Red-Black Trees

Manolis Koubarakis

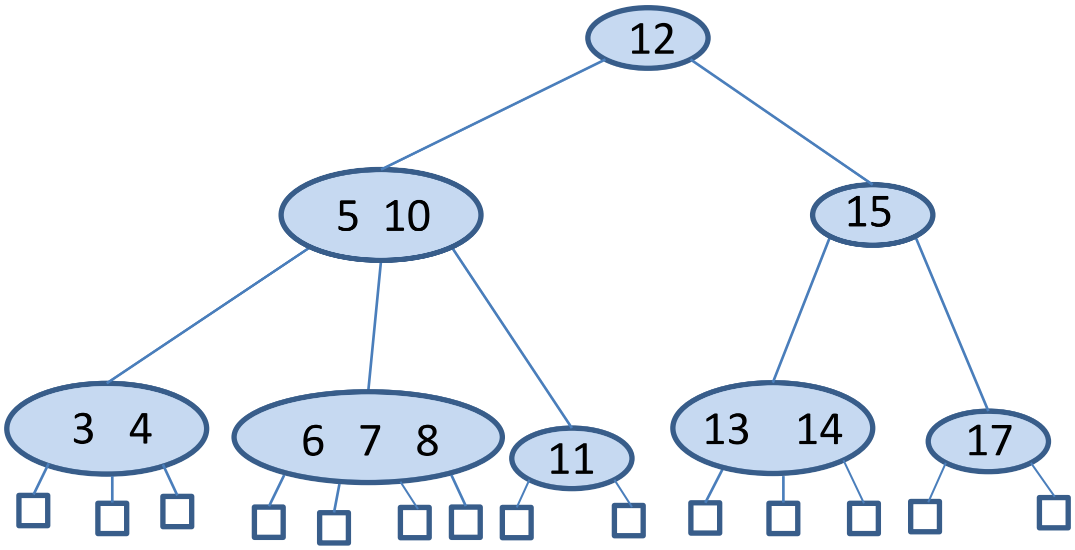
Red-Black Trees

- AVL trees and (2,4) trees have very nice properties, but:
  - AVL trees might need many rotations after a removal
  - (2,4) trees might require many split or fusion operations after an update
- **Red-black trees** are a data structure which requires only $\boldsymbol{O(1)}$ structural changes after an update in order to remain balanced.

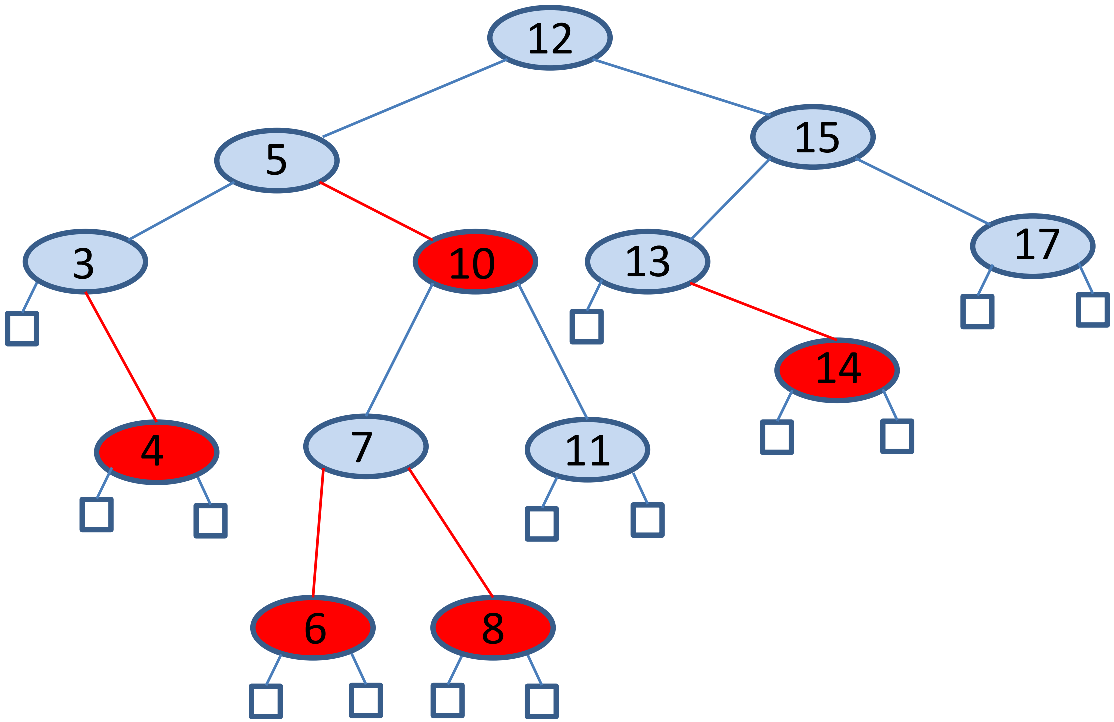
Definition

- A **red-black tree** is a binary search tree with nodes colored red and black in a way that satisfies the following properties:
  - **Root Property**: The root is black.
  - **External Property**: Every external node is black.
  - **Internal Property**: The children of a red node are black.
  - **Depth Property**: All the external nodes have the same **black depth**, defined as the number of black ancestors minus one (recall that a node is an ancestor of itself).

Example (2,4) Tree



Corresponding Red-Black Tree

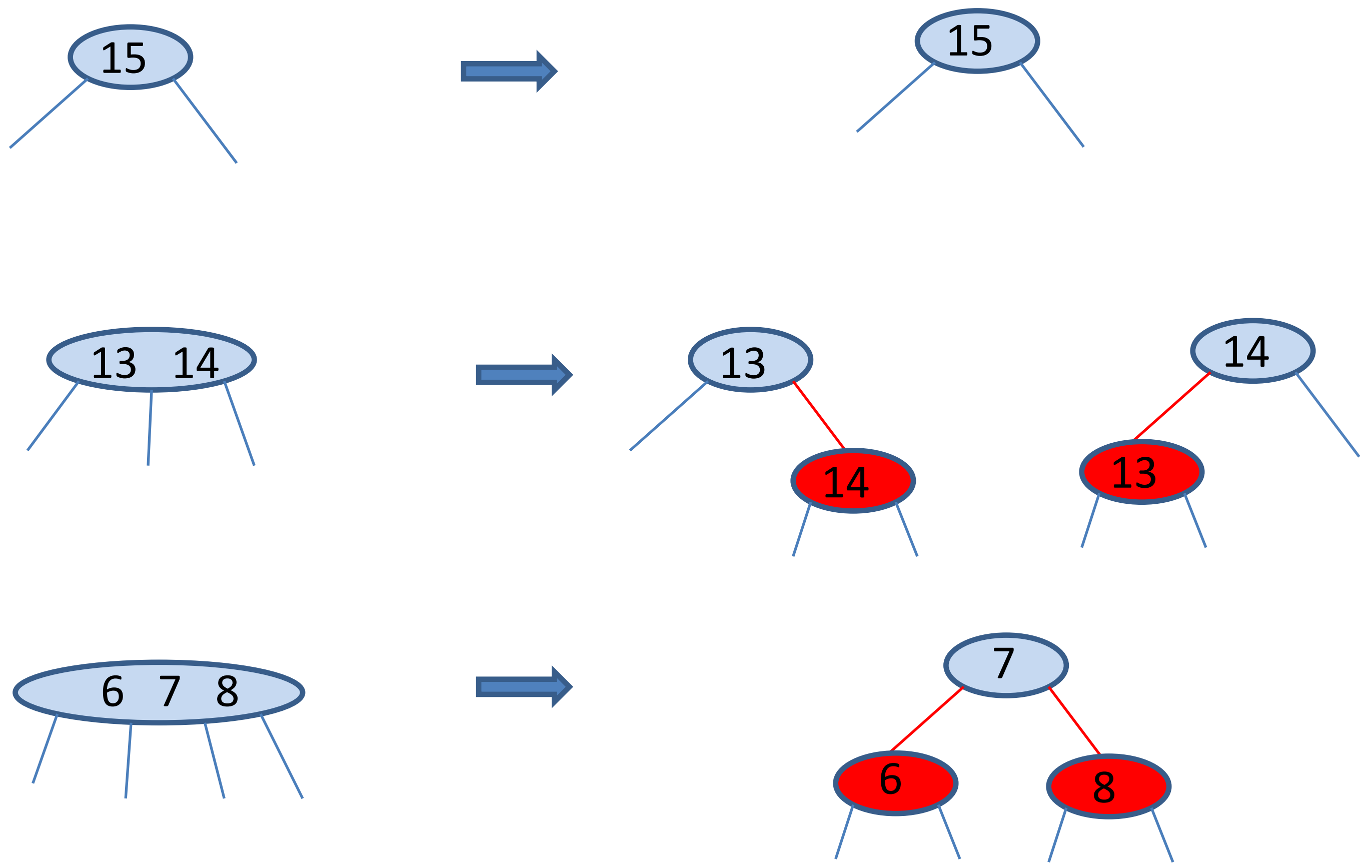


In our figures, we use **light blue color instead of black**.

(2,4) Trees vs. Red-Black Trees

- Given a red-black tree, we can construct a corresponding (2,4) tree by merging every red node v into its parent and storing the entry from v at its parent.
- Given a (2,4) tree, we can transform it into a red-black tree by performing the following transformations for each internal node v:
  - If v is a 2-node, then keep the (black) children of v as is.
  - If v is a 3-node, then create a new red node w, give v's first two (black) children to w, and make w and v's third child be the two children of v (the symmetric operation is also possible; see next slide).
  - If v is a 4-node, then create two new red nodes w and z, give v's first two (black) children to w, give v's last two (black) children to z, and make w and z be the two children of v.

(2,4) Trees vs. Red-Black Trees (cont'd)



Proposition

- The height of a red-black tree storing n entries is $O(\log n)$.
- Proof?

Proof

- Let T be a red-black tree storing n entries, and let h be the height of T. We will prove the following:

$$\log(n+1) \leq h \leq 2\log(n+1)$$

- Let d be the common black depth of all the external nodes of T. Let T' be the (2,4) tree associated with T, and let h' be the height of T'.
- Because of the correspondence between red-black trees and (2,4) trees, we know that $h' = d$.
- Hence, $d = h' \leq \log(n+1)$ by the proposition for the height of (2,4) trees. By the internal node property of red-black trees, we have $h \leq 2d$. Therefore, $h \leq 2\log(n+1)$.

Proof (cont'd)

- The other inequality, $\log(n+1) \leq h$ follows from the properties of proper binary trees and the fact that T has n internal nodes.

Updates

- Performing update operations in a red-black tree is similar to the operations of binary search trees, but we must additionally take care not to destroy the color properties.
- For an update operation in a red-black tree T, it is important to keep in mind the correspondence with a (2,4) tree T' and the relevant update algorithms for (2,4) trees.

Insertion

- Let us consider the insertion of a new entry with key k into a red-black tree T.
- We search for k in T until we reach an external node of T, and we replace this node with an internal node z, storing (k, i) and having two external-node children.
- If z is the root of T, we color z black, else we color z red. We also color the children of z black.
- This operation corresponds to inserting (k, i) into a node of the (2,4) tree T' with external-node children.
- This operation preserves the root, external, and depth properties of T, but **it might violate the internal property.**

Insertion (cont'd)

- Indeed, if z is not the root of T and the parent v of z is red, then we have **a parent and a child that are both red.**
- In this case, by the root property, v cannot be the root of T.
- By the internal property (which was previously satisfied), the parent u of v must be black.
- Since z and its parent are red, but z's grandparent u is black, we call this violation of the internal property a **double red** at node z.

