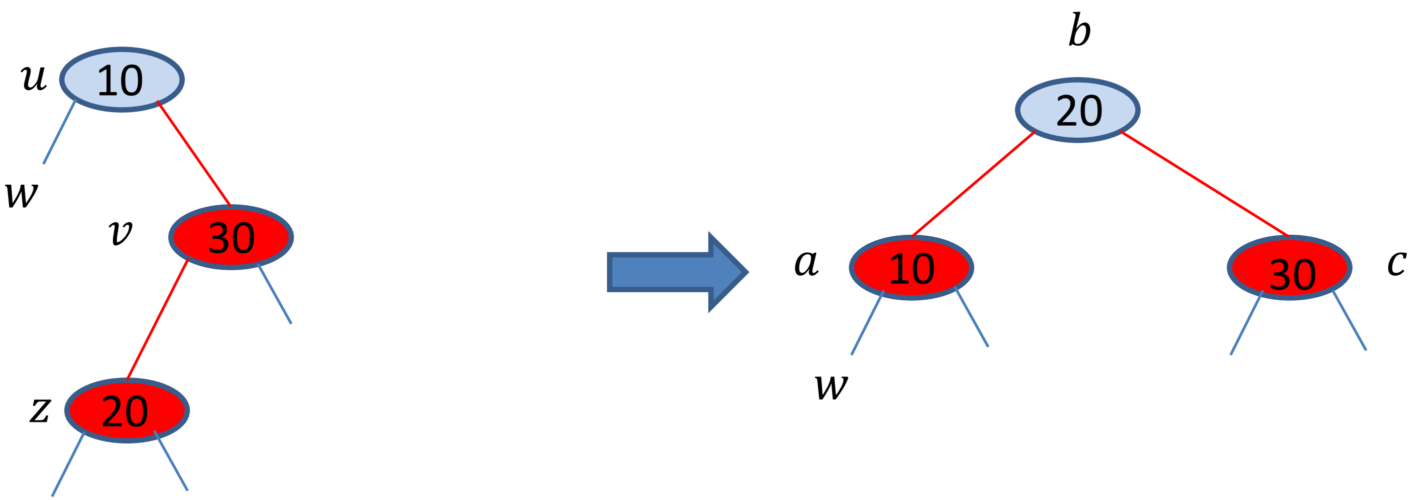
- **Double left-right rotation** at v and u (first a left rotation at v then a right rotation at u).

Trinode Restructuring Graphically (cont'd)



- **Left rotation** at u

Trinode Restructuring Graphically (cont'd)

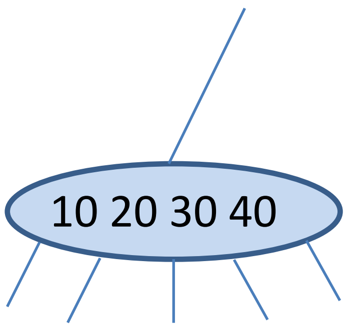


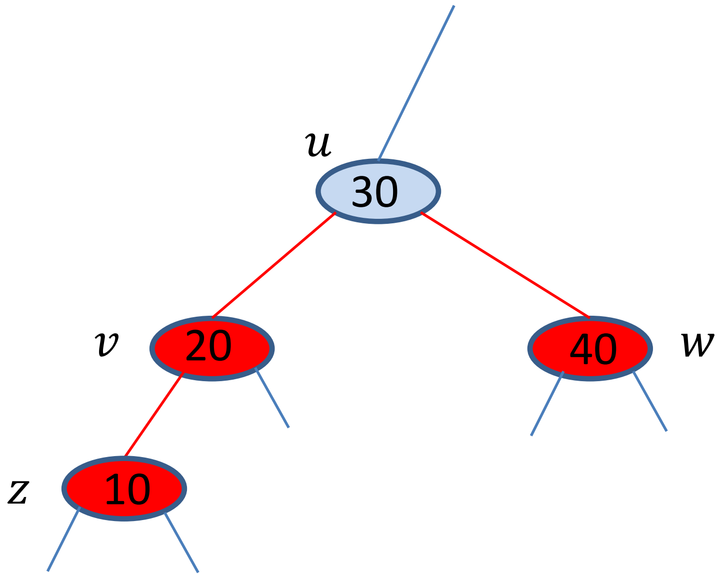
- **Double right-left rotation** at v and u (first a right rotation at v then a left rotation at u).

Insertion (cont'd)

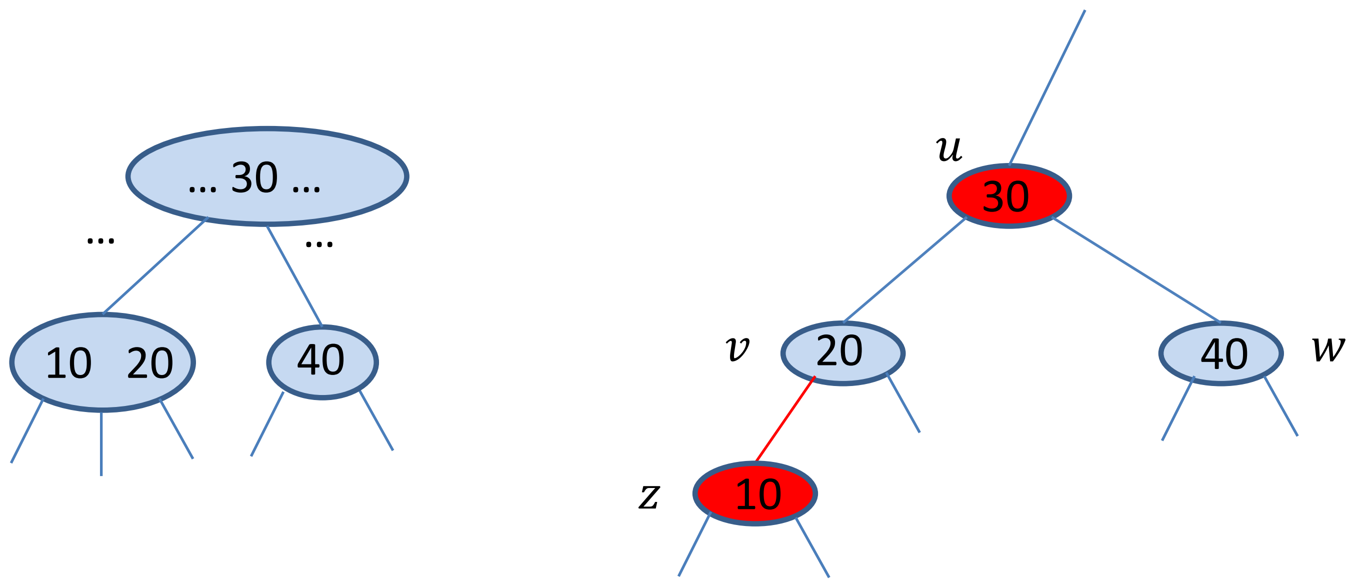
- **Case 2: the sibling w of v is red**. In this case, the double red denotes an **overflow** in the corresponding (2,4) tree T'.
- To fix the problem, we perform **the equivalent of a split operation**. Namely, we do a **recoloring (αναχρωματισμό)**: we color v and w black and their parent u red (unless u is the root, in which case it is colored black).

Overflow





Recoloring



Recoloring vs. Trinode Restructuring

- The **trinode restructuring** operation involves a local restructuring of the tree (implemented by pointer manipulation) and changes in color.
- **Recoloring** only needs changes in color and the structure of the tree does not change.
- The term “recoloring” should **not** be used in the case of trinode restructuring although colors change in that case too.

Insertion (cont'd)

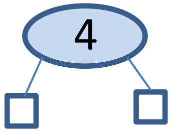
- It is possible that, after such a recoloring, the double red problem **reappears** at u (if u has a red parent). Then, we repeat the consideration of the two cases.
- Thus, a recoloring either eliminates the double red problem at node z or propagates it to the grandparent u of z.
- We continue going up T performing recoloring until we finally resolve the double red problem (either with a final recoloring or a trinode restructuring).
- Thus, the number of recolorings caused by insertion is no more than half the height of tree T (why?), that is, no more than $\log(n+1)$ by the proposition we have proved about the height of a red-black tree.

Example

- Let us now see some examples of insertions in an initially empty red-black tree.

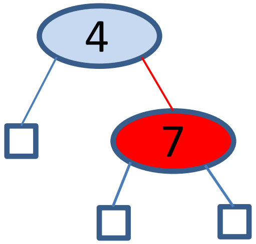
Initial Empty Tree

Insert 4



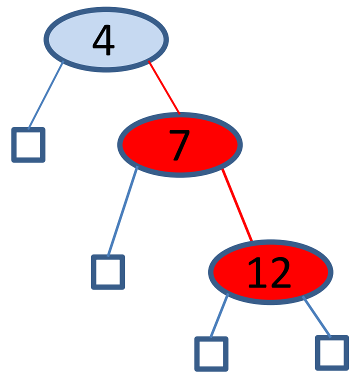
- Easy.

Insert 7



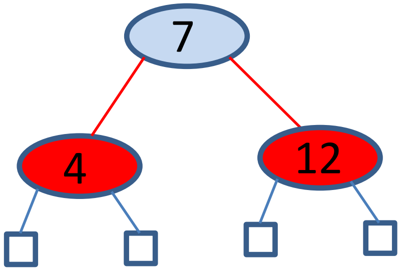
- Easy.

Insert 12 – Double Red

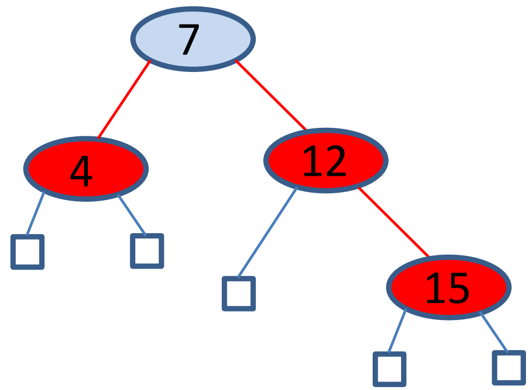


- We are in Case 1. We will do a trinode restructuring (left rotation at 4).

After Restructuring

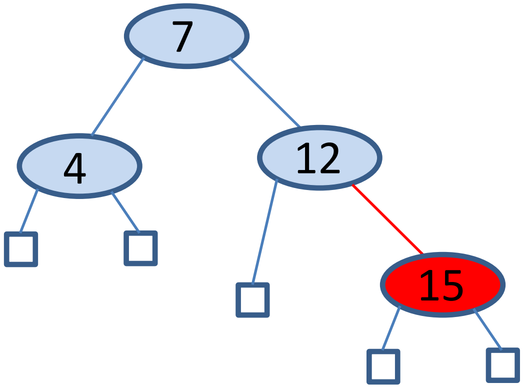


Insert 15 – Double Red

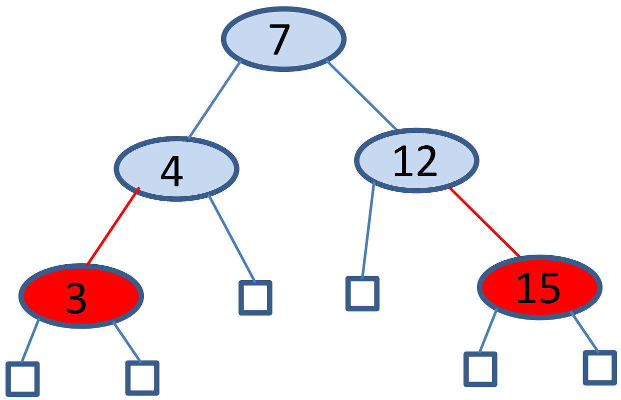


- We are in Case 2. We will do a recoloring.

After Recoloring

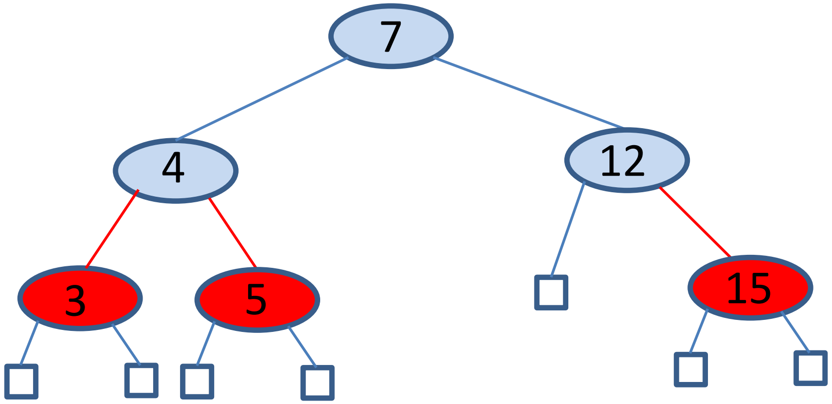


Insert 3



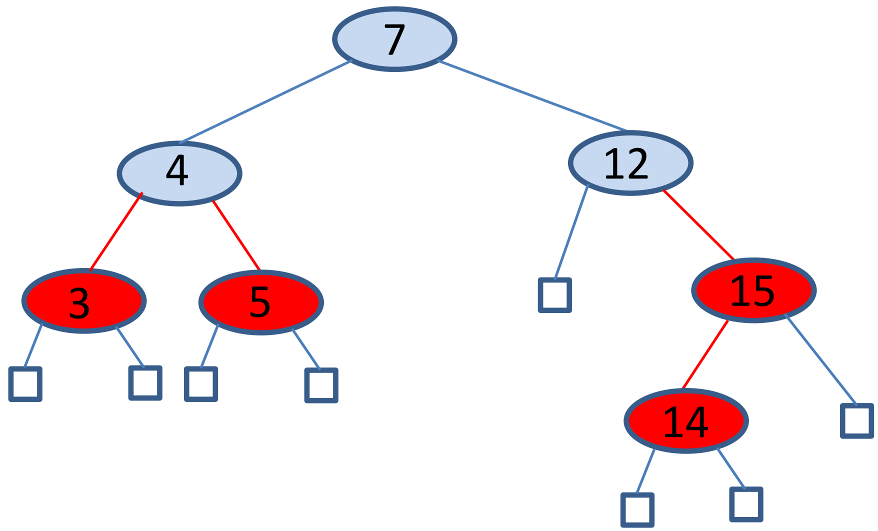
- Easy.

