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ON THE BOX DIMENSION FOR A CLASS OF NONAFFINE FRACTAL INTERPOLATION FUNCTIONS

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Abstract

We present lower and upper bounds for the box dimension of the graphs of certain nonaffine fractal interpolation functions by generalizing the results that hold for the affine case.

Key Words *Fractal, Box dimension, Iterated function system*

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1 Introduction

There has been great interest in the calculation of the box dimension of fractal interpolation functions because of their potential utility in approximation theory and in computer graphics.

In the case of equally spaced interpolation points, Hardin and Massopust[7] computed the box dimension of certain self-affine functions in one dimension. Later, Barnsley et. al. in [3] showed how the class of one-dimensional interpolation functions can be usefully widened by considering the projections of the graphs of higher-dimensional self-affine functions, which he named hiddenvariable fractal interpolation functions. The construction of space-filling curves using these hidden-variable fractal interpolation functions is considered in [6]. The determination of the conditions that a vertical scaling factor must obey to effectively model an arbitrary function and the introducton of the polar fractal interpolation functions as a fractal interpolation method of a nonaffine character are considered in [4].

Here our aim is to estimate the box dimension of the graphs of certain nonaffine functions in one dimension so as to generalize the results in [3] to the nonaffine case. Finally, some examples of how one can use the proposed bounds to estimate and, in some cases, to compute the box dimension of the fractal functions mentioned earlier are given.

2 Iterated Function Systems

Within Fractal Geometry, the method of iterated function systems introduced by Hutchinson[9] and popularised by Barnsley et. al. [2] and Demko et. al. [5], provide a framework for encoding and generating a large class of fractal images.

Let $X, Y \subset \mathbf{R}^d$. A function $f: X \to Y$ is called a Lipschitz function if

$$|f(x) - f(y)| \leqslant c|x - y|$$

for all $x, y \in X$ and for some constant $c \geqslant 0$. A Lipschitz function is a contraction with contractivity factor c, if $c < 1$. The function f is called a bi-Lipschitz function if

$$c_1|x - y| \leqslant |f(x) - f(y)| \leqslant c_2|x - y|$$

for all $x, y \in X$ and some constants $0 < c_1 \leqslant c_2 < \infty$. An iterated function system, or IFS for short, may be considered as a pair consisting of a closed subset X of $\mathbf{R}^d$ and a finite collection of continuous mappings $w_n: X \to X$, $n = 1, 2, \cdots, N$. It is often convenient to write an IFS formally as $\{X; w_1, w_2, \cdots, w_N\}$ or, somewhat more briefly, as $\{X; w_{1-N}\}$.

We introduce the associated map of subsets $W: \mathscr{H}(X) \to \mathscr{H}(X)$, given by

$$W(E) = \bigcup_{n=1}^{N} w_n(E) \qquad \text{for} \qquad E \in \mathscr{H}(X),$$

where $\mathscr{H}(X)$ is the metric space of all nonempty compact subsets of X endowed with the Hausdorff metric. The map W is called the collage map to alert us to the fact that $W(E)$ is formed as a union or collage of sets. Sometimes $\mathscr{H}(X)$ is referred to as the "space of fractals in X" (but note that not all members of X are fractals).

If w_n are contractions with corresponding contractivity factors $s_n, n = 1, 2, \cdots, N$, the IFS is termed hyperbolic and the map W is itself a contraction with contractivity factor $s = \max\{s_1, s_2, \cdots, s_N\}$ (ref. [1], Theorem 7.1, p. 81). In what follows we abbreviate by f^k the k-fold composition $f \circ f \circ \cdots \circ f$.

The attractor of a hyperbolic IFS is the unique set A for which $\lim\limits_{k \to \infty} W^k(E) = A$ for every starting set E. The term attractor is chosen to suggest the movement of E towards A under repeated applications of W. Note that A is also the unique set in $\mathscr{H}(X)$ which is not changed by W, i.e., $W(A)=A$, and from this important perspective it is often called the invariant set of the IFS.

A transformation w is affine in $\mathbf{R}^d$ if it may be represented by a matrix B and a translation t as $w(X)=Bx+t$, or, in the case of $\mathbf{R}^2$,

$$w=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a & b\\c & s\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}d\\e\end{bmatrix}.$$

The code of w is the 6-tuple (a,b,c,s,d,e), and the code of an IFS is a table whose rows are the codes of $w_1,w_2,\cdots,w_N$. We refer the interested reader to [1] or [8].