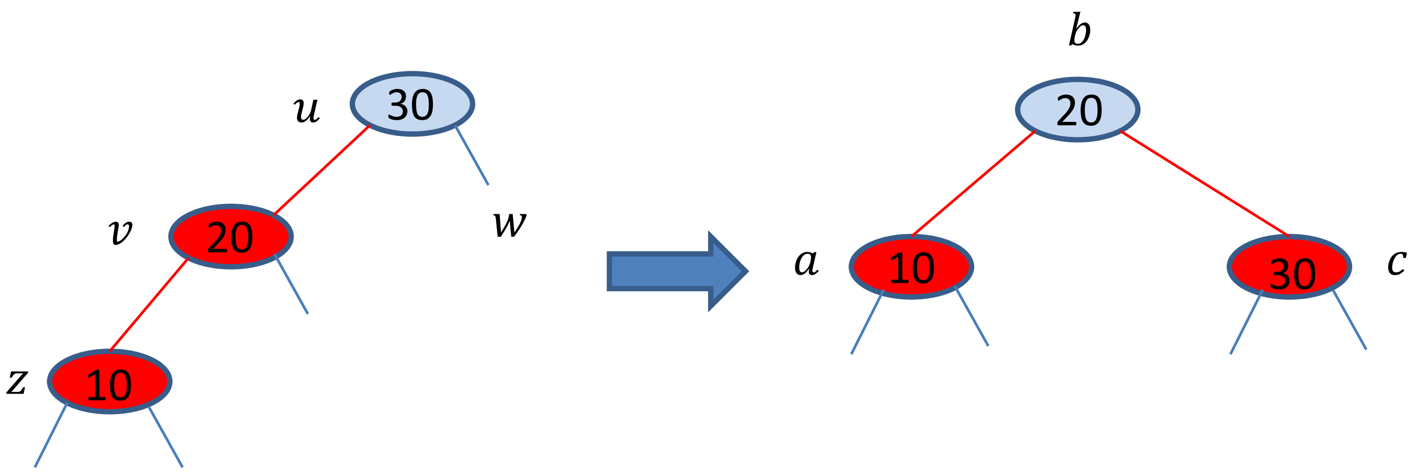
Insertion (cont'd)

- To remedy a double red, we consider two cases.
- **Case 1: the sibling w of v is black**. In this case, the double red denotes the fact that we have created in our red-black tree T a **malformed** replacement for a corresponding 4-node of the (2,4) tree T', which has as its children the four black children of u, v and z.
- Our malformed replacement has one red node (v) that is the parent of another red node (z) while we want it to have **two red nodes as siblings** instead.
- To fix this problem, we perform a **trinode restructuring** of T as follows.

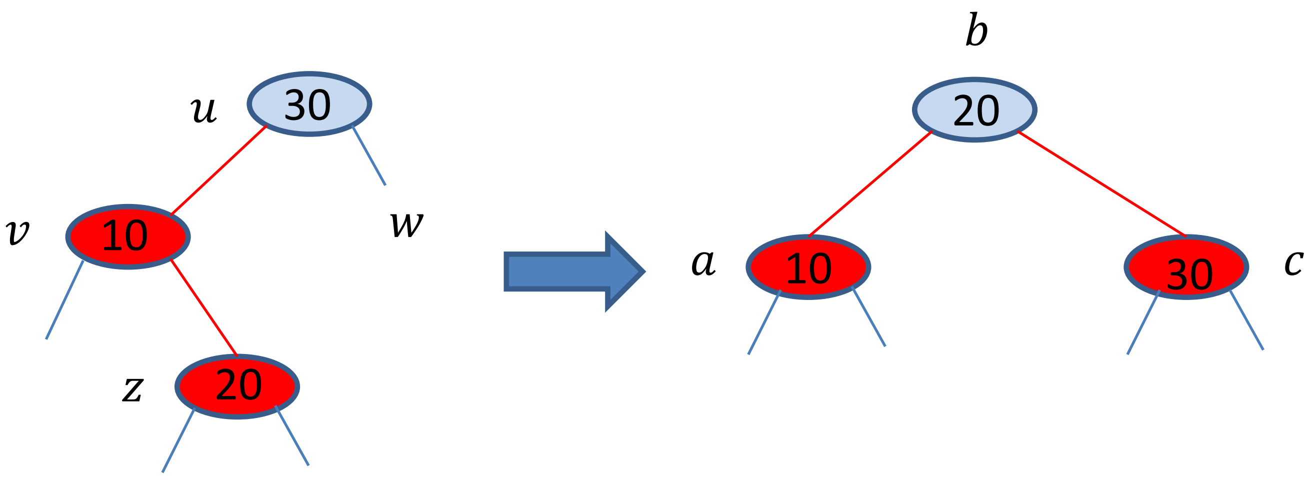
Trinode Restructuring

- Take node z, its parent v, and grandparent u, and temporarily relabel them as a, b and c, in left-to-right order, so that a, b and c will be visited in this order by an **inorder** tree traversal.
- Replace the grandparent u with the node labeled b, and make nodes a and c the children of b keeping inorder relationships unchanged.
- After restructuring, we color b black and we color a and c red. Thus, the restructuring **eliminates** the double red problem.

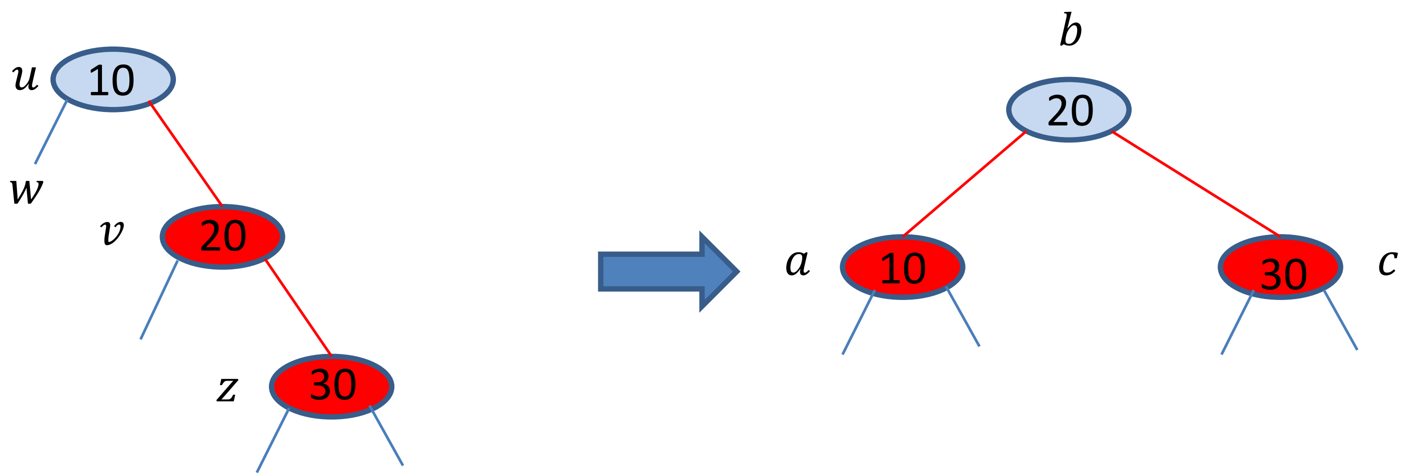
Trinode Restructuring Graphically



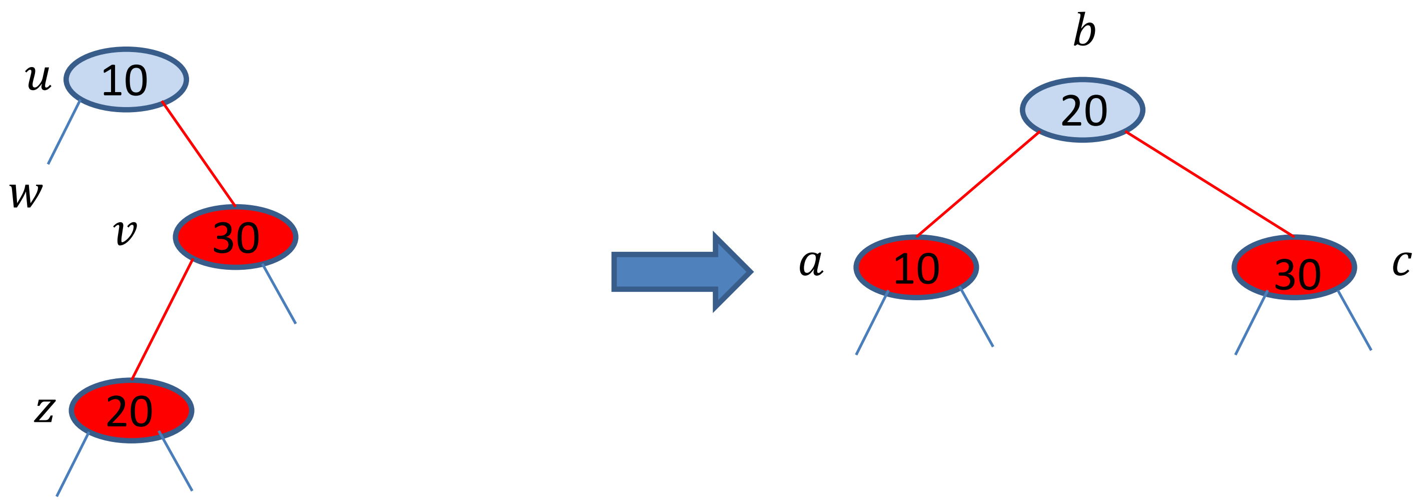
Trinode Restructuring Graphically (cont'd)



Trinode Restructuring Graphically (cont'd)



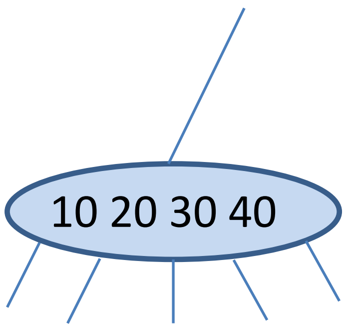
Trinode Restructuring Graphically (cont'd)

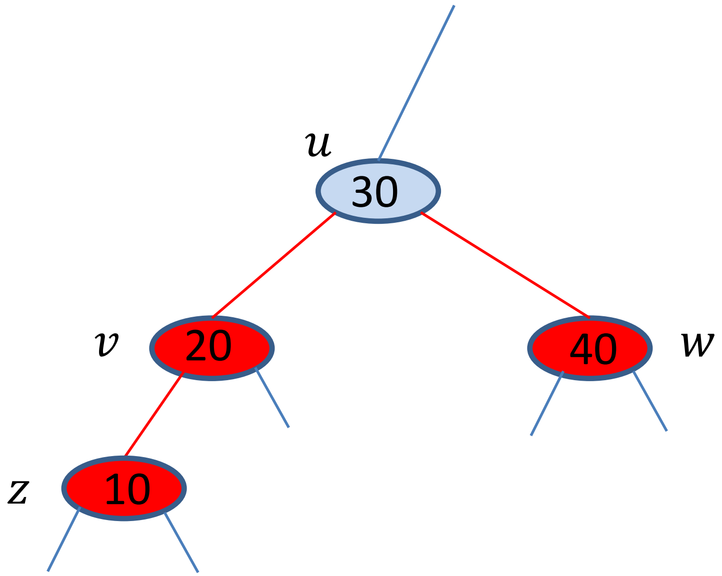


Insertion (cont'd)