Insert 5



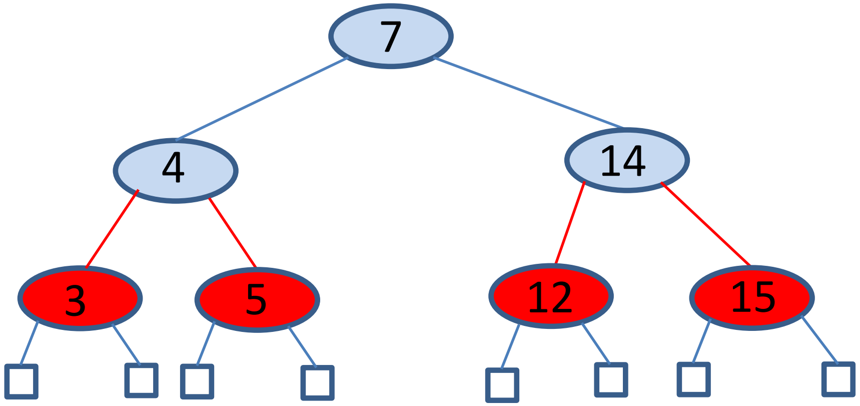
- Easy.

Insert 14 – Double Red

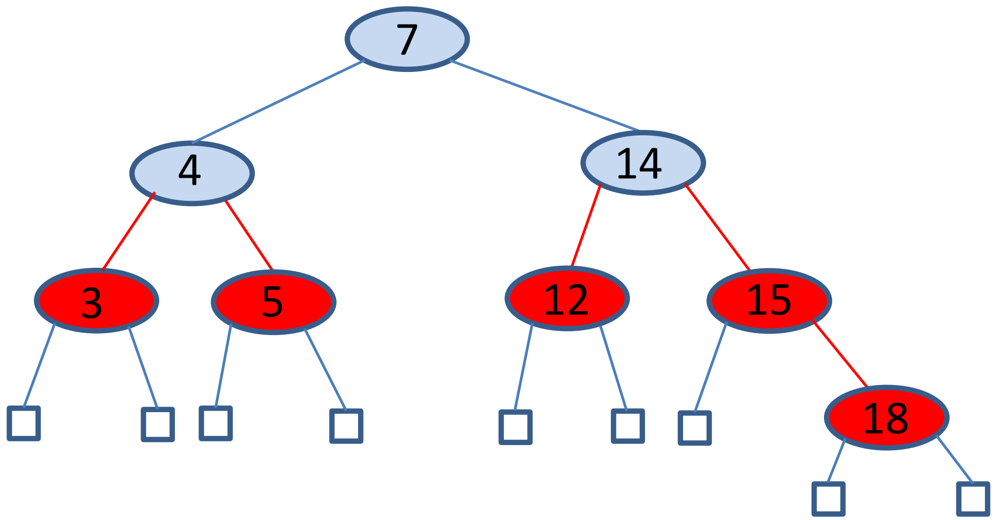


- We are in Case 1. We will do a trinode restructuring (double right-left rotation at 15 and 12).

After Restructuring

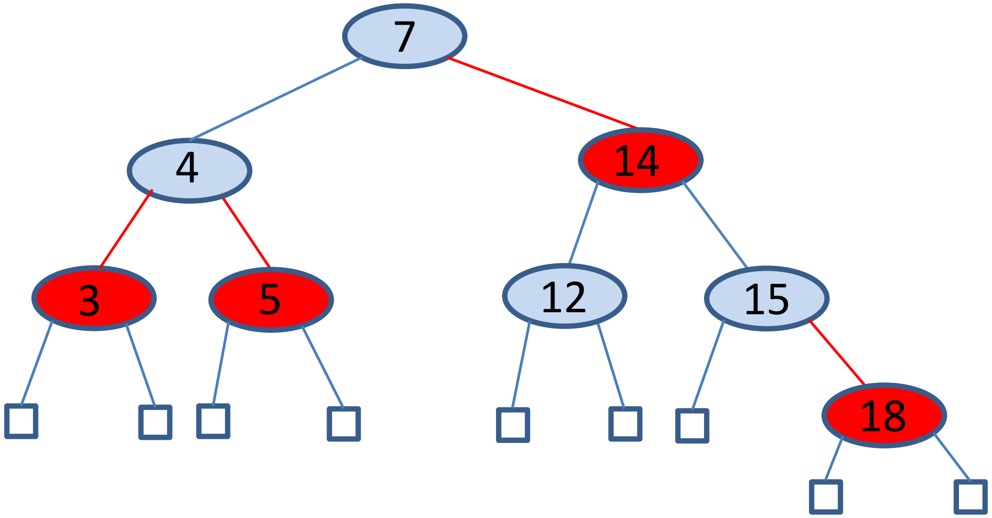


Insertion of 18 – Double Red

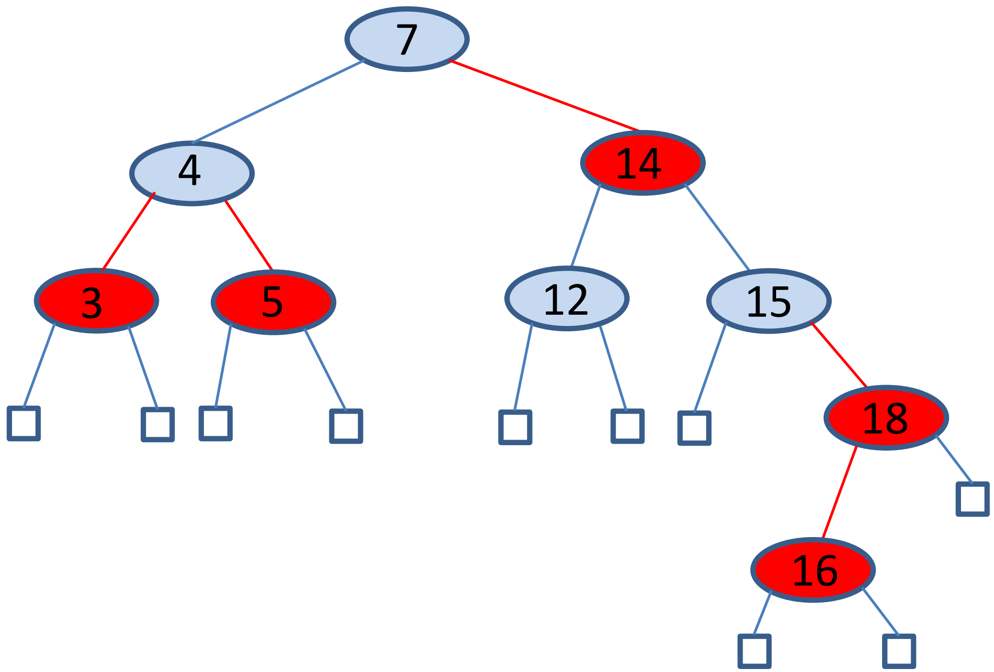


- We are in Case 2. We will do a recoloring.

After Recoloring

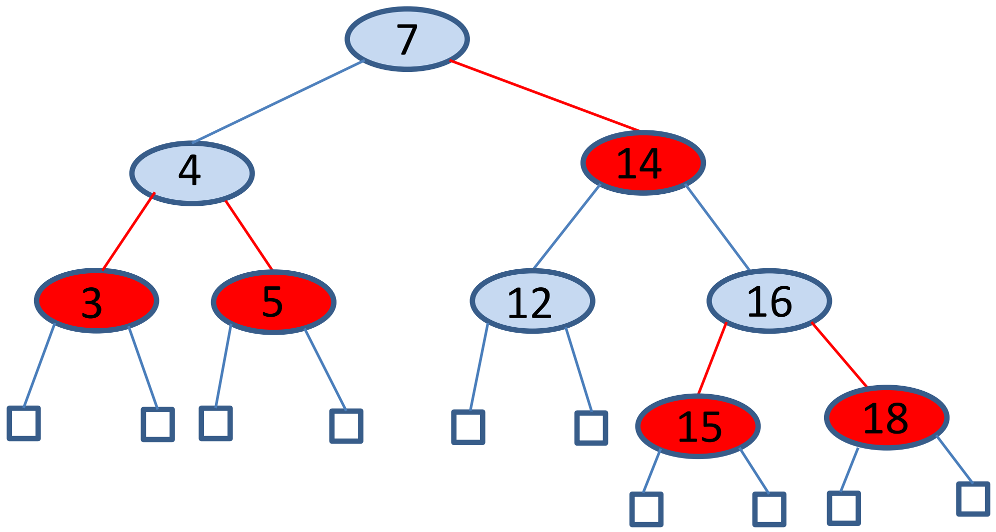


Insertion of 16 – Double Red

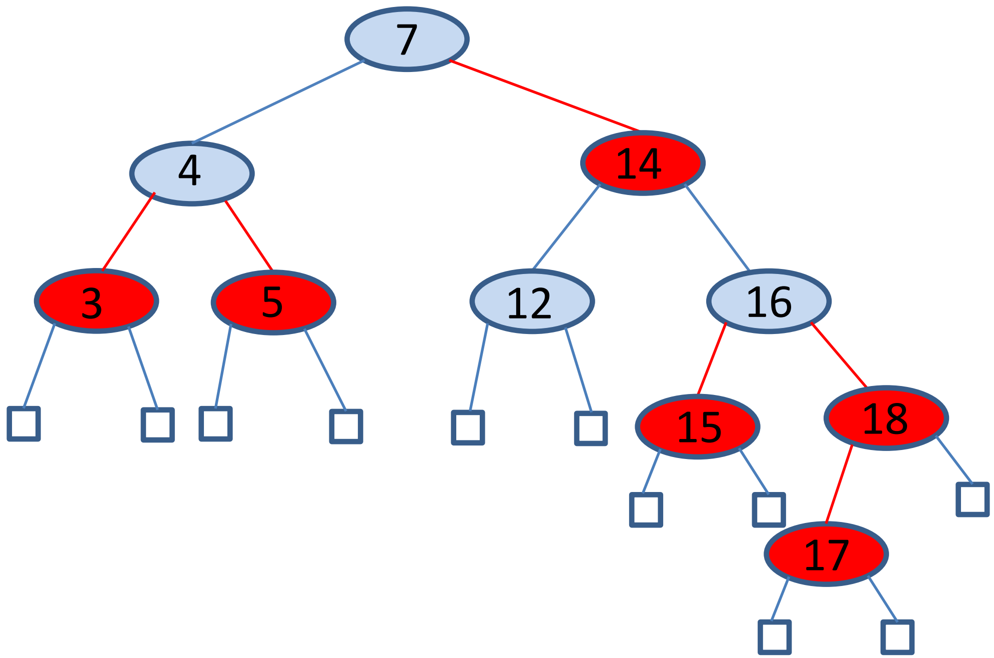


- We are in Case 1. We will do a trinode restructuring (double right-left rotation at 18 and 15).