3 Fractal Interpolation Functions

Let f be a continuous real function defined on the real closed interval $I=[0,1]$. Further, let $0=x_0<x_1<\cdots<x_{N-1}<x_N=1$ be a partition of I, where $x_0,x_1,\cdots,x_N$ are $N+1$ distinct points. It is not assumed that these points are equidistant. The funciton f is called an interpolation function corresponding to the set of data $\{(x_i,y_i)\in \mathrm{I}\times\mathbf{R}: i=0,1,\cdots,N\}$, if $f(x_i)=y_i$ for all $i=0,1,\cdots,N$. We shall write for brevity $f(x_i)=f_i, i=0,1,\cdots,N$. The points (x_i,f_i) are called the interpolation points. We say that the function f interpolates the data and that (the graph of) f passes through the interpolation points. Let us denote by $G_f=\{(x,f(x)):x\in I\}$ the graph of a function f on I. Throughout this section we will work in the complete metric space $K=I\times\mathbf{R}$ with respect to the Euclidean or to some other equivalent metric.

Let N be a positive integer greater than 1. Set $I_n=[x_{n-1},x_n]$ and define $L_n:I\to I_n$ by

$$L_n(x)=a_nx+d_n,$$

where the real numbers a_n,d_n, for $n=1,2,\cdots,N$, are chosen to ensure that $L_n(I)=I_n$. Thus, for $n=1,2,\cdots,N$,

$$a_n=\frac{x_n-x_{n-1}}{x_N-x_0}=x_n-x_{n-1}>0,$$

$$d_n=\frac{x_Nx_{n-1}-x_0x_n}{x_N-x_0}=x_{n-1}.$$

Since $N\geqslant 2, 0<a_n<1, L_n$ are contractive homeomorphisms for $n=1,2,\cdots,N$ with contractivity factor $A=\max\{a_n:n=1,2,\cdots,N\}$.

Now define $M_n:K\to\mathbf{R}$, by

$$M_n(x,y)=c_ng_n(x)+s_nh_n(y)+e_n,$$

where $g_n:I\to\mathbf{R}$ and $h_n:\mathbf{R}\to\mathbf{R}$ are continuous functions such that

$$|g_n(x)-g_n(x')|\leqslant m_n|x-x'|$$

with $g_n(x_N) \neq g_n(x_0)$ and

$$l_n |y - y'| \leqslant |h_n(y) - h_n(')| \leqslant r_n |y - y'|,$$

where $x, x' \in I$, $y, y' \in \mathbf{R}$ and $l_n, r_n > 0$ are constants for $n = 1, 2, \cdots, N$.

The real constants c_n and e_n depend on the adjustable real parameters s_n and chosen so that

$$M_n(x_0, f_0) = f_{n-1}, \qquad M_n(x_N, f_N) = f_n.$$

Thus,

$$c_n = \frac{f_n - f_{n-1}}{g_n(x_N) - g_n(x_0)} - s_n \frac{h_n(f_N) - h_n(f_0)}{g_n(x_N) - g_n(x_0)},$$

$$e_n = \frac{g_n(x_N) f_{n-1} - g_n(x_0) f_n}{g_n(x_N) - g_n(x_0)} - s_n \frac{g_n(x_N) h_n(f_0) - g_n(x_0) h_n(f_N)}{g_n(x_N) - g_n(x_0)}$$

for $n = 1, 2, \cdots, N$. The mapping $M_n, n = 1, 2, \cdots, N$ are Lipschitz with respect to the first variable, with Lipshitz constant $|c_n|$ and bi-Lipschitz with respect to the second variable, with constants $|s_n| l_n$, $|s_n| r_n$.

Now define the functions $w_n: K \to K$ by

$$w_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_n(x) \\ M_n(x, y) \end{bmatrix}$$

for all $(x, y) \in K$ and $n = 1, 2, \cdots, N$. Then the IFS is of the form $\{K; w_{1-N}\}$, where the maps are of the special structure

$$w_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_n x \\ c_n g_n(x) + s_n h_n(y) \end{bmatrix} + \begin{bmatrix} d_n \\ e_n \end{bmatrix}$$

and a_n, c_n, s_n, d_n, e_n are real numbers for $n = 1, 2, \cdots, N$. We refer to s_n as the vertical scaling factor of the transformation w_n, which must obey

$$w_n \begin{bmatrix