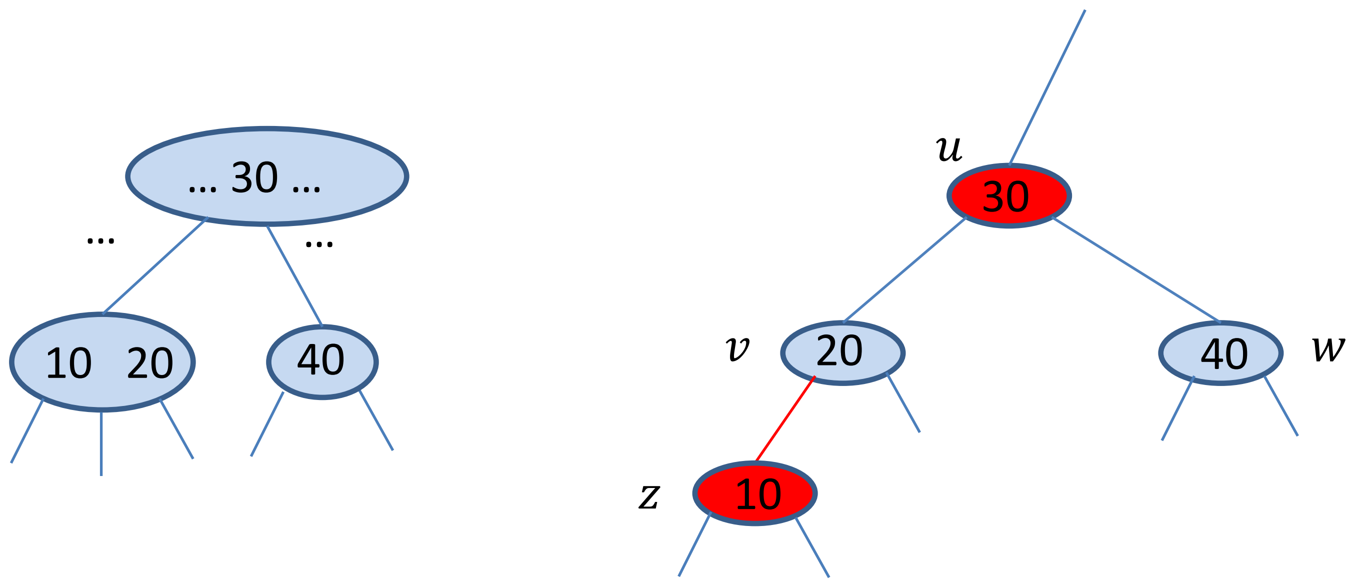
- **Case 2: the sibling $\boldsymbol{w}$ of $\boldsymbol{v}$ is red**. In this case, the double red denotes an overflow in the corresponding (2,4) tree T'. To fix the problem, we perform the equivalent of a split operation. Namely, we do a **recoloring**: we color v and w black and their parent u red (unless u is the root, in which case it is colored black).
- It is possible that, after such a recoloring, the double red problem **reappears** at u (if u has a red parent). Then, we repeat the consideration of the two cases.
- Thus, a recoloring either eliminates the double red problem at node z or propagates it to the grandparent u of z.
- We continue going up T performing recoloring until we finally resolve the double red problem (either with a final recoloring or a trinode restructuring).
- Thus, the number of recolorings caused by insertion is no more than half the height of tree T, that is, no more than $\log(n+1)$ by the previous proposition.

Recoloring





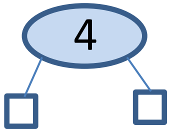
Recoloring (cont'd)



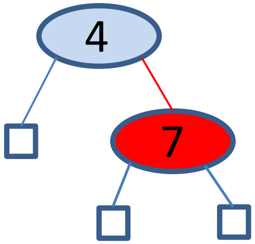
Example

- Let us now see some examples of insertions in an initially empty red-black tree.