After Restructuring

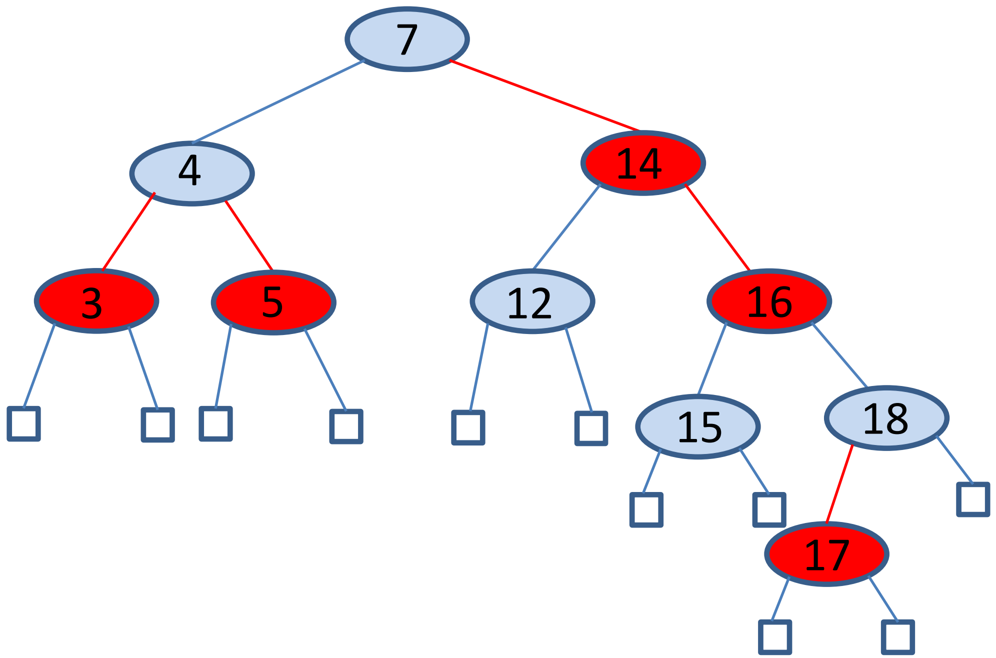


Insertion of 17 – Double Red



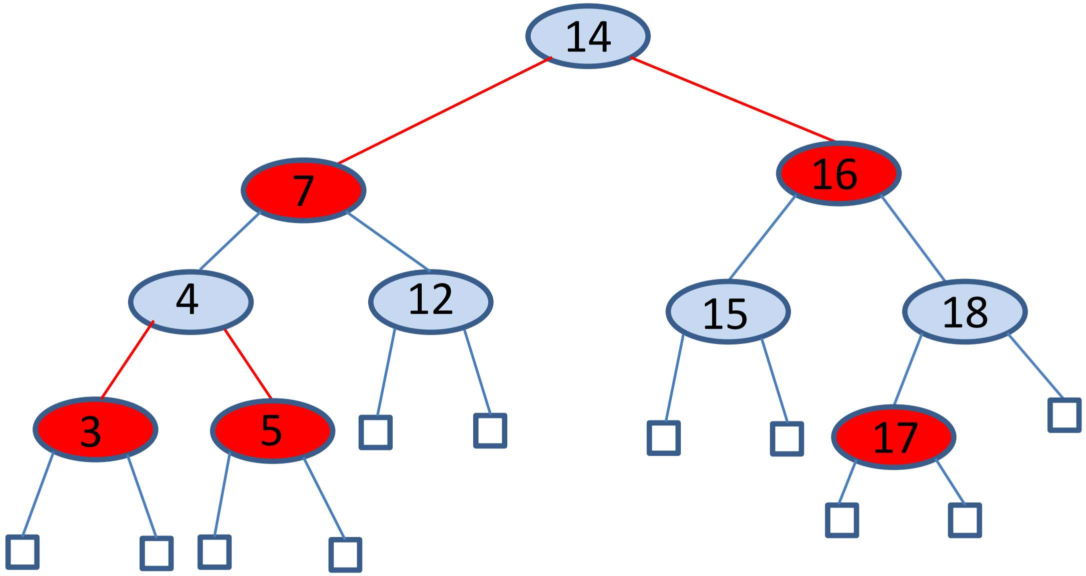
- We are in Case 2. We will do a recoloring.

After Recoloring – Double Red



- We are in Case 1. We will do a trinode restructuring (left rotation at 7).

After Restructuring



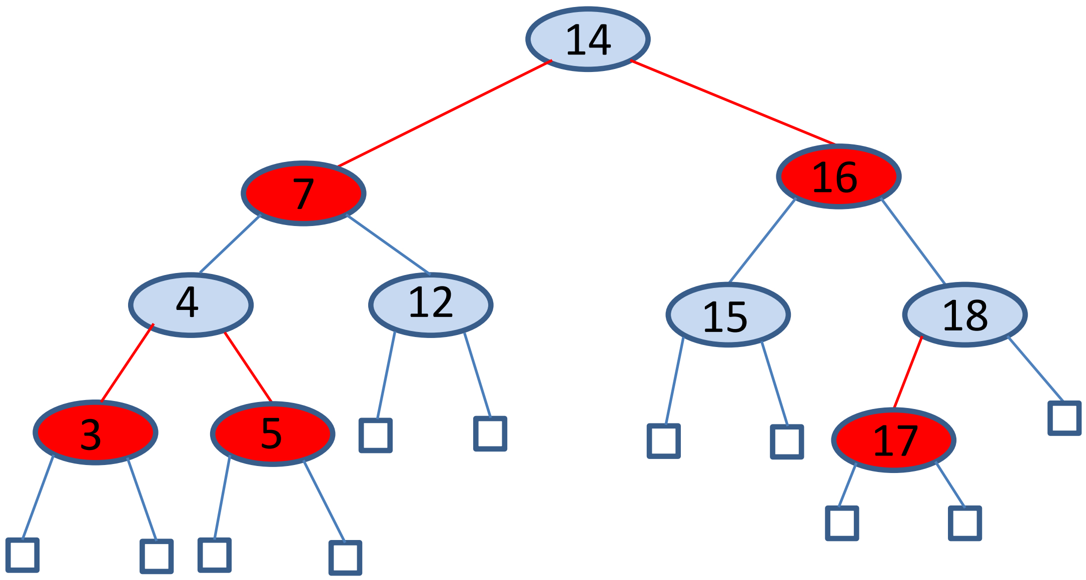
Proposition

- The insertion of a key-value entry in a red-black tree storing n entries can be done in $\boldsymbol{O}\,(\mathbf{log}\,\boldsymbol{n})$ **time** and requires $\boldsymbol{O}\,(\mathbf{log}\,\boldsymbol{n})$ **recolorings** and **one trinode restructuring**.

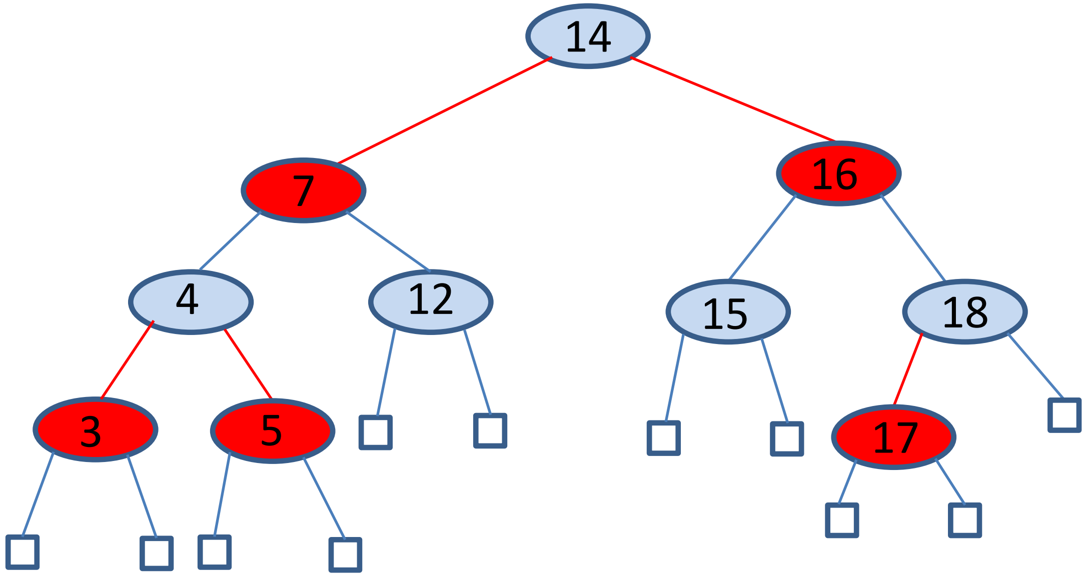
Removal

- Let us now present how to remove an entry with key k from a red-black tree T.
- Let us first see a few examples of removal from a given red-black tree.

Initial Tree

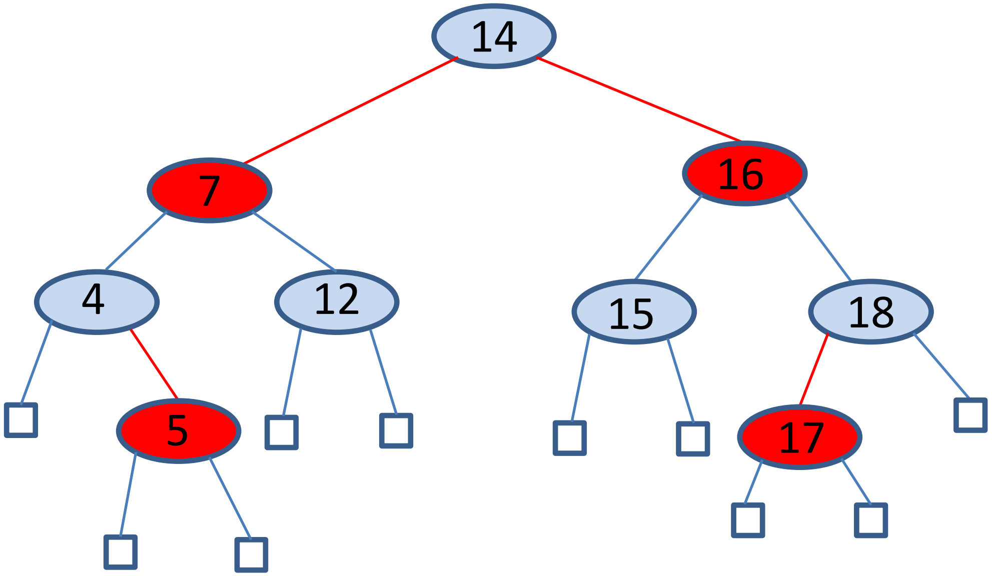


Remove 3

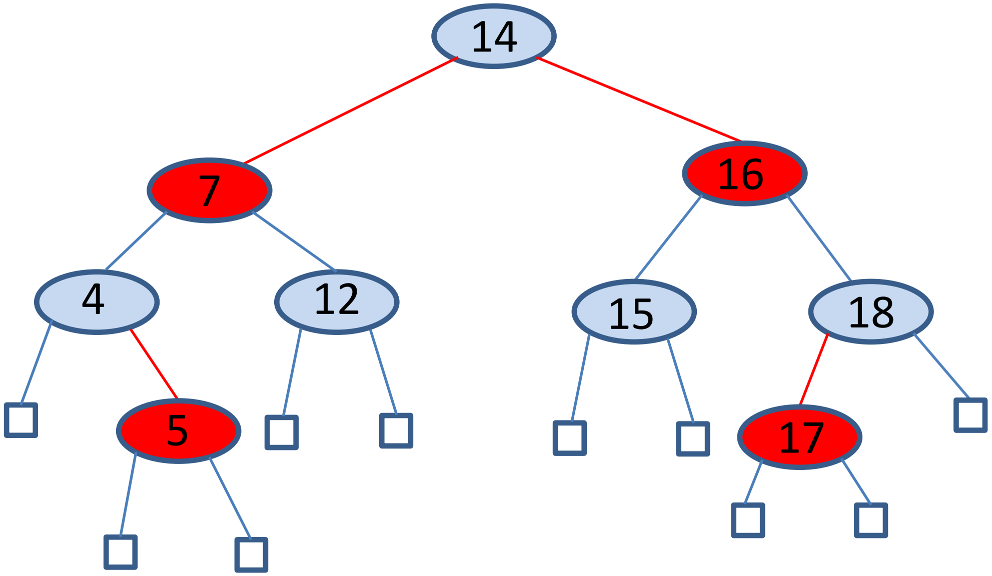


- Easy.

After Removing 3

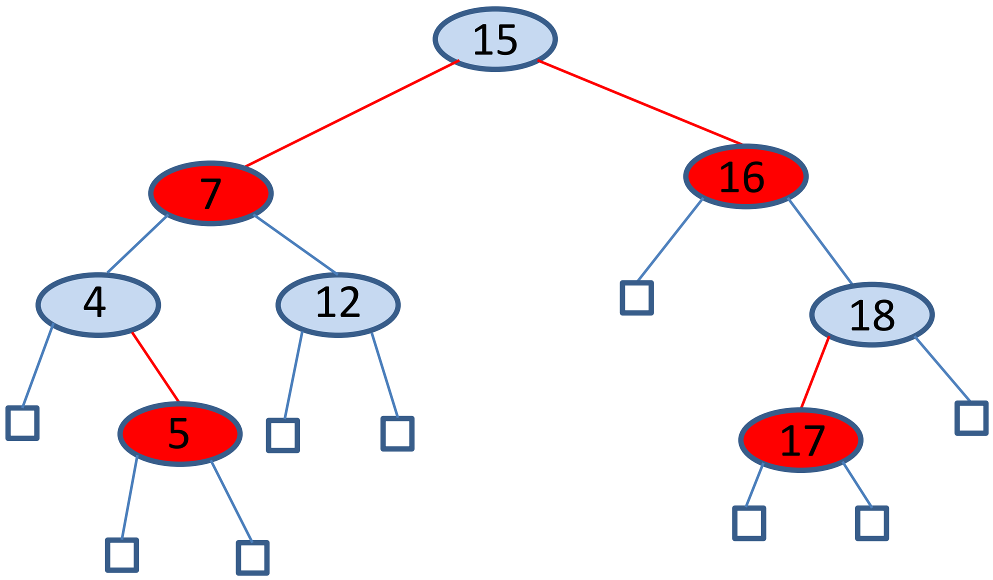


Remove 14



- To remove 14, we find the key which follows 14 in the natural order of keys (15), move this key to the position of 14 and delete it from the tree.