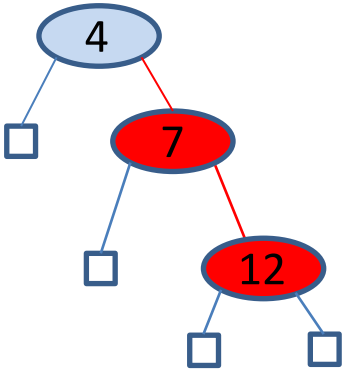
Insert 4



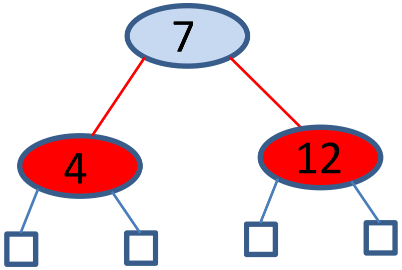
Insert 7



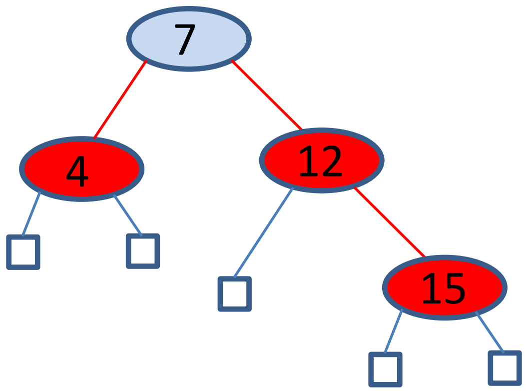
Insert 12 – Double Red



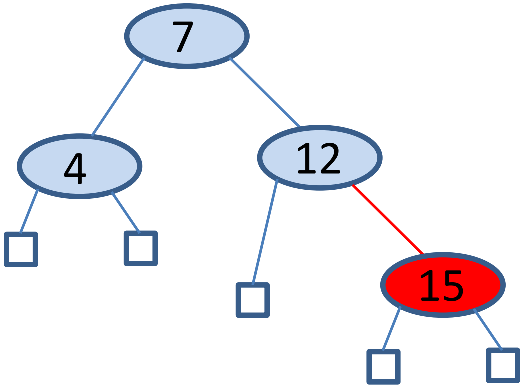
After Restructuring



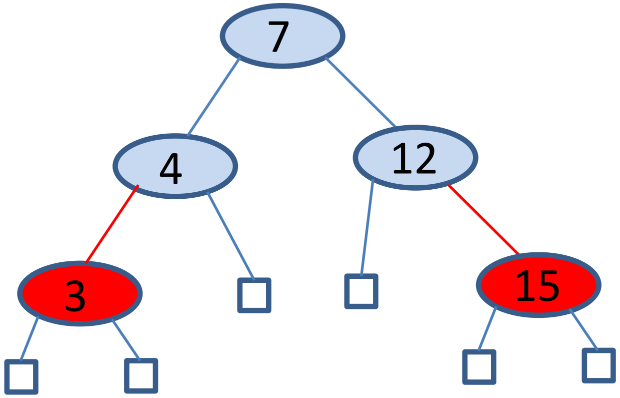
Insert 15 – Double Red



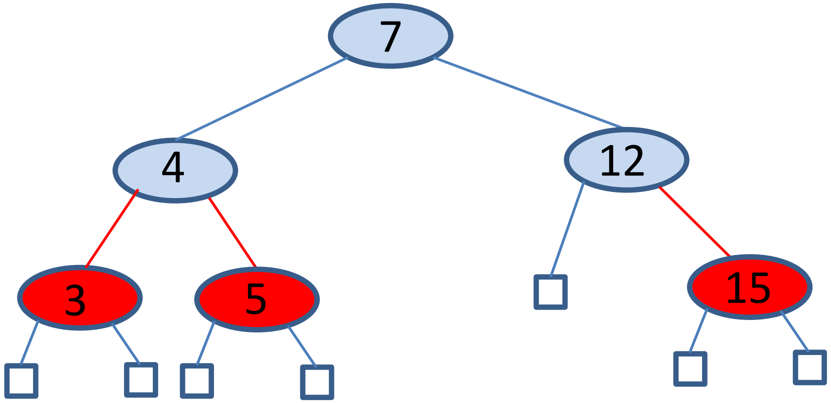
After Recoloring



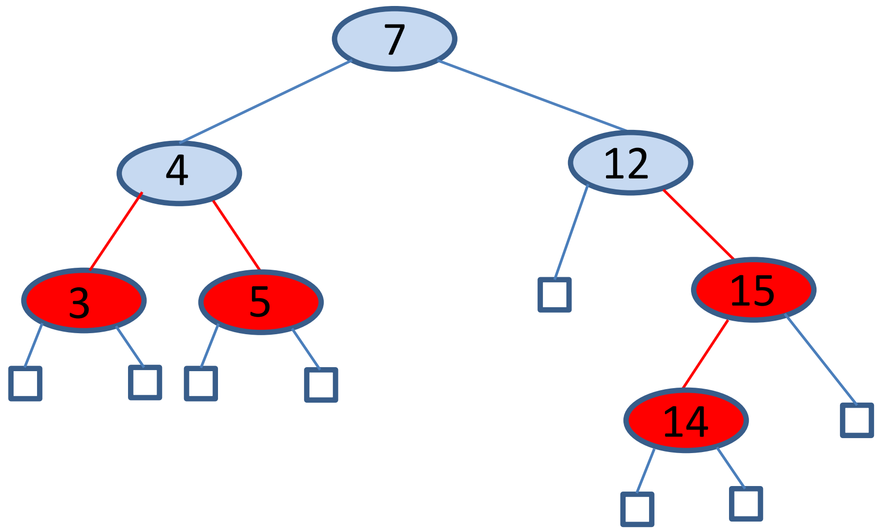
Insert 3



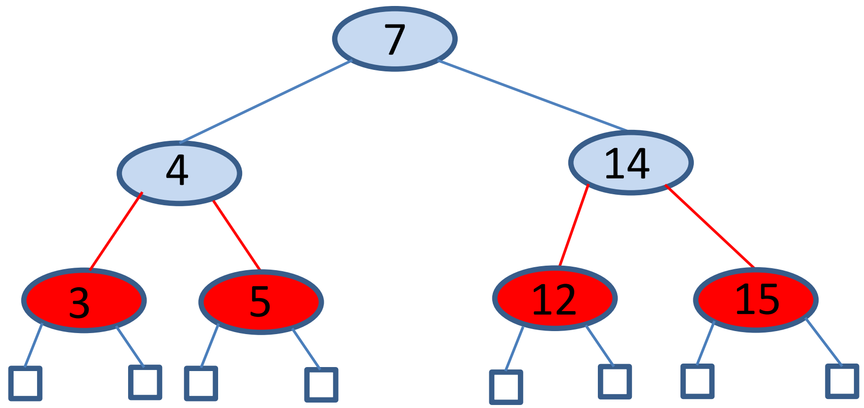
Insert 5



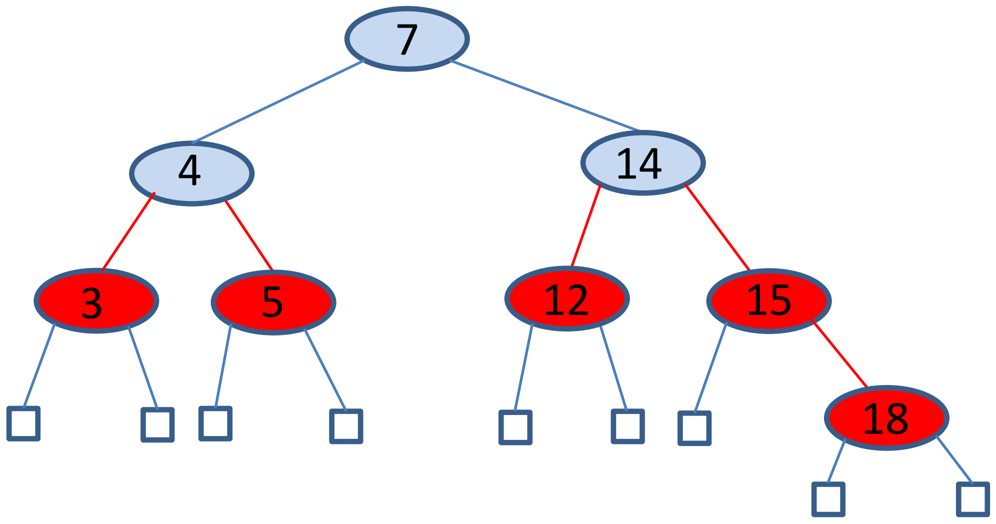
Insert 14 – Double Red



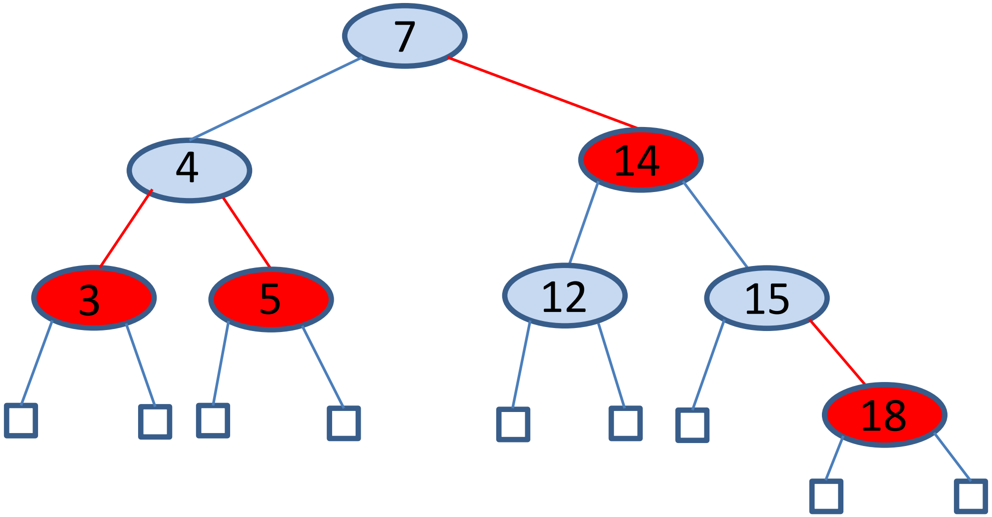
After Restructuring



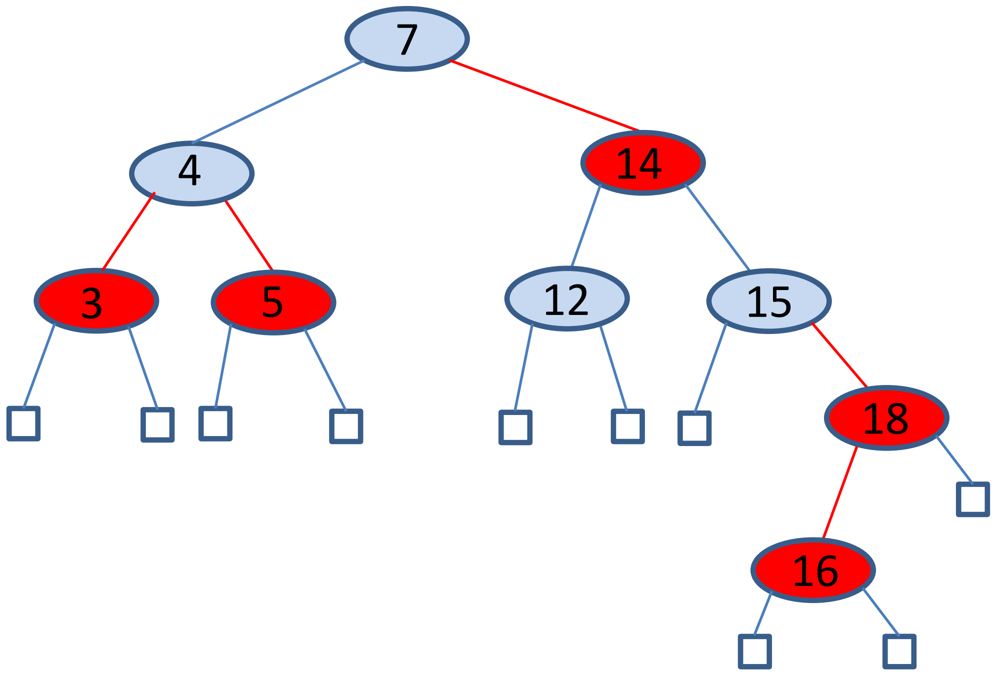
Insertion of 18 – Double Red



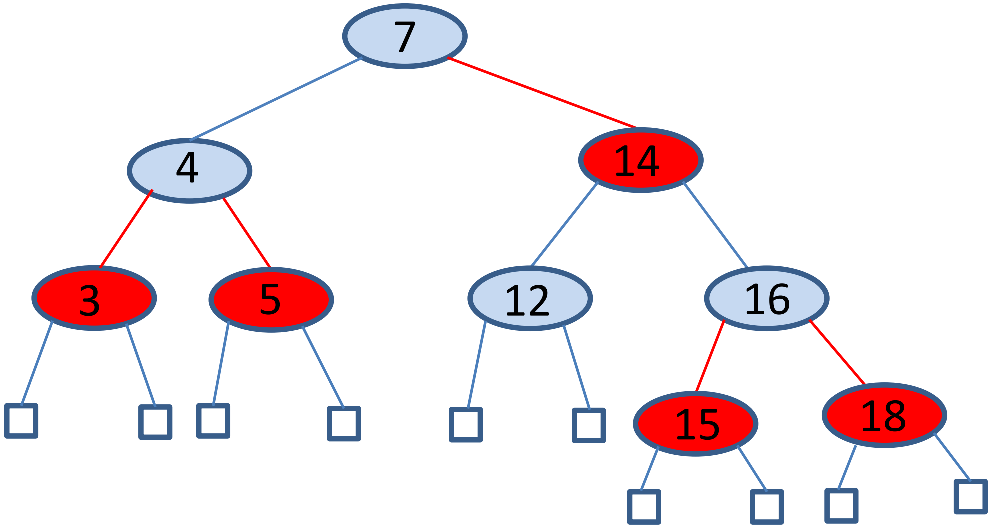
After Recoloring



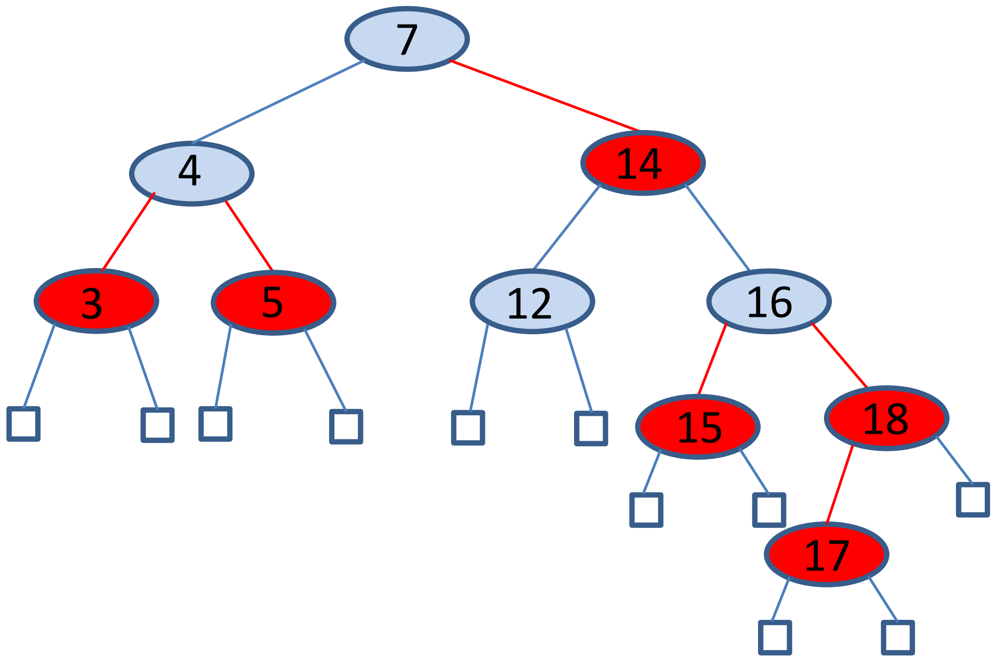
Insertion of 16 – Double Red



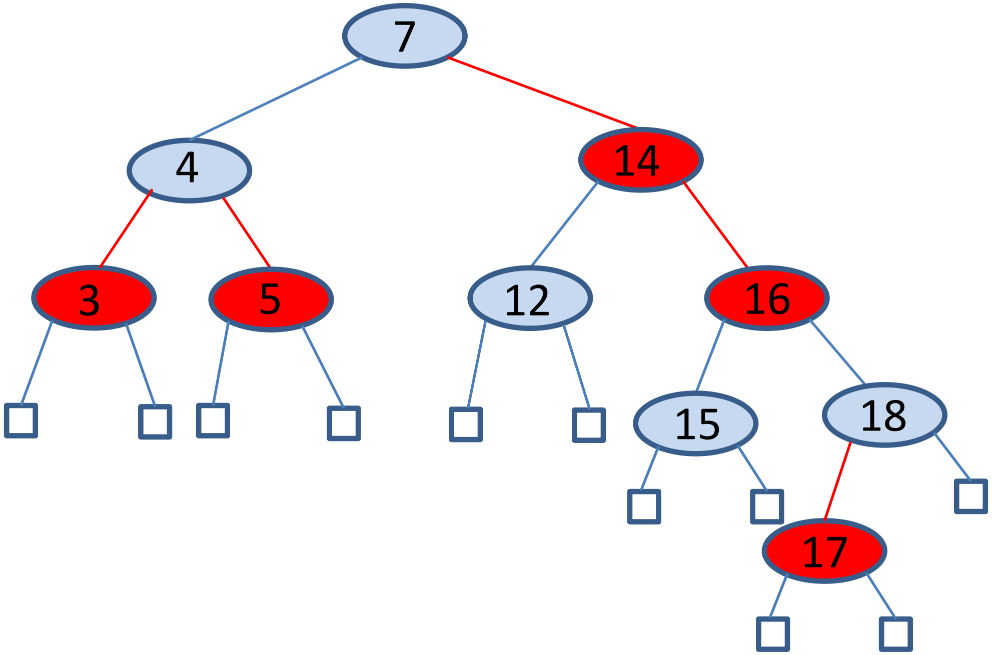
After Restructuring



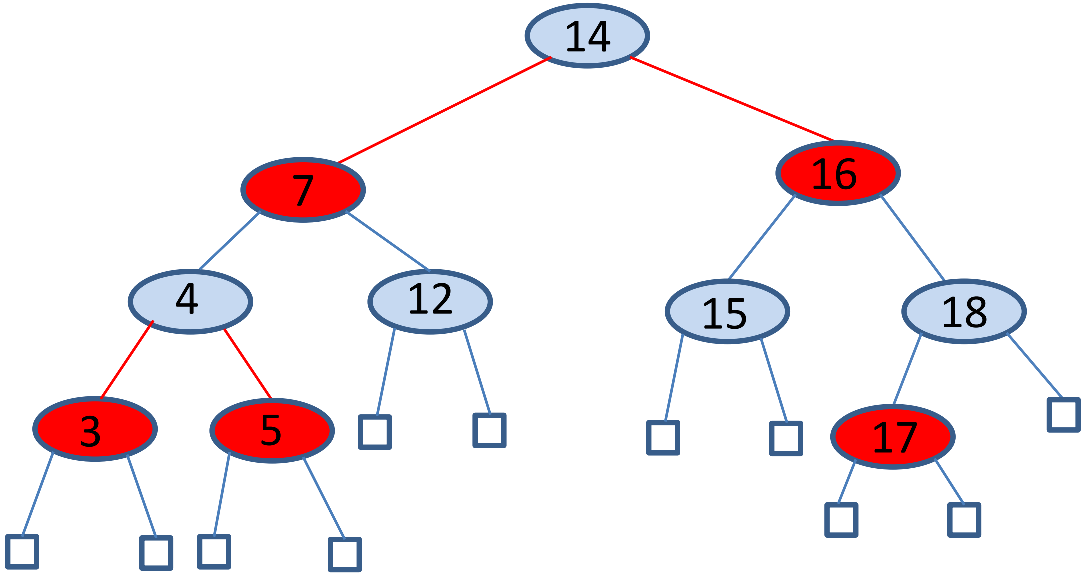
Insertion of 17 – Double Red



After Recoloring – Double Red



After Restructuring



Proposition

- The insertion of a key-value entry in a red-black tree storing n entries can be done in $O\ (\log n)$ time and requires $O\ (\log n)$ recolorings and one trinode restructuring.

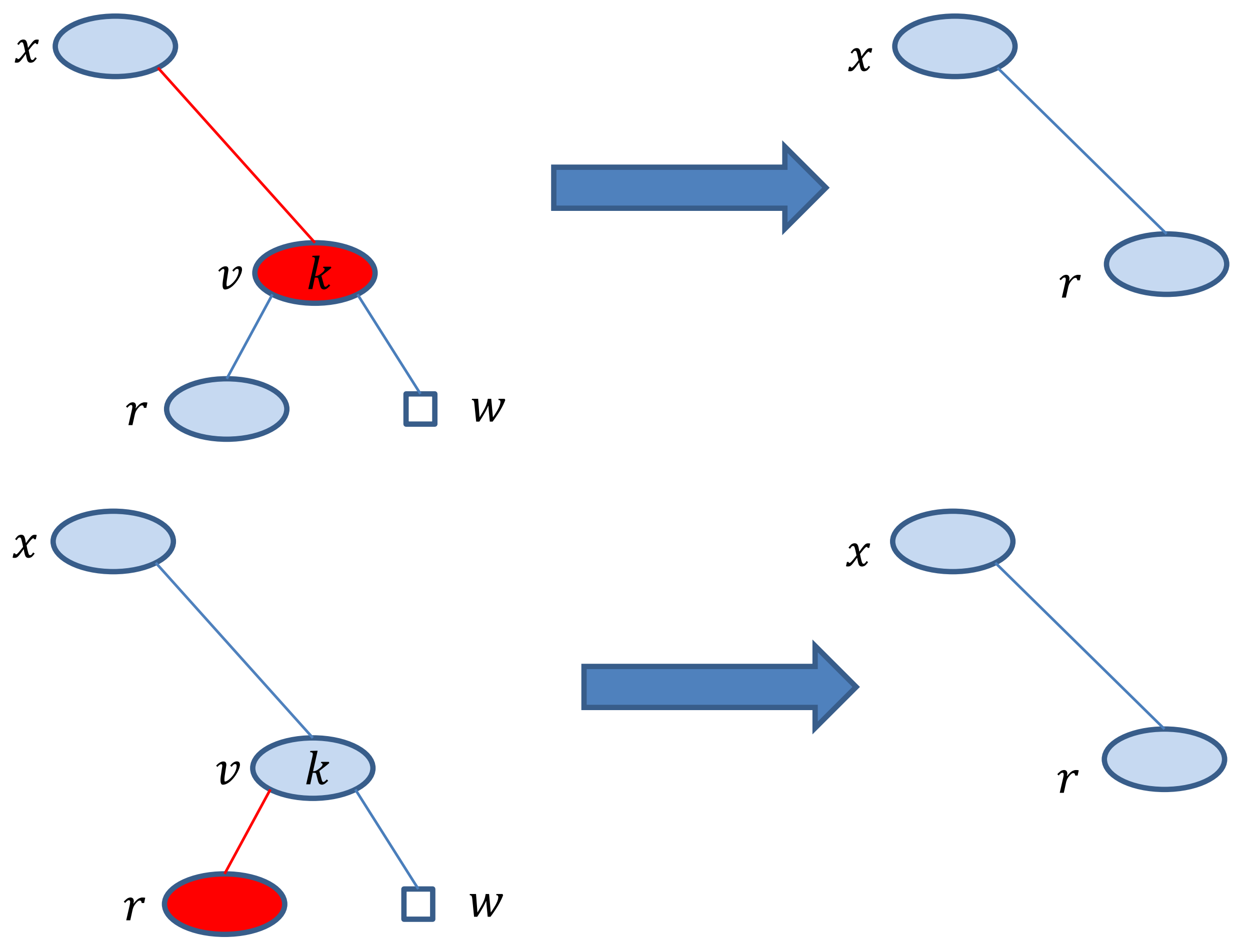
Removal

- Let us now remove an entry with key k from a red-black tree T.
- We proceed like in a binary tree search searching for a node u storing such an entry.
- If u does not have an external-node child, we find the internal node v following u in the inorder traversal of T. This node has an external-node child. We move the entry at v to u, and perform the removal at v.
- Thus, we may consider only the removal of an entry with key k stored at a node v with an external-node child w.

Removal (cont'd)

- To remove the entry with key k from a node v of T with an external-node child w, we proceed as follows.
- Let r be the sibling of w and x the parent of v. We remove nodes v and w, and make r a child of x.
- If v was red (hence r is black) or r is red (hence v was black), we color r black and we are done.

Graphically



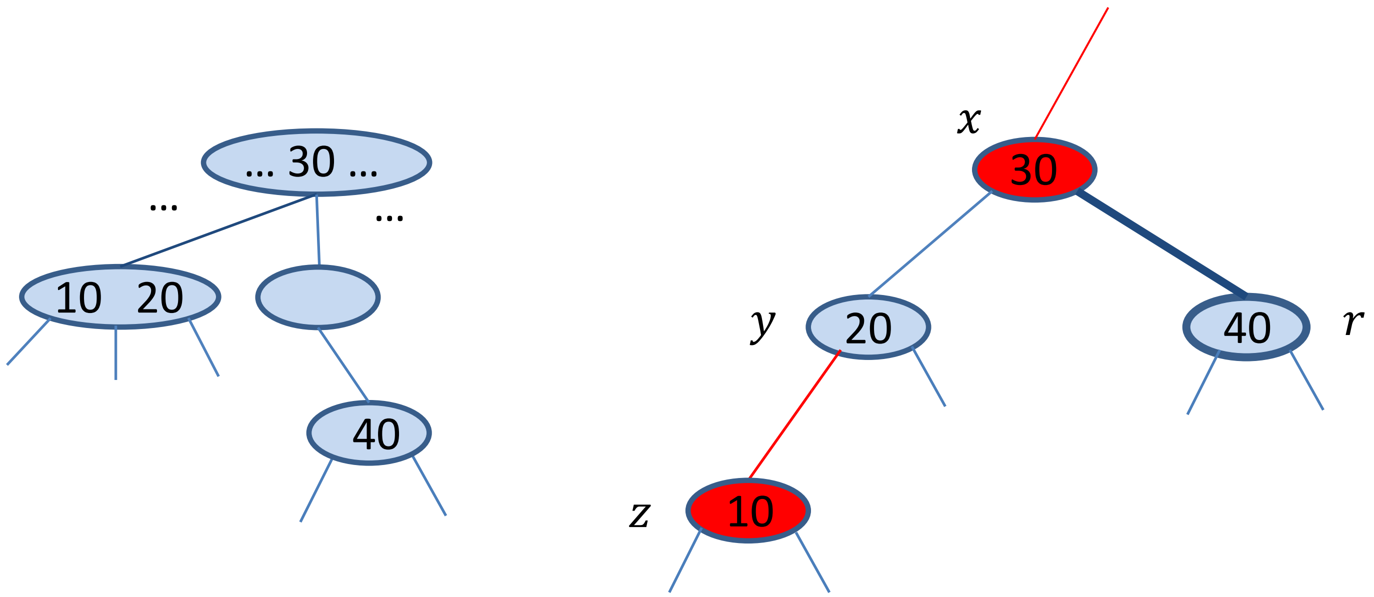
Removal (cont'd)

- If, instead, r is black and v is black, then, to preserve the **depth property**, we give r a fictitious **double black** color.
- We now have a color violation, called the **double black problem**.
- A double black in T denotes an **underflow** in the corresponding (2,4) tree T'.
- To remedy the double-black problem at r, we proceed as follows.

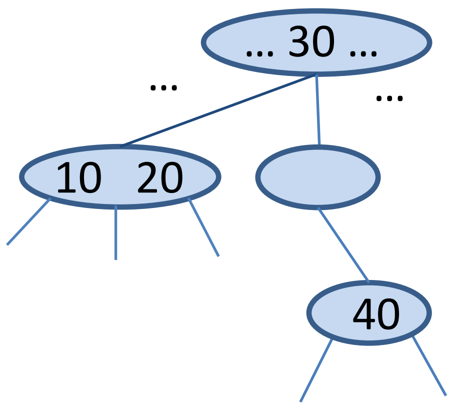
Removal (cont'd)

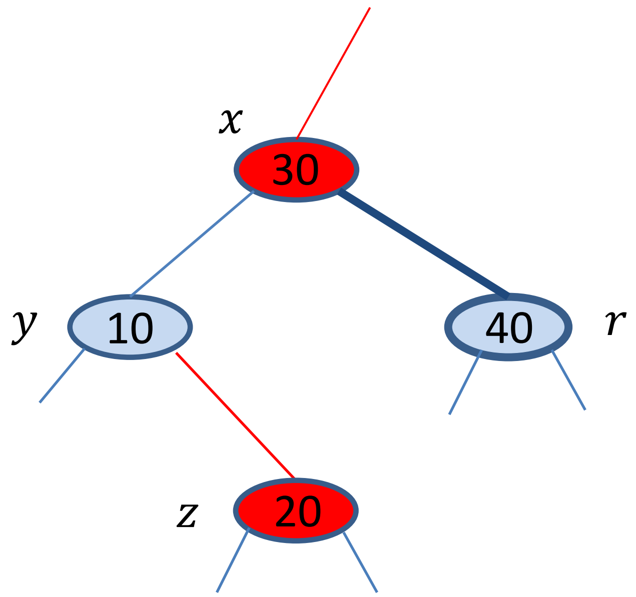
- **Case 1: the sibling y of r is black and has a red child z.**
- Resolving this case corresponds to a **transfer** operation in the (2,4) tree T'.
- We perform a **trinode restructuring**: we take the node z, its parent y, and grandparent x, we label them temporarily left to right as a, b and c, and we replace x with the node labeled b, making it parent of the other two nodes.
- We color a and c black, give b the former color of x, and color r black.
- This trinode restructuring eliminates the double black problem.