After Moving Key 15 and Deleting Its Node



- The resulting tree is no longer a red-black tree because **the depth property has been violated** for the external-node child of node with key 16.

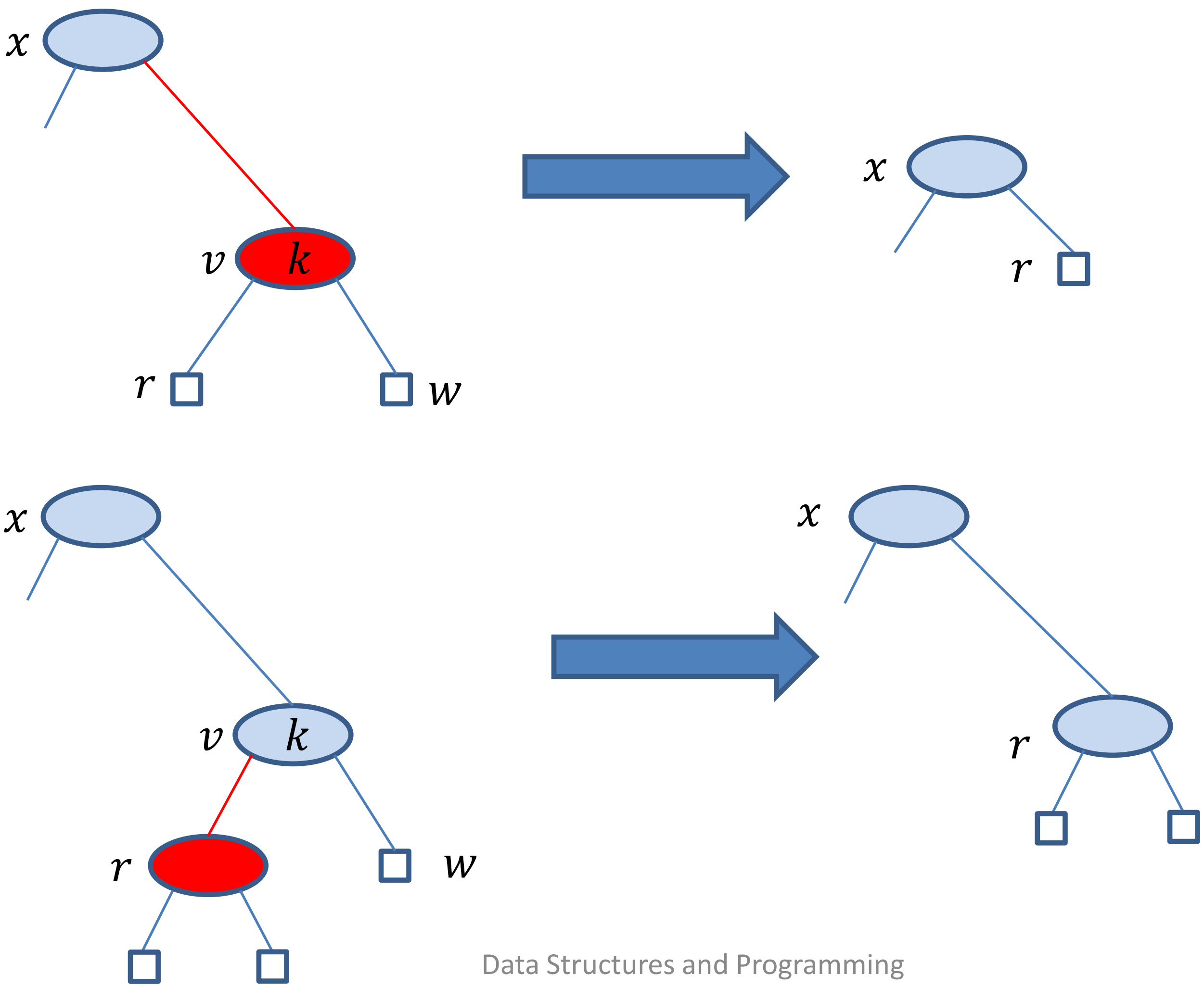
Removal (cont’d)

- Let us now discuss the details of the algorithm for removing an entry with key k from a red-black tree T.
- We proceed like in a binary tree search searching for a node u storing such an entry.
- **If $\boldsymbol{u}$ does not have an external-node child**, we find the internal node v following u in the inorder traversal of T. This node has an external-node child. We move the entry at v to u, and perform the removal at v.
- Thus, **we may consider only the removal of an entry with key $\boldsymbol{k}$ stored at a node $\boldsymbol{v}$ with an external-node child $\boldsymbol{w}$.**

Removal (cont'd)

- To remove the entry with key k from a node v of T with an external-node child w, we proceed as follows.
- Let r be the sibling of w and x the parent of v. We remove nodes v and w, and make r a child of x.
- If $\boldsymbol{v}$ **was red (hence** $\boldsymbol{r}$ **is black)** then none of the properties of red-black trees is violated and we are done.
- If $\boldsymbol{r}$ **is red (hence** $\boldsymbol{v}$ **was black)** then the depth property is violated. In this case **we need to color** $\boldsymbol{r}$ **black** to restore the depth property.
- These two cases are shown graphically on the next slide. Note that there are also their symmetric cases when v is the left child of x (and similarly for r and w).

Graphically



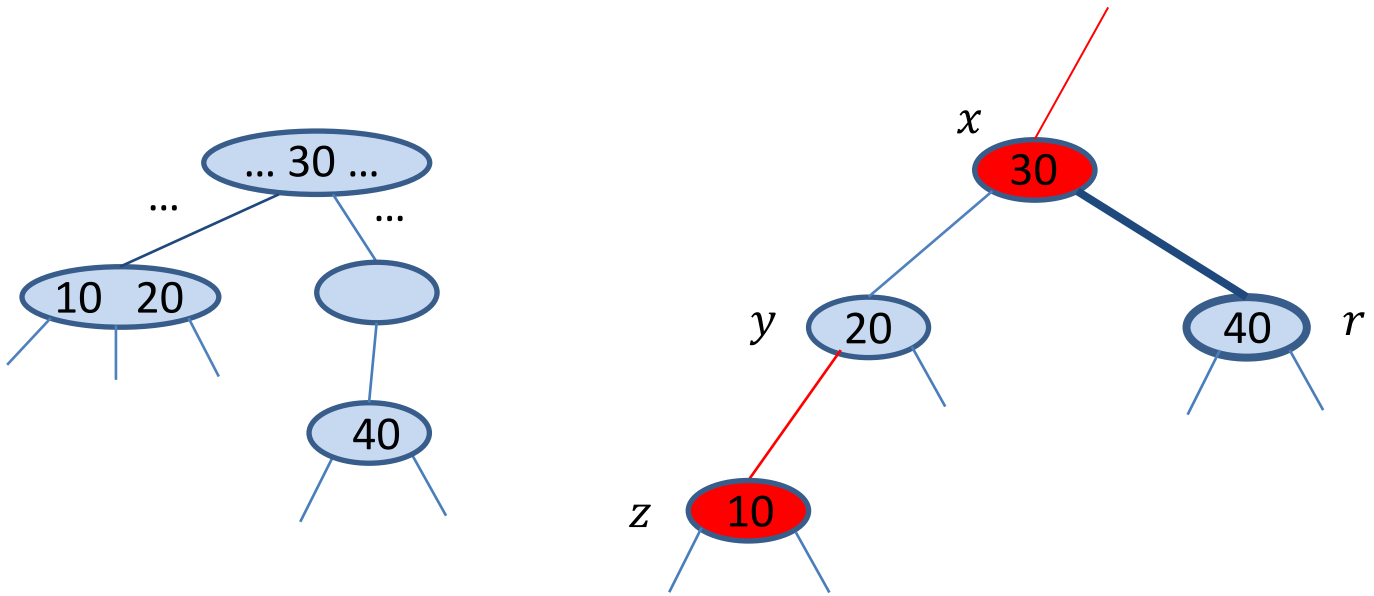
Removal (cont’d)

- Finally, if $\boldsymbol{r}$ **is black and** $\boldsymbol{v}$ **is black** then we have a violation of the depth property again.
- In this case, to preserve the depth property, we give r a fictitious **double black** color.
- We now have a color violation, called the **double black problem**.
- A double black in T denotes an **underflow** in the corresponding (2,4) tree T'.
- To remedy the double-black problem at r, we proceed as follows.
- We will have **3 cases** depending on the color of sibling y of r in the tree resulting from the deletion of v and the color of y’s children.

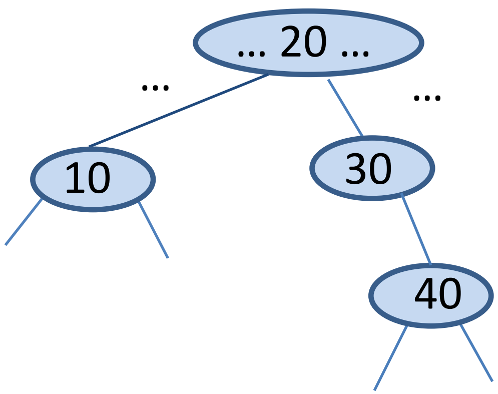
Removal (cont'd)

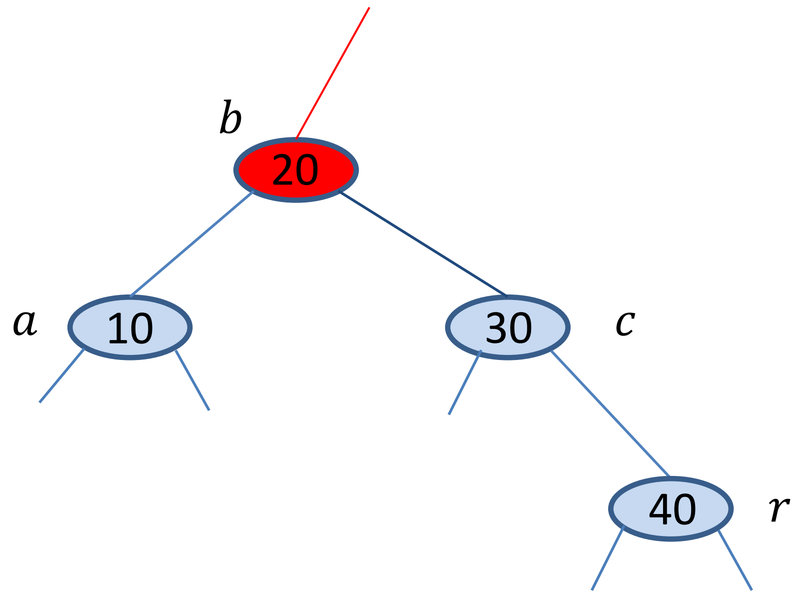
- **Case 1: the sibling y of r is black and has a red child z.**
- Resolving this case corresponds to a **transfer** operation in the (2,4) tree T'.
- We perform a **trinode restructuring**: we take the node z, its parent y, and grandparent x, we label them temporarily left to right as a, b and c, and we replace x with the node labeled b, making it parent of the other two nodes.
- We color a and c black, give b the former color of x, and color r black.
- This trinode restructuring eliminates the double black problem because the path $b - c - r$ now contains **two black nodes.**

Example of Case 1

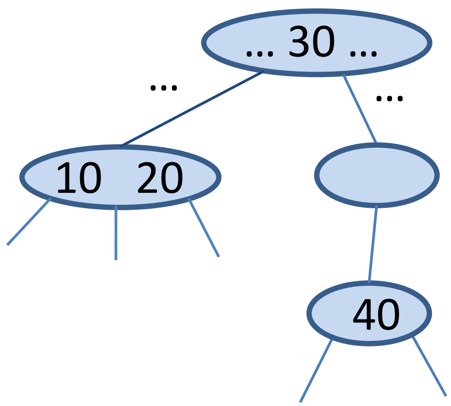


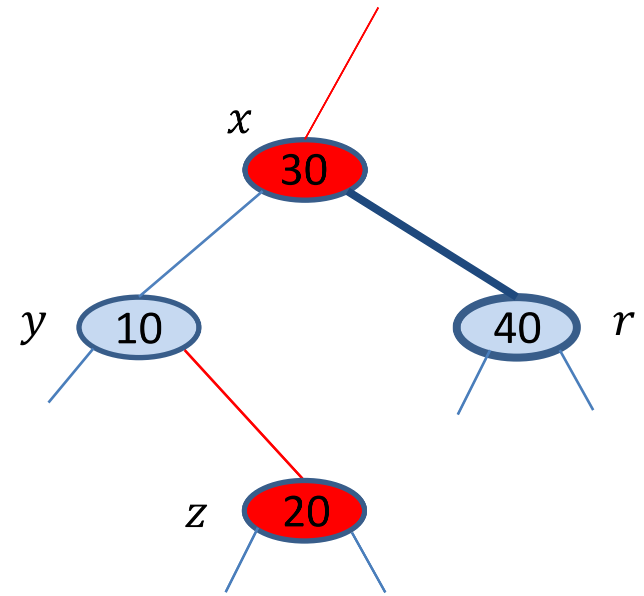
After the Restructuring (Right Rotation at x)



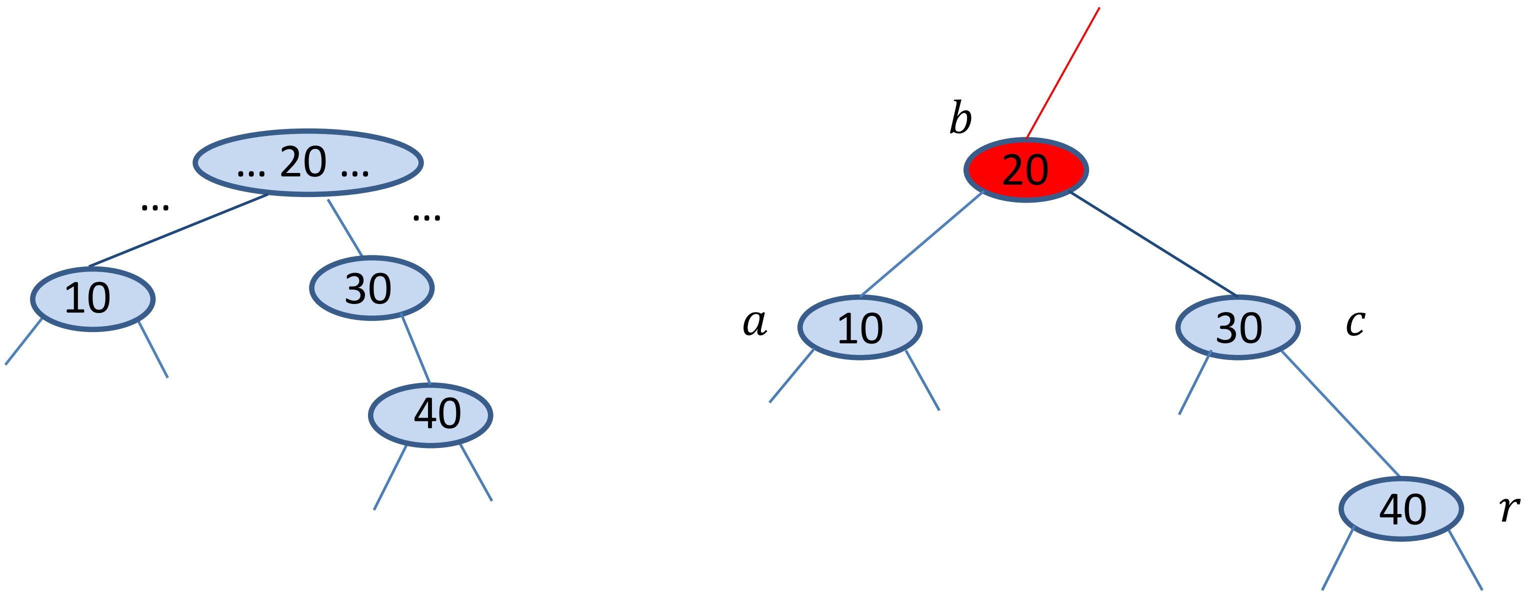


Alternative Example of Case 1





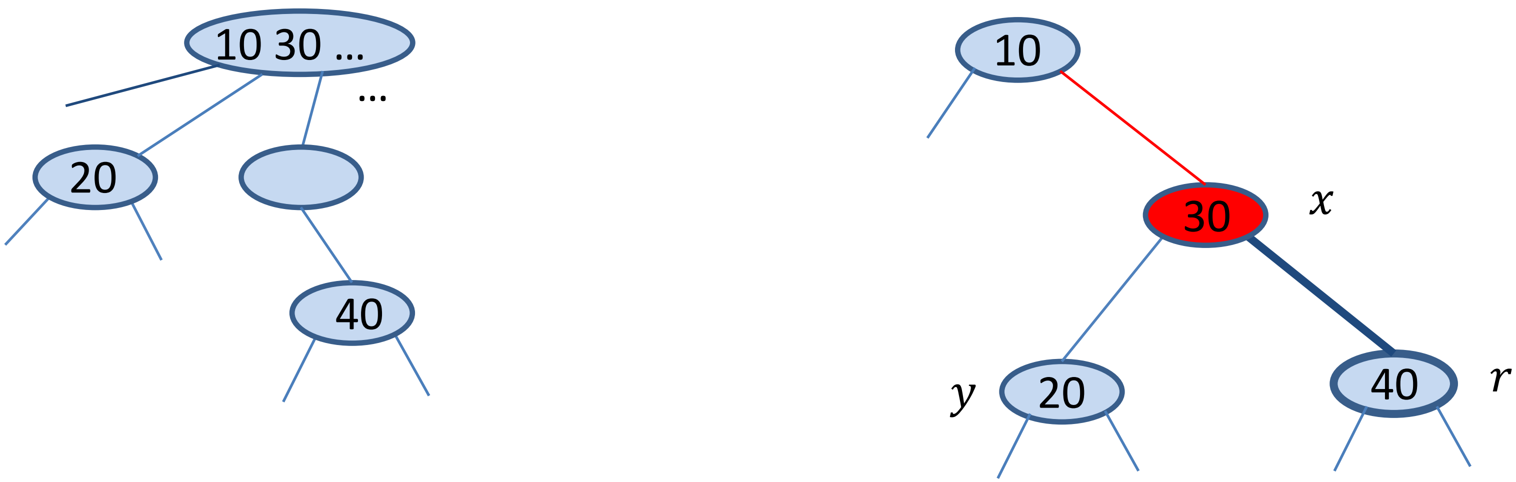
After the Restructuring (Double Left-Right Rotation at y and x)



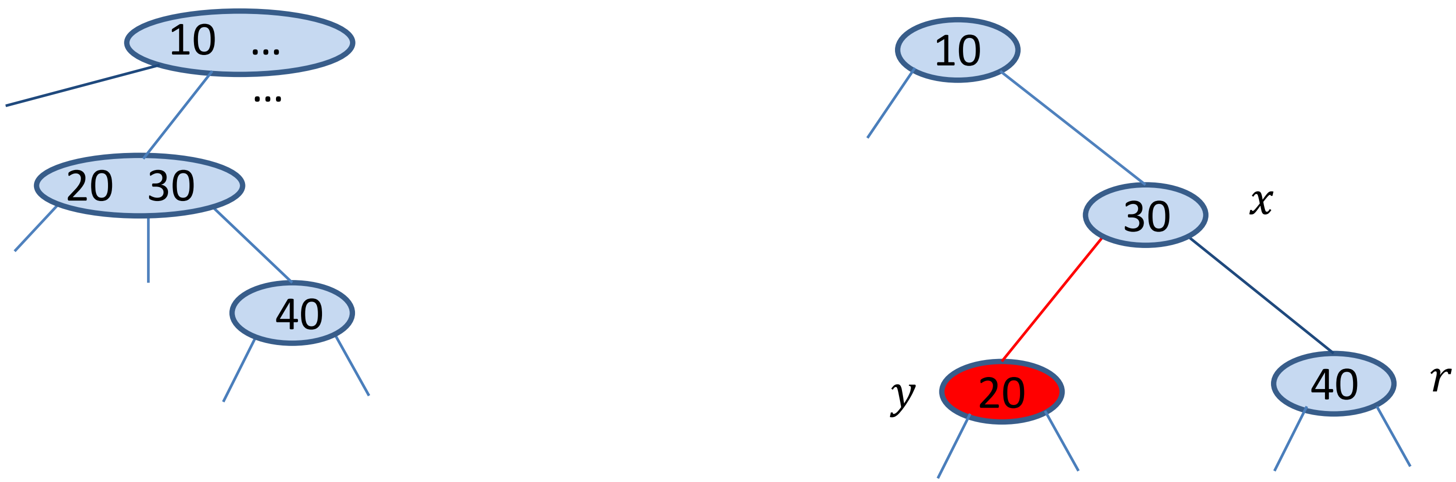
Removal (cont'd)

- **Case 2: the sibling y of r is black and both children of y are black**.
- Resolving this case corresponds to a **fusion** operation in the corresponding (2,4) tree T'.
- We do a **recoloring**: we color r black, we color y red, and, if x is red, we color it black; otherwise, we color x **double black**.
- Hence, after this recoloring, the double black problem might reappear at the parent x of r. We then repeat consideration of these three cases at x.

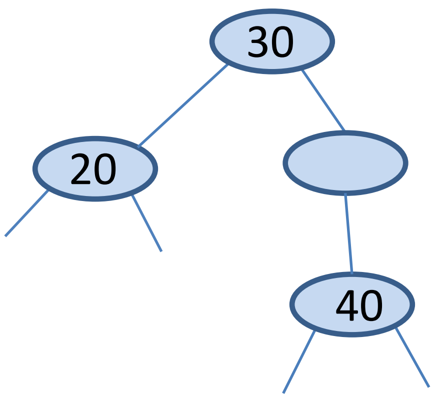
Recoloring a Red-Black Tree that Fixes the Double Black Problem

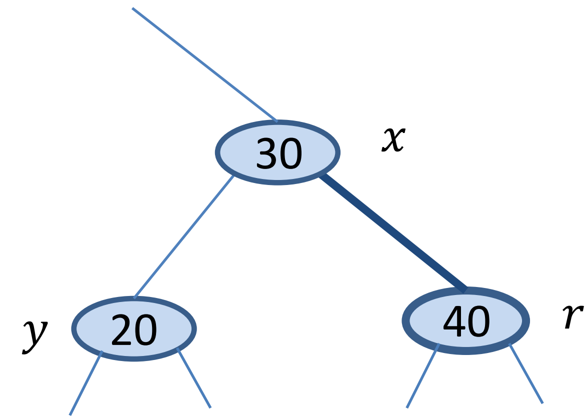


After the Recoloring

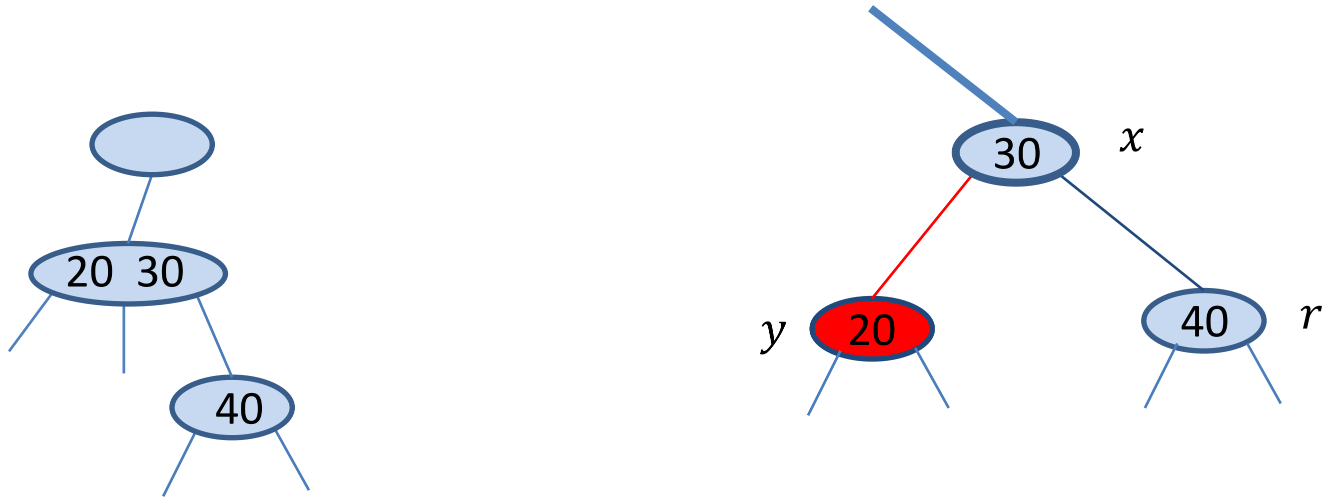


Recoloring a Red-Black Tree that Propagates the Double Black Problem





After the Recoloring



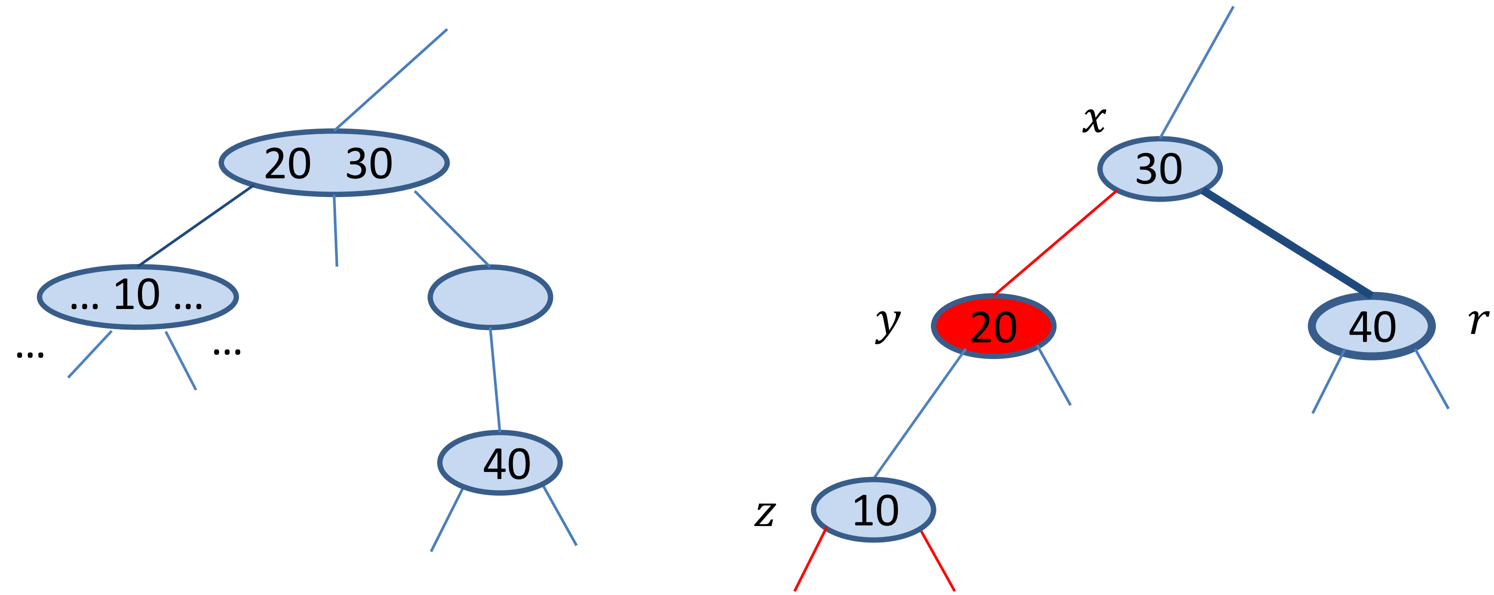
Removal (cont'd)