Restructuring a Red-Black Tree to Remedy the Double Black Problem

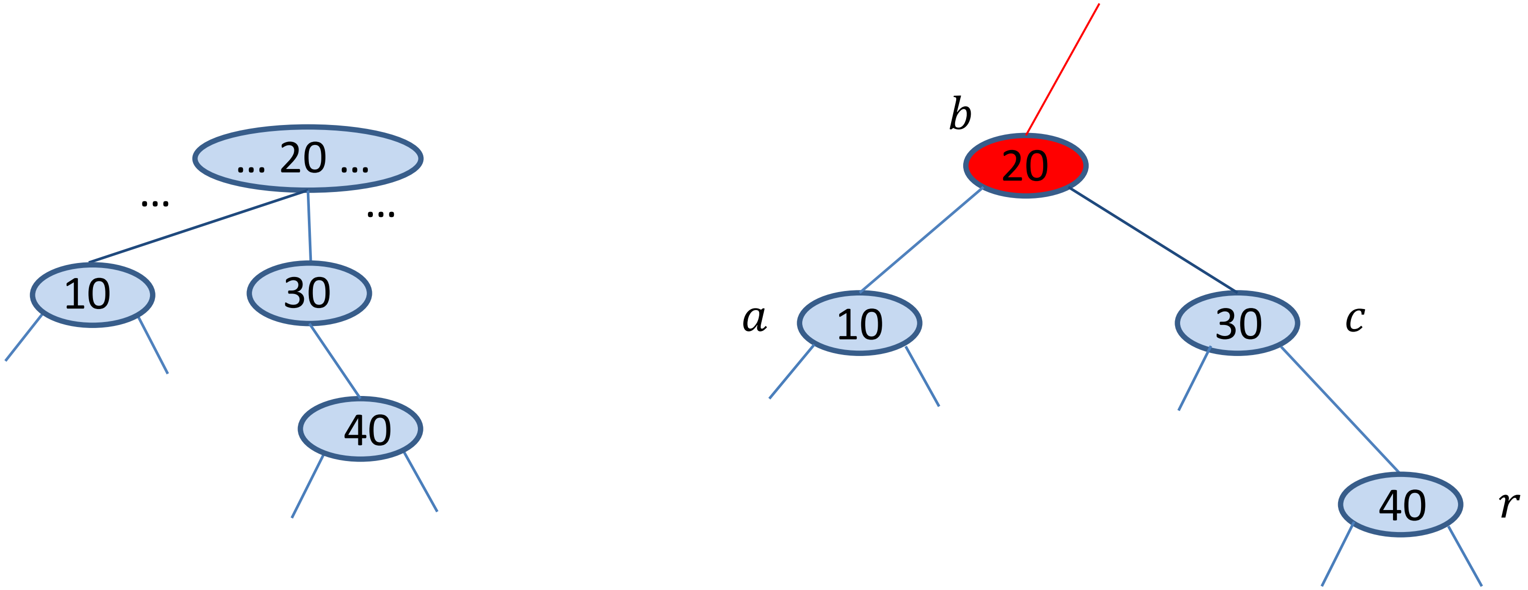


Restructuring (cont'd)





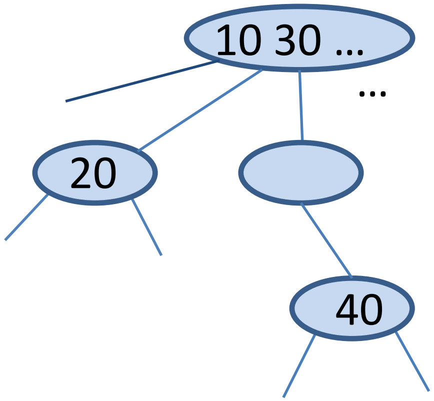
After the Restructuring

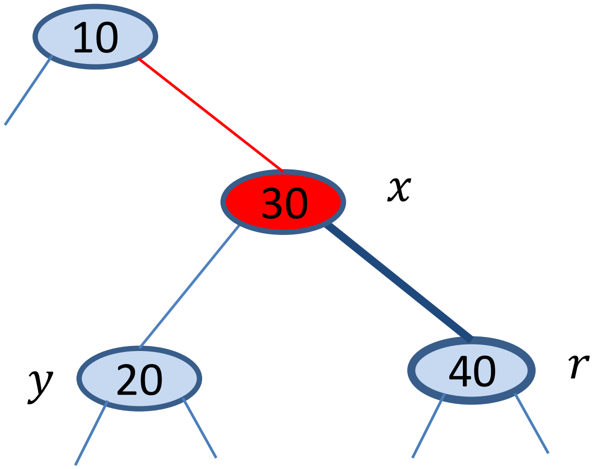


Removal (cont'd)

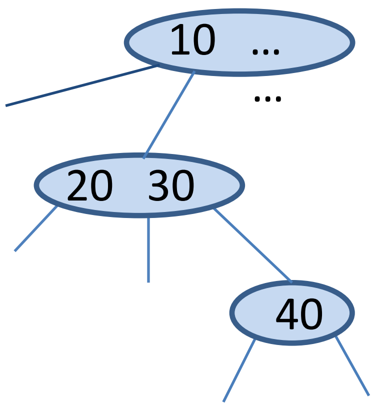
- **Case 2: the sibling y of r is black and both children of y are black**.
- Resolving this case corresponds to a **fusion** operation in the corresponding (2,4) tree T'.
- We do a **recoloring**: we color r black, we color y red, and, if x is red, we color it black; otherwise, we color x **double black**.
- Hence, after this recoloring, the double black problem might reappear at the parent x of r. We then repeat consideration of these three cases at x.

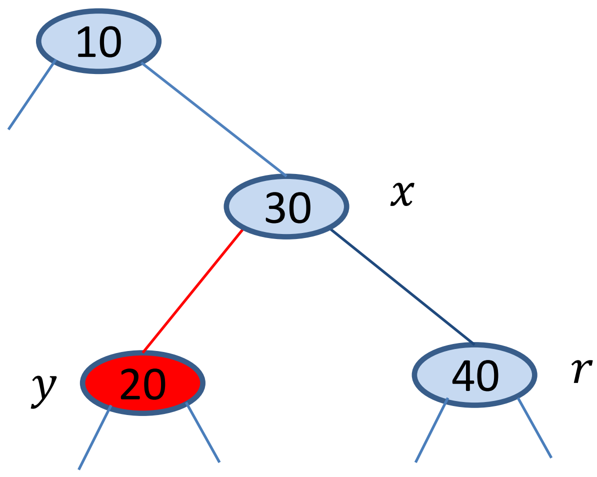
Recoloring a Red-Black Tree that Fixes the Double Black Problem



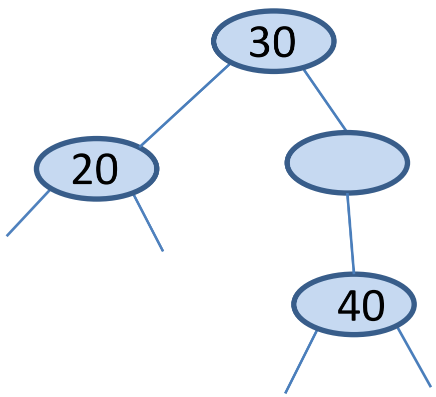


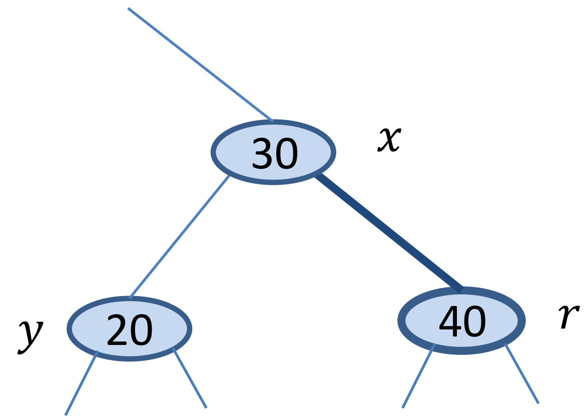
After the Recoloring



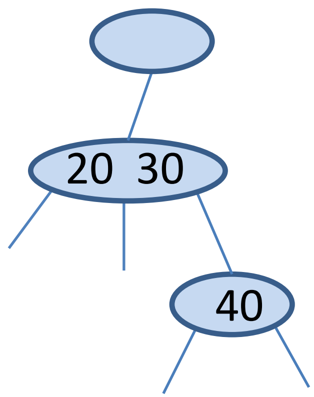


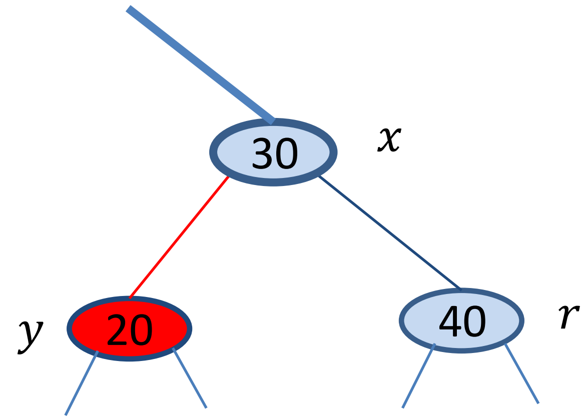
Recoloring a Red-Black Tree that Propagates the Double Black Problem





After the Recoloring





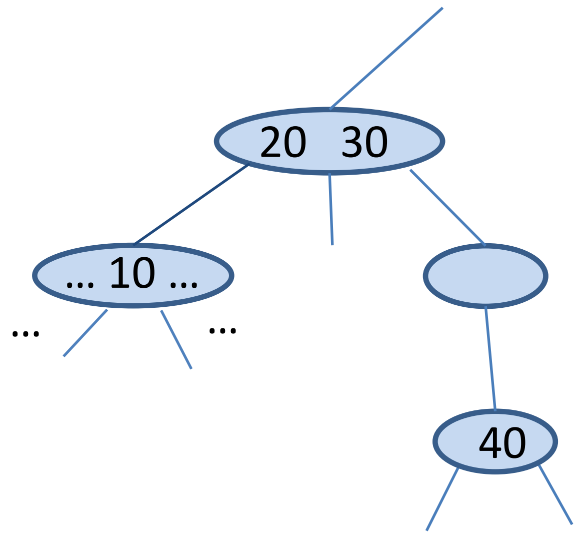
Removal (cont’d)

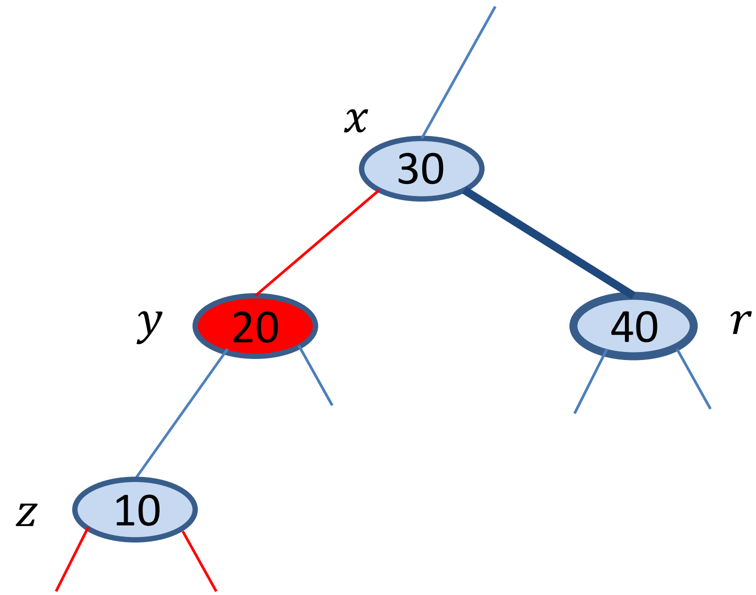
- **Case 3: the sibling y of r is red.**
- In this case, we perform an **adjustment operation** as follows.
- If y is the right child of x, let z be the right child of y; otherwise, let z be the left child of y.
- Execute the trinode restructuring operation which makes y the parent of x.
- Color y black and x red.

Removal (cont'd)

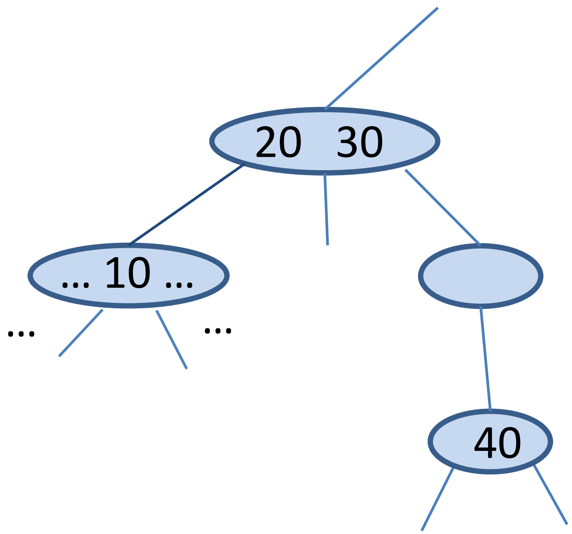
- An adjustment corresponds to choosing a different representation of a 3-node in the (2,4) tree T'.
- After the adjustment operation, the sibling of r is black, and either Case 1 or Case 2 applies, with a different meaning of x and y.
- Note that if Case 2 applies, the double black problem cannot reappear.
- Thus, to complete Case 3 we make one more application of either Case 1 or Case 2 and we are done.
- Therefore, **at most one adjustment** is performed in a removal operation.