- **Case 3: the sibling y of r is red.**
- In this case, we perform an **adjustment operation (πράξη προσαρμογής)** as follows.
- If y is the right child of x, let z be the right child of y; otherwise, let z be the left child of y.
- Execute the trinode restructuring operation which makes y the parent of x.
- Color y black and x red.

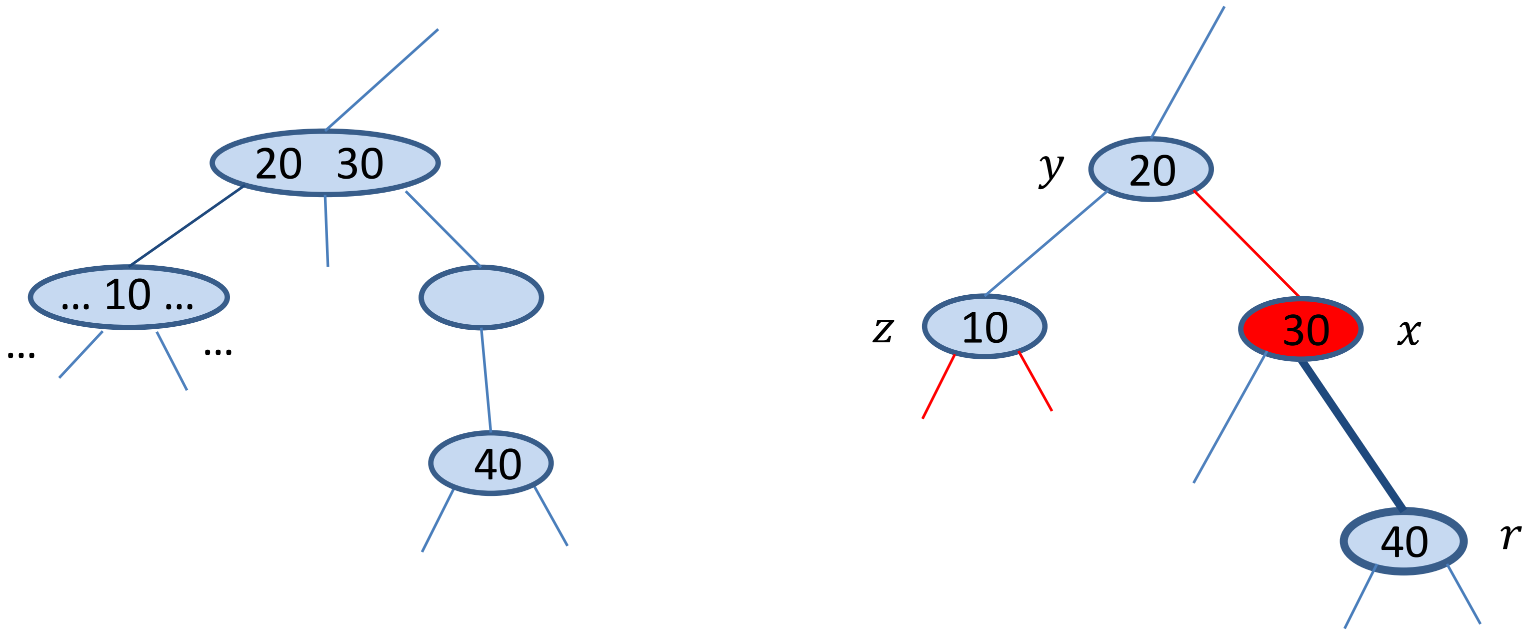
Removal (cont'd)

- **An adjustment corresponds to choosing in the red-black tree T a different representation of a 3-node** from the corresponding (2,4) tree T'.
- After the adjustment operation, the sibling of r is black, and either Case 1 or Case 2 applies, with a different meaning of x and y.
- Note that if Case 2 applies, the double black problem cannot reappear because the parent of r is red.
- Thus, to complete Case 3 we make one more application of either Case 1 or Case 2 and we are done.
- Therefore, **at most one adjustment** is performed in a removal operation.

Adjustment of a Red-Black Tree in the Presence of a Double Black Problem



After the Adjustment (Right Rotation at x)



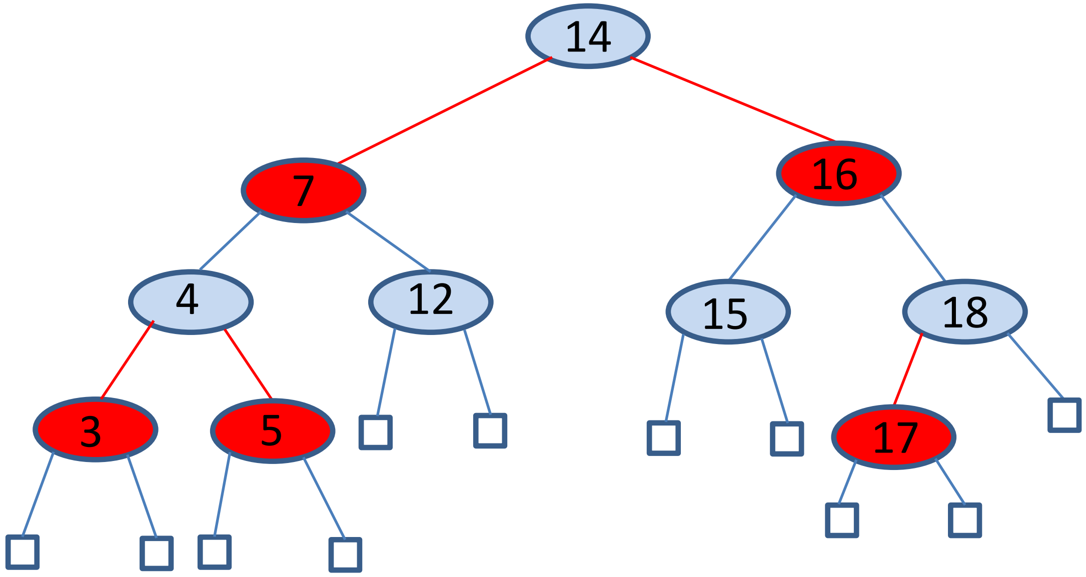
Removal (cont'd)

- The algorithm for removing an entry from a red-black tree with n entries takes $\boldsymbol{O(\log n)}$ **time** and performs $\boldsymbol{O(\log n)}$ **recolorings** and **at most one adjustment plus one additional trinode restructuring**.

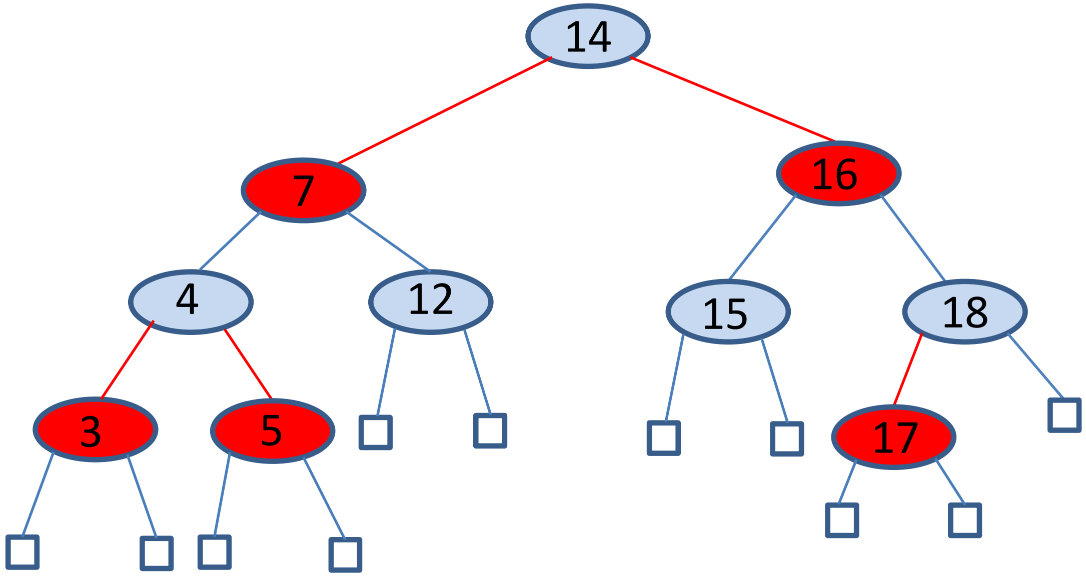
Example

- Let us now see a few removals from a given red-black tree.

Initial Tree

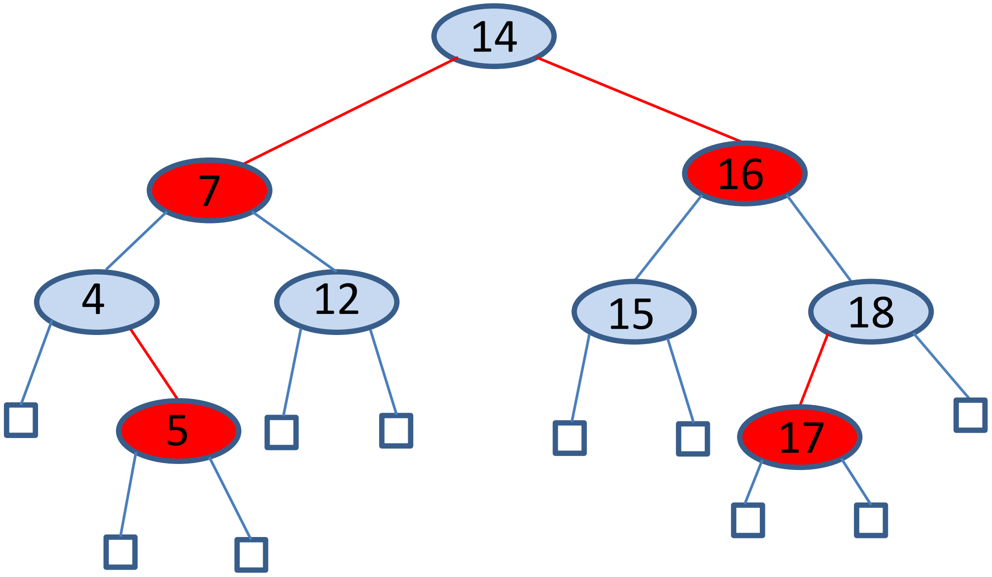


Remove 3

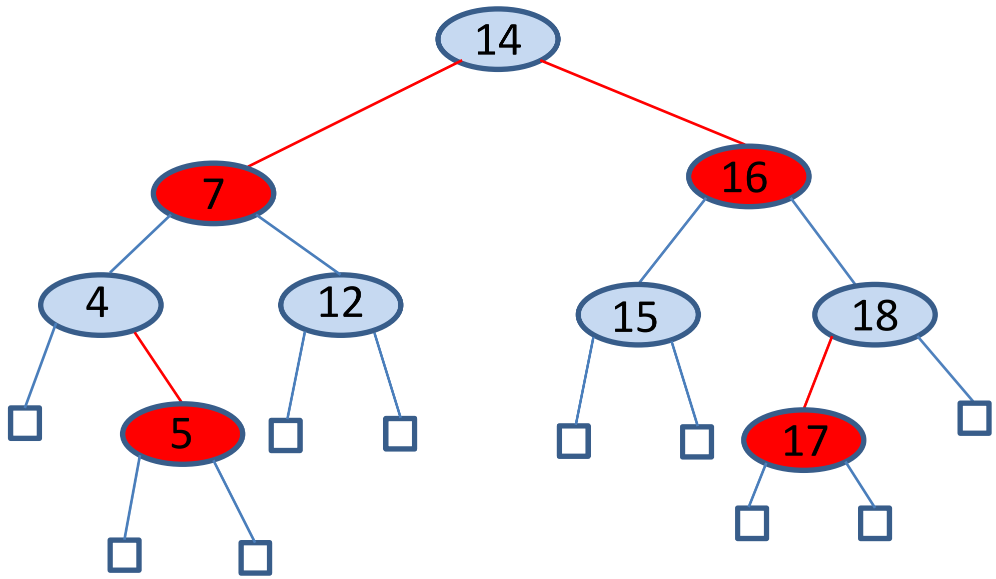


- Easy.

After Removing 3

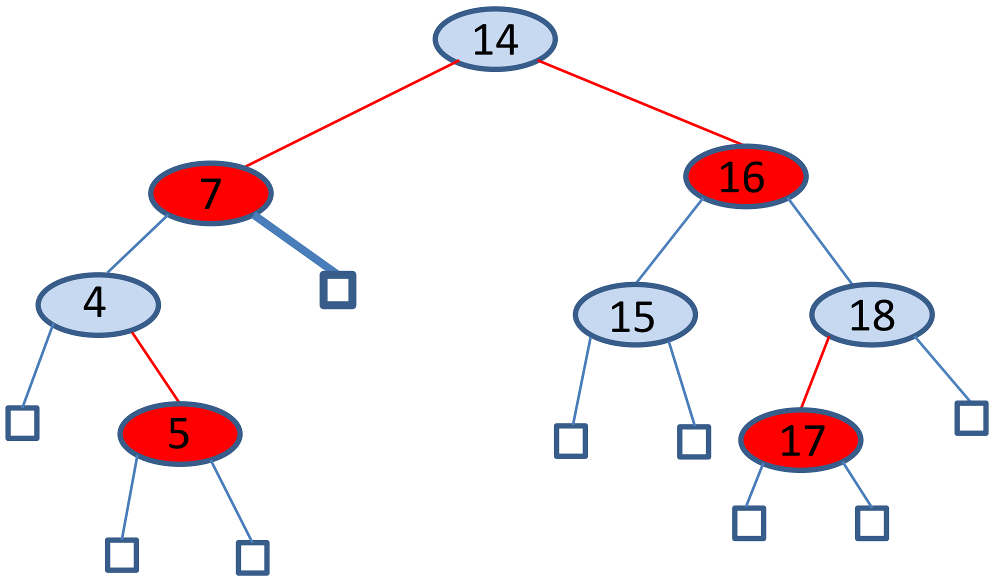


Remove 12

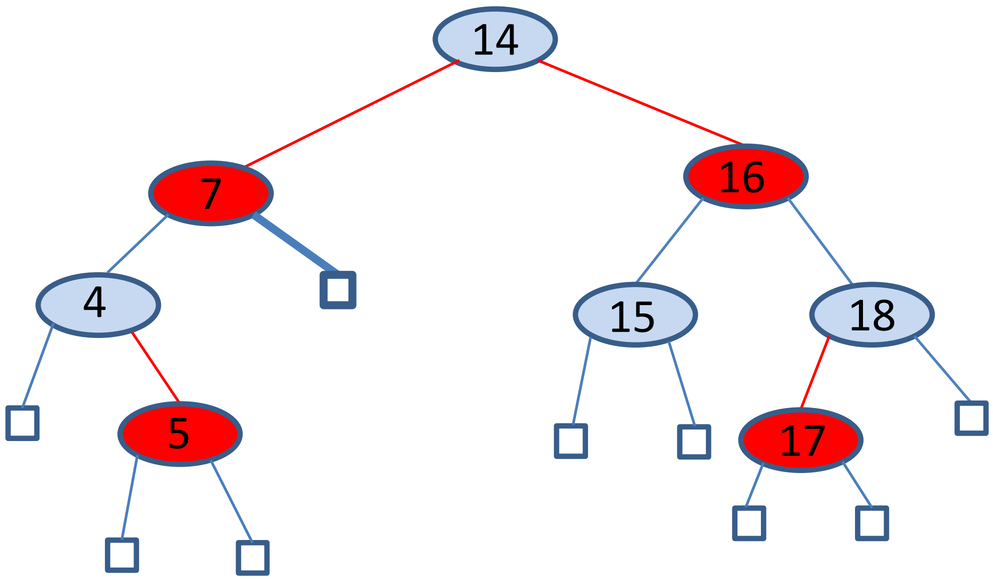


- A black node is removed hence a double black will be created.

After Removing 12 – Double Black

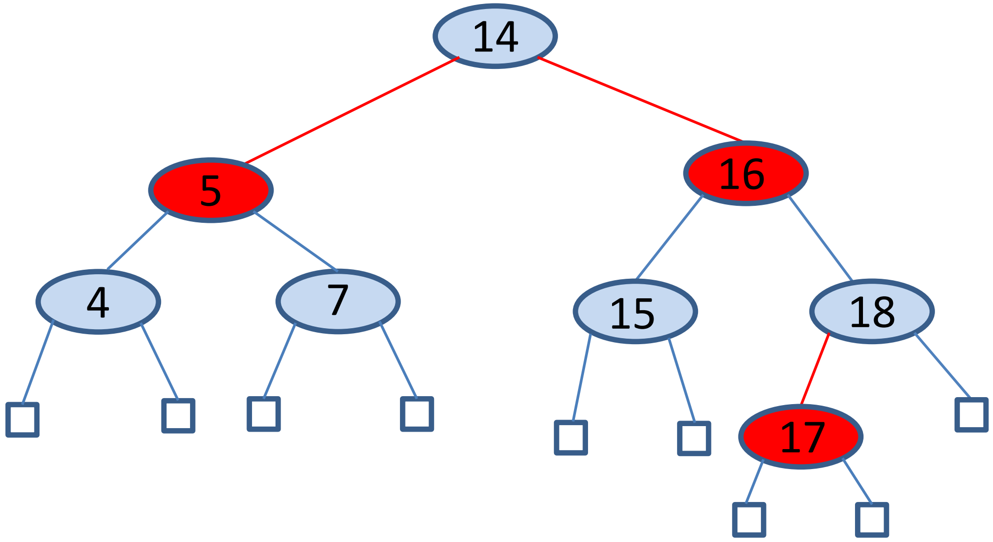


Double Black

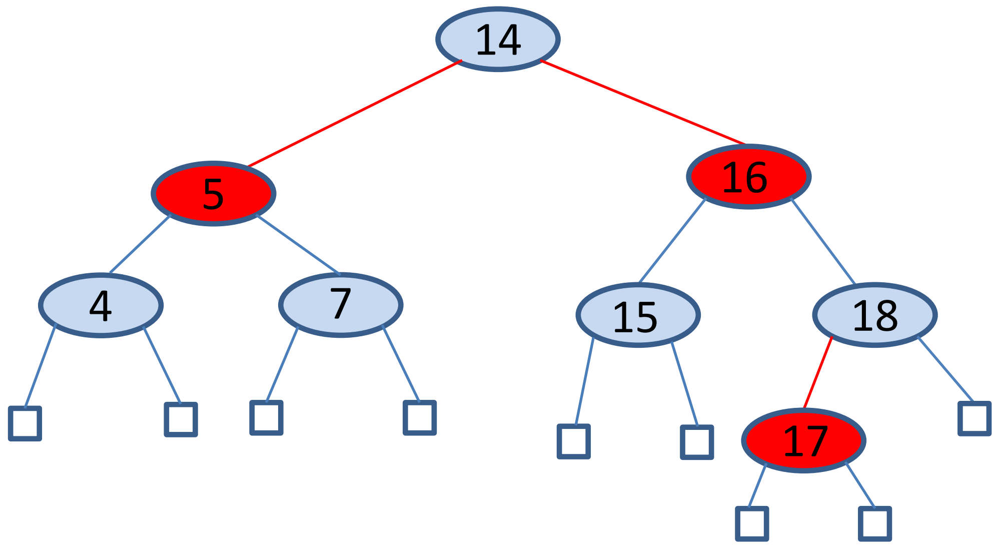


- We are in Case 1. We need to do trinode restructuring (double left-right rotation at 4 and 7).

After Restructuring

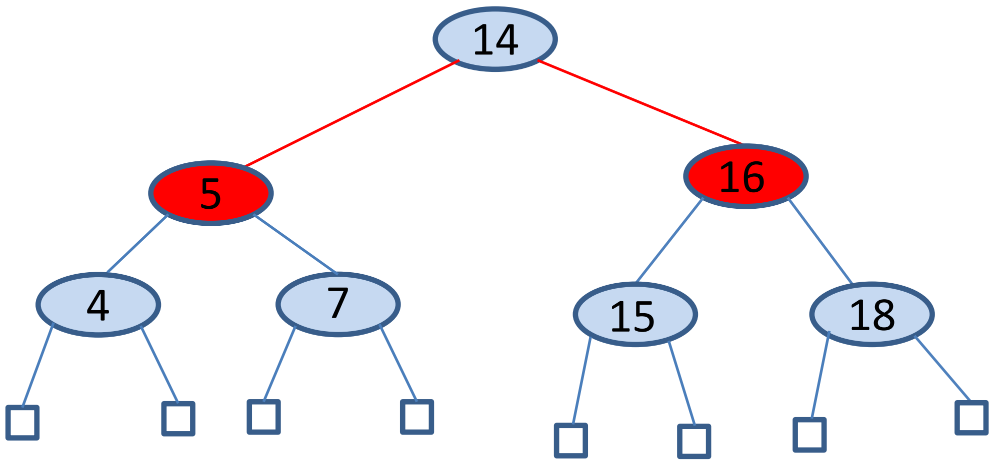


Remove 17

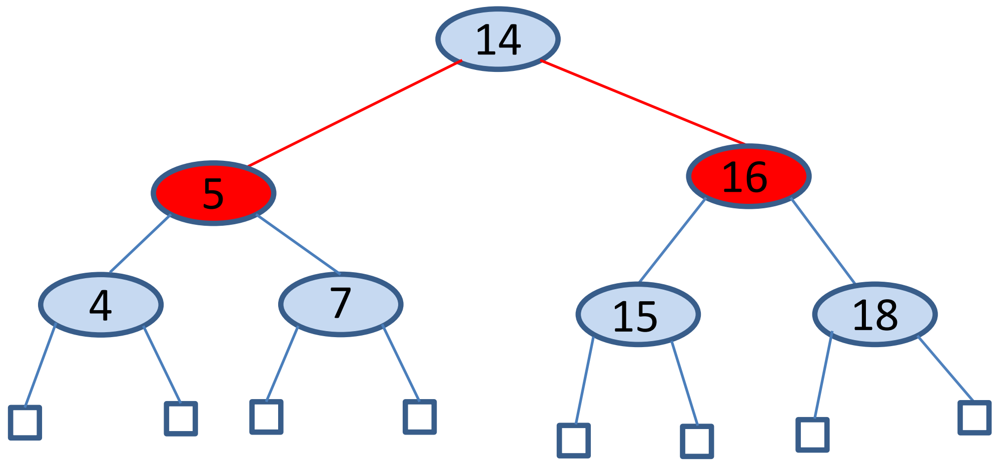


- Easy.

After Removing 17

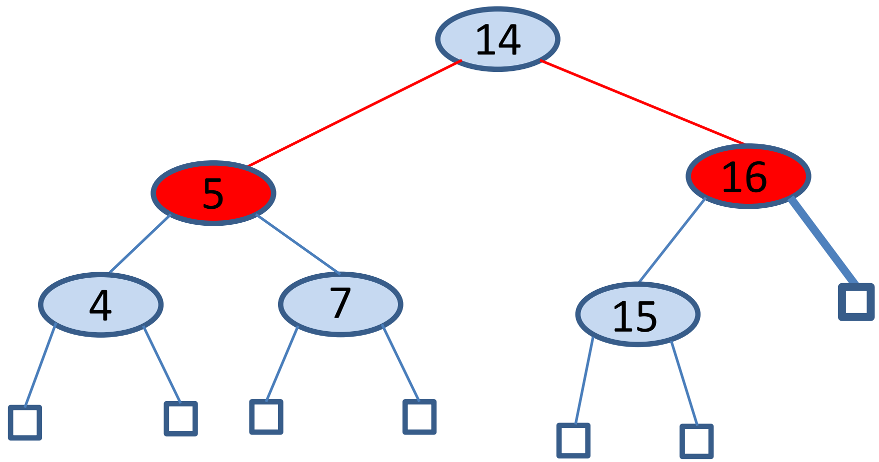


Remove 18



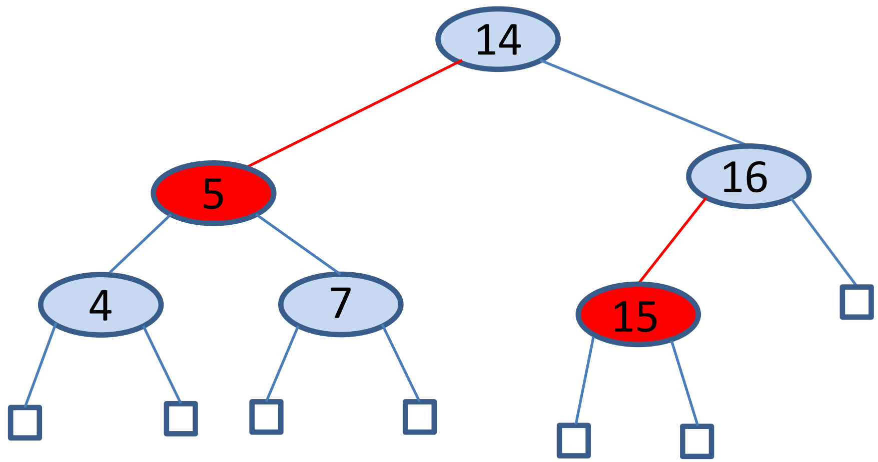
- A black node is removed hence a double black is created.