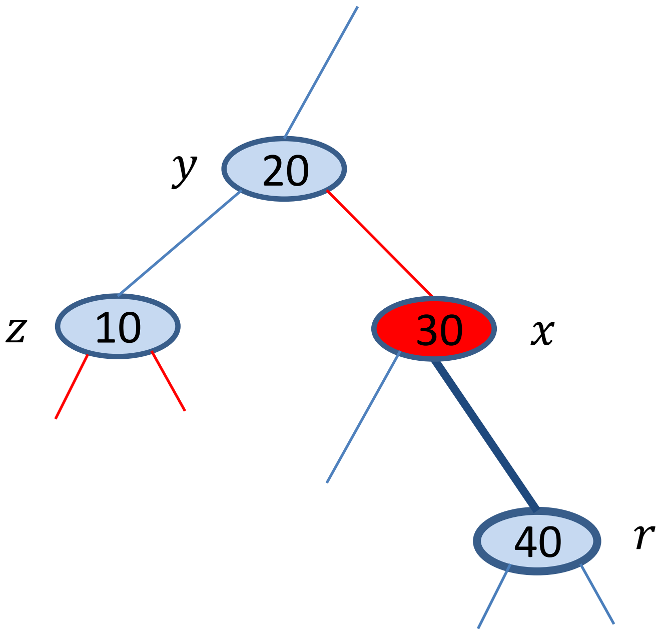
Adjustment of a Red-Black Tree in the Presence of a Double Black Problem





After the Adjustment





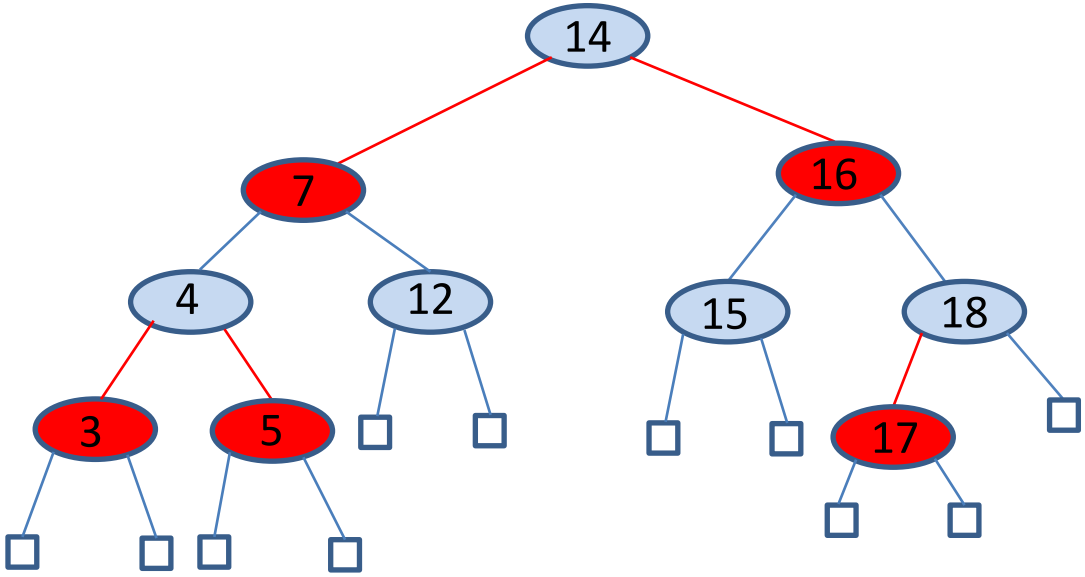
Removal (cont'd)

- The algorithm for removing an entry from a red-black tree with n entries takes $O(\log n)$ time and performs $O(\log n)$ recolorings and at most one adjustment plus one additional trinode restructuring.

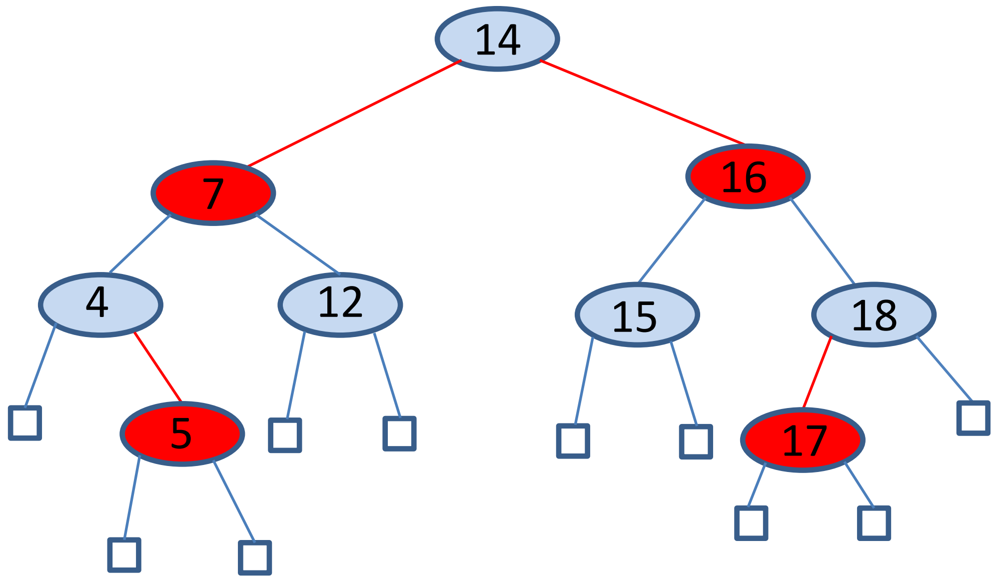
Example

- Let us now see a few removals from a given red-black tree.