After Removing 18 – Double Black

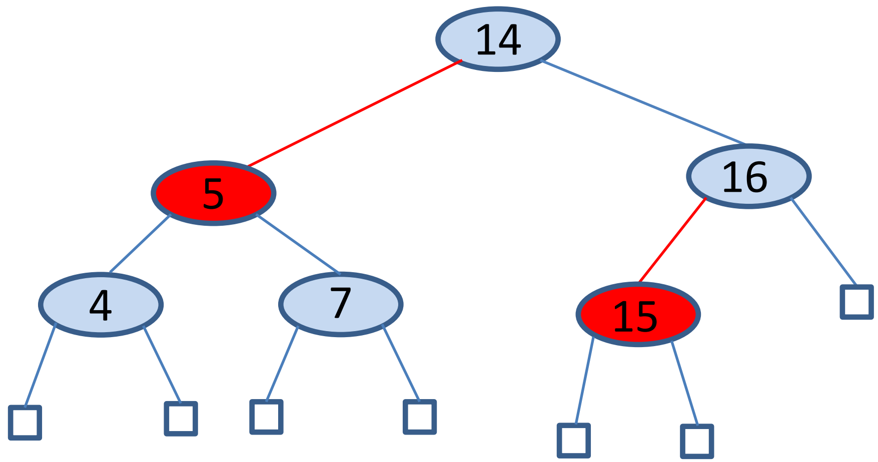


- We are in Case 2. We will do a recoloring.

After Recoloring

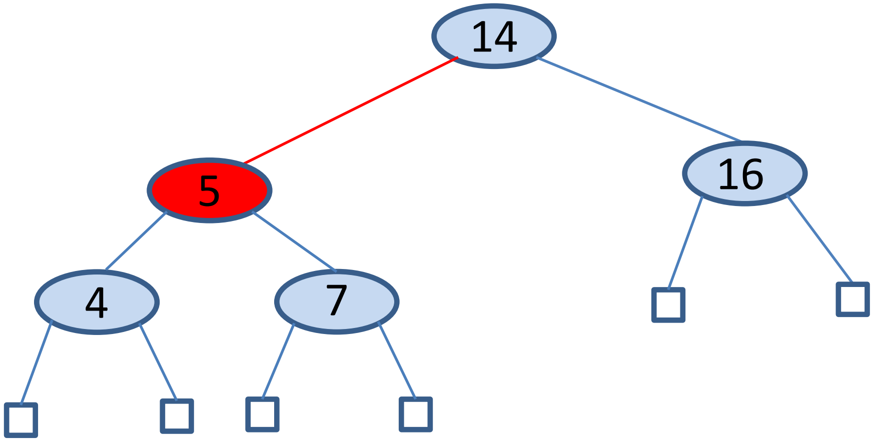


Remove 15

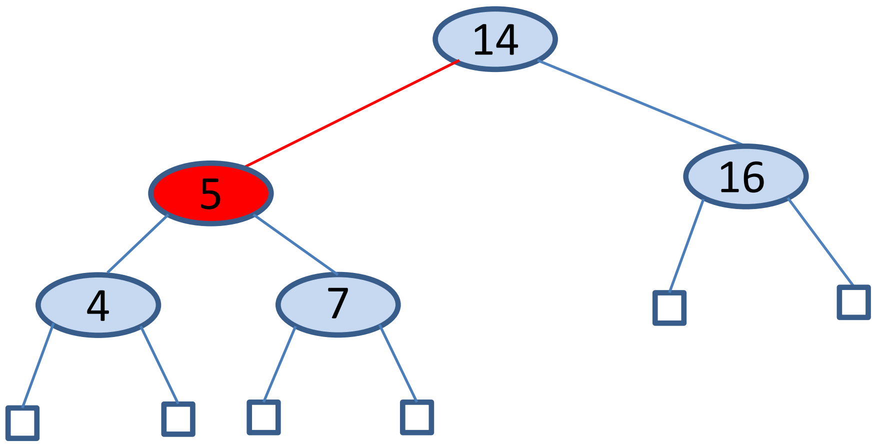


- Easy.

After Removing 15

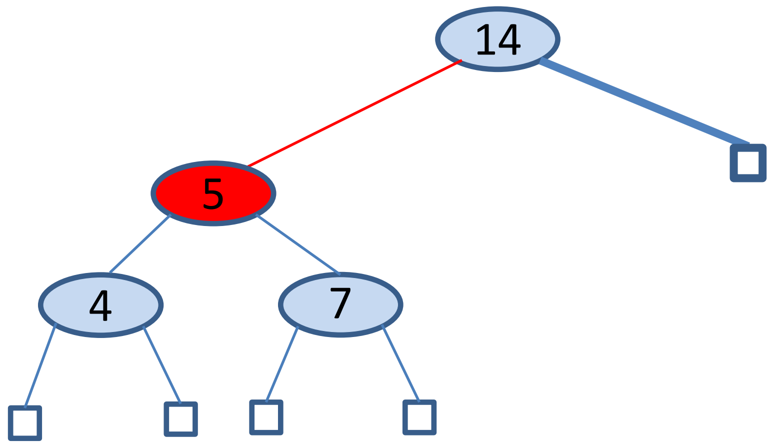


Remove 16



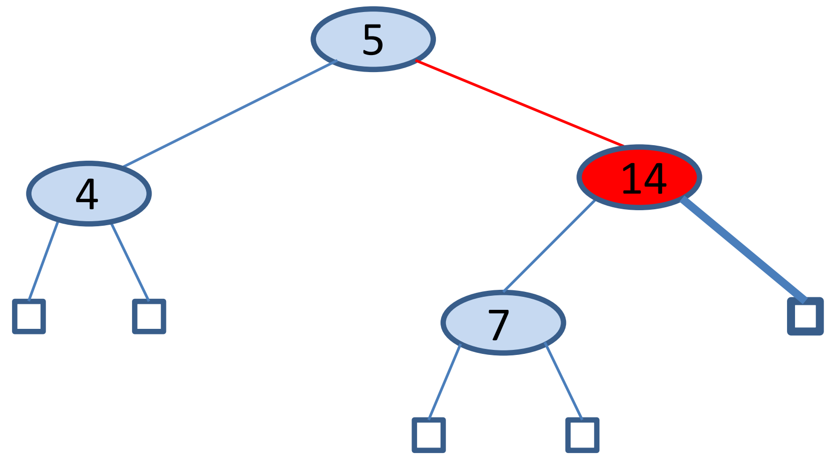
- A black node is removed hence a double black will be created.

After Removing 16 – Double Black



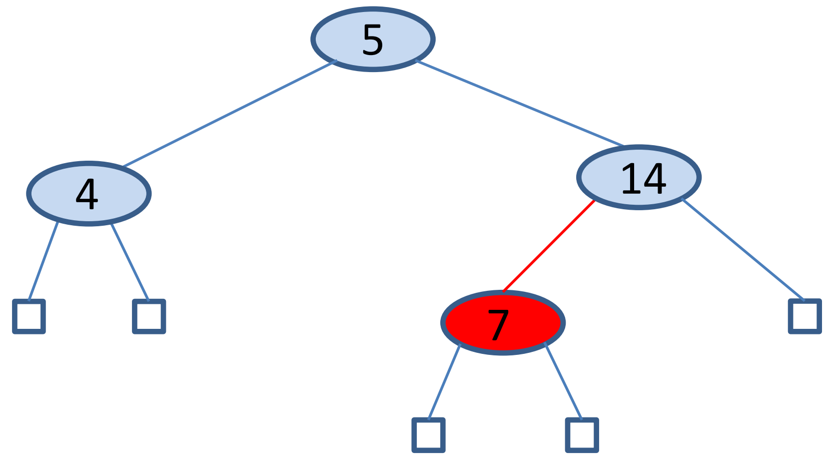
- We are in Case 3. We will do an adjustment (right rotation at 14).

After the Adjustment – Double Black

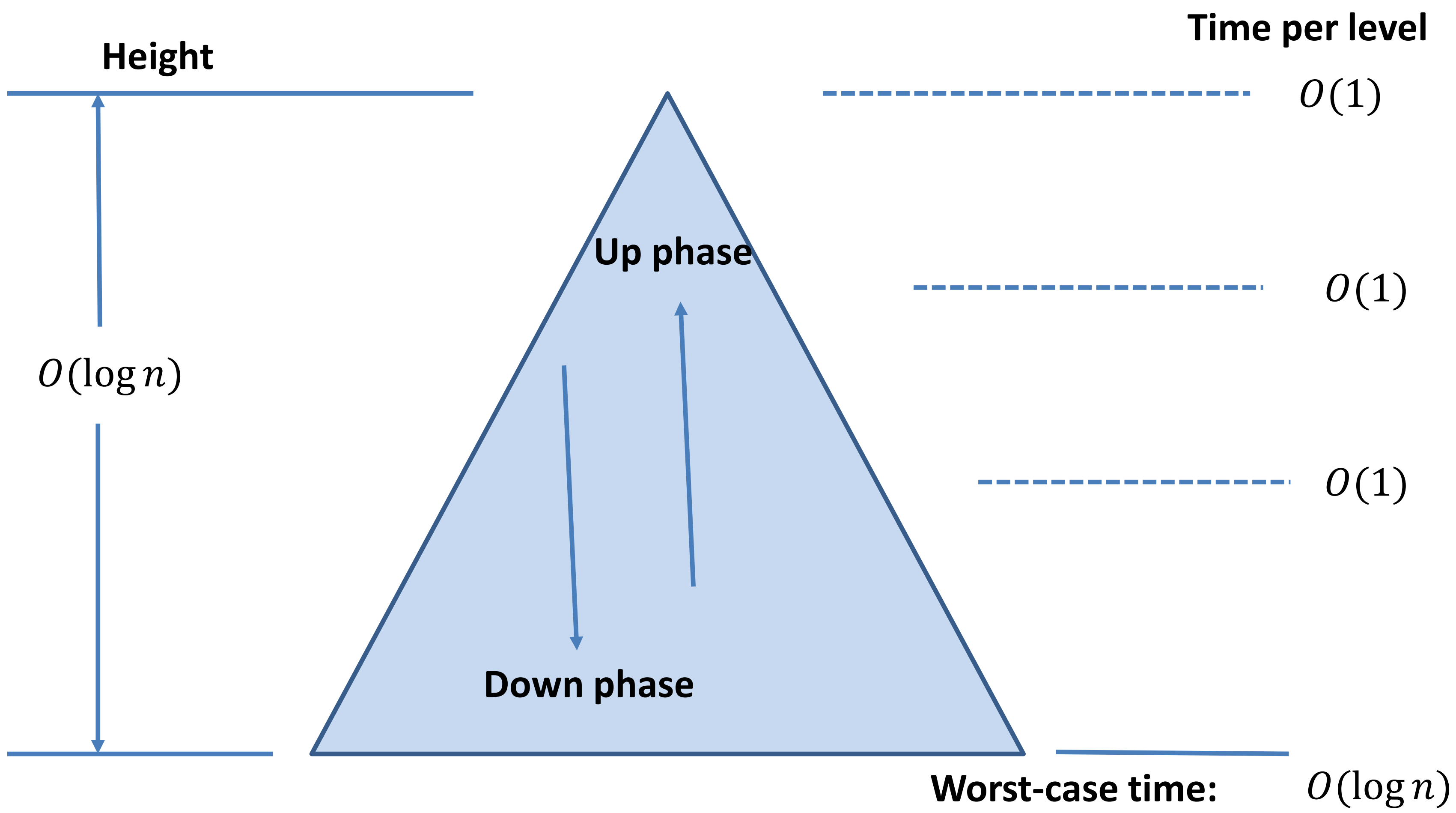


- We are in Case 2. Will do a recoloring.

After the Recoloring



Complexity of Operations in a Red-Black Tree



Summary

- The red-black tree data structure is **slightly more complicated** than its corresponding (2,4) tree.
- However, the red-black tree has the conceptual advantage that only a **constant number of trinode restructurings** are ever needed to restore the balance after an update.

Readings

- M. T. Goodrich, R. Tamassia and Michael H. Goldwasser. *Data Structures and Algorithms in Java. 6th* edition. John Wiley and Sons, 2014.
  - Section 11.6
- R. Sedgewick. Αλγόριθμοι σε C.
  - Κεφ. 13.4