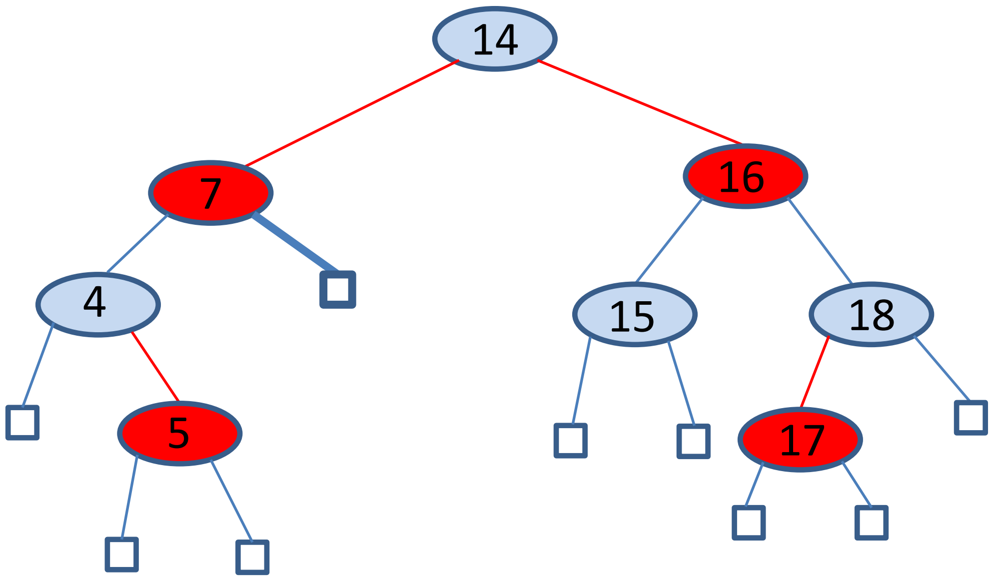
Initial Tree



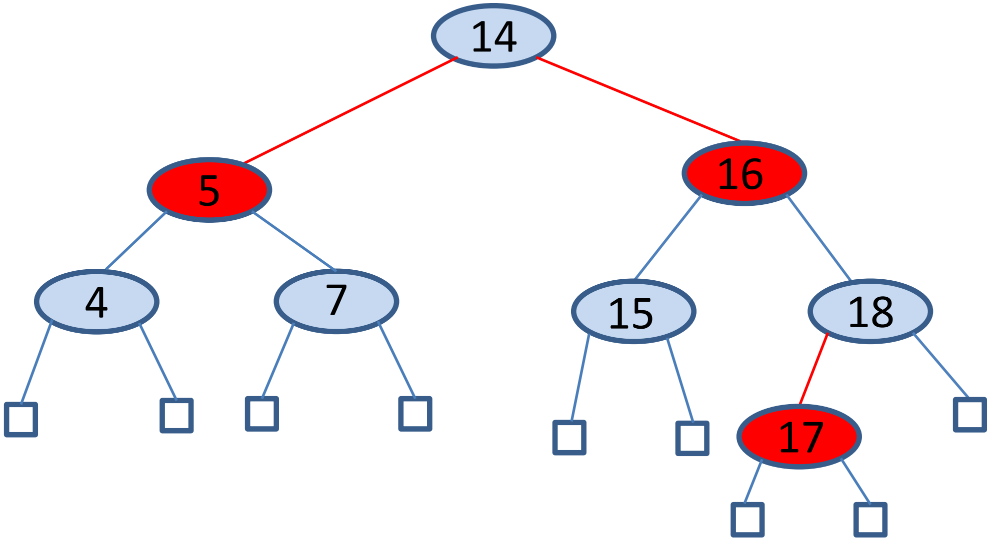
Remove 3



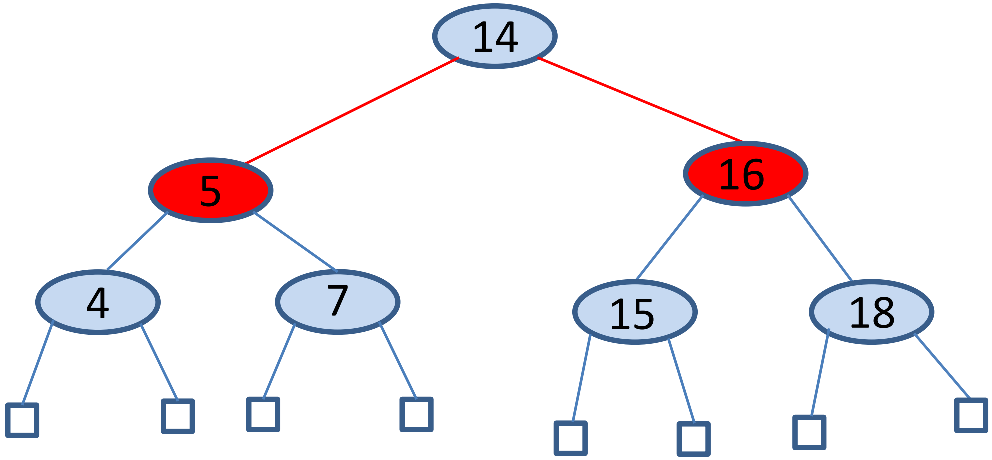
Remove 12 – Double Black



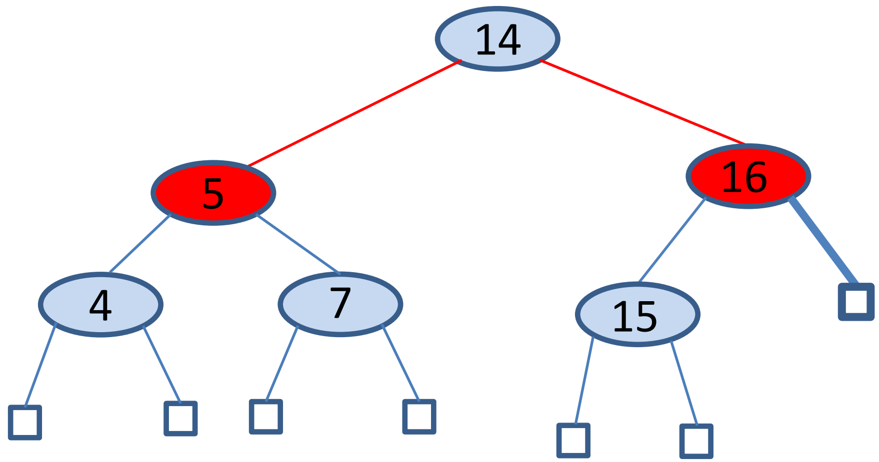
After Restructuring



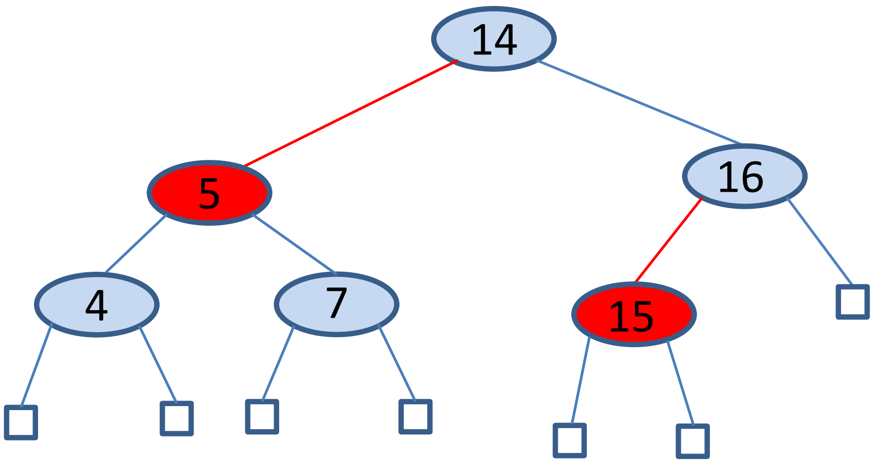
Remove 17



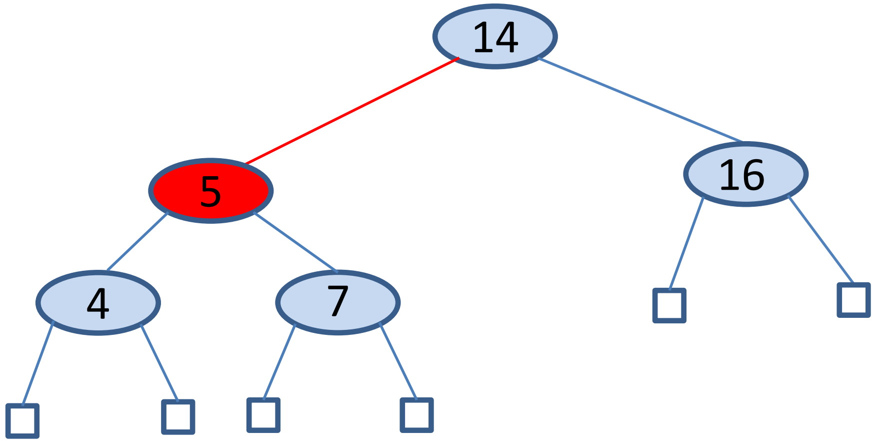
Remove 18 – Double Black



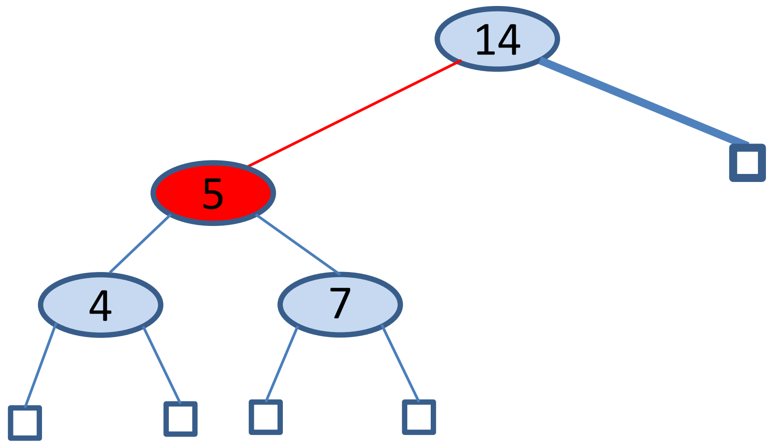
After Recoloring



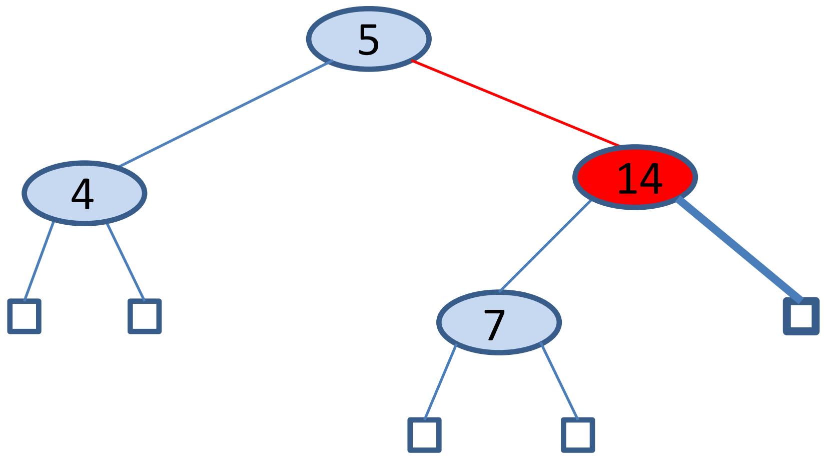
Remove 15



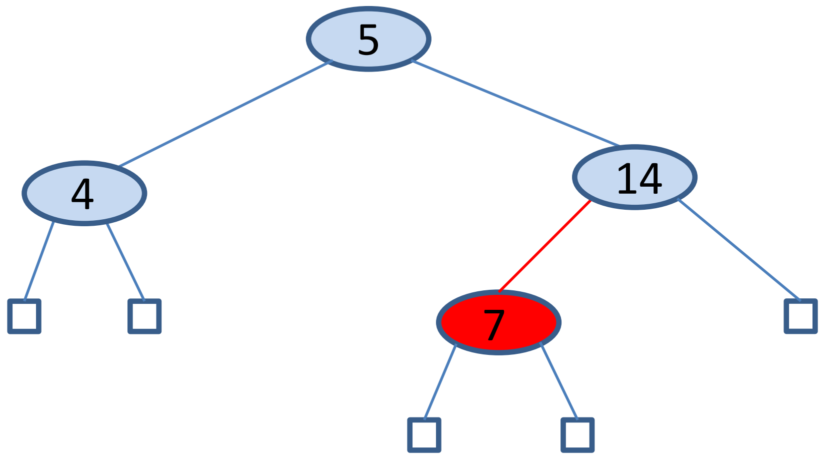
Remove 16 – Double Black



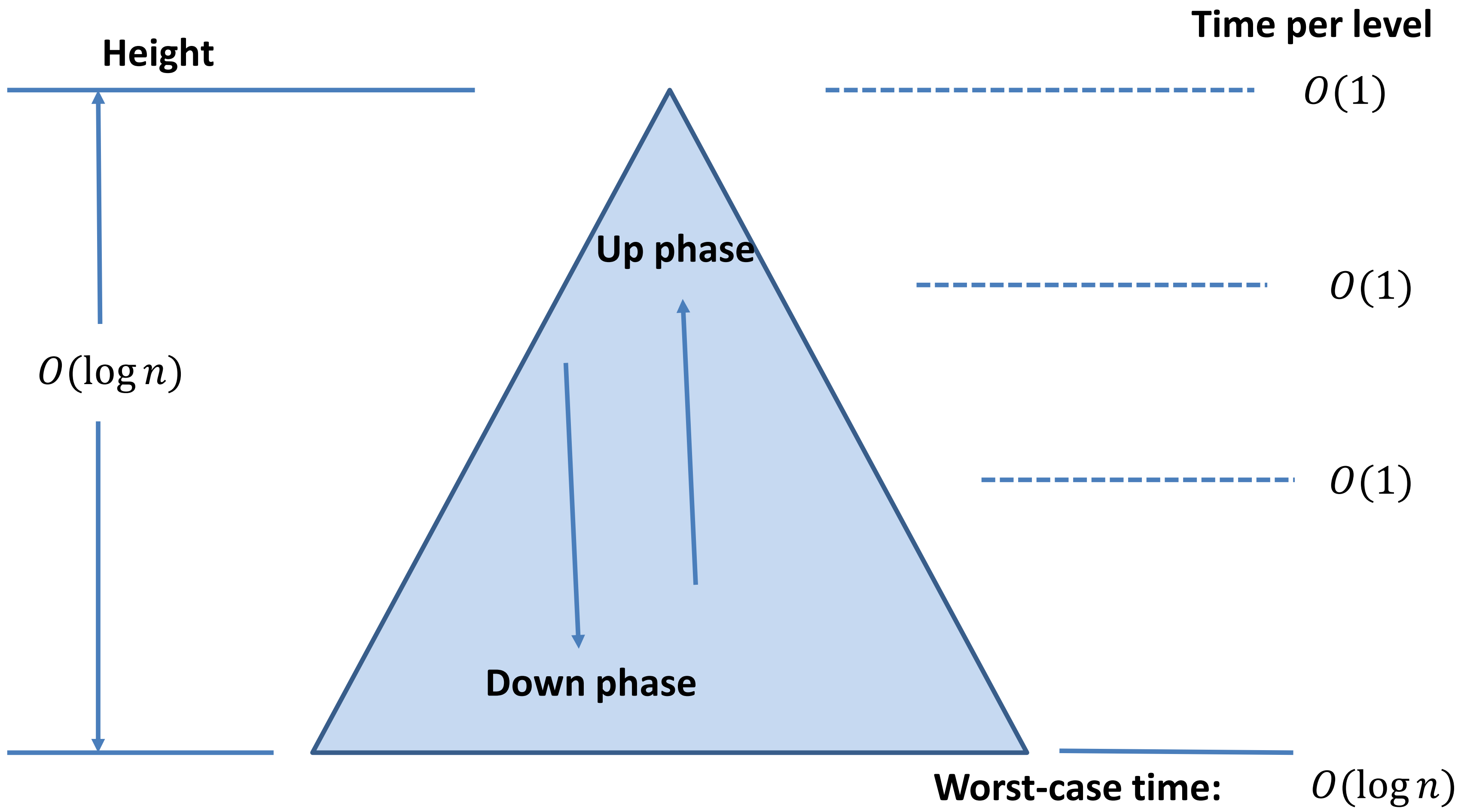
After the Adjustment – Double Black



After the Recoloring



Complexity of Operations in a Red-Black Tree



Summary

- The red-black tree data structure is **slightly more complicated** than its corresponding (2,4) tree.
- However, the red-black tree has the conceptual advantage that only a **constant number of trinode restructurings** are ever needed to restore the balance after an update.

Readings

- M. T. Goodrich, R. Tamassia and D. Mount. *Data Structures and Algorithms in C++.* 2nd edition. John Wiley.
  - Section 10.5
- M. T. Goodrich, R. Tamassia. *Δομές Δεδομένων και Αλγόριθμοι σε Java*. 5η έκδοση. Εκδόσεις Δίαυλος.
  - Κεφ. 10.5
- R. Sedgewick. Αλγόριθμοι σε C.
  - Κεφ. 13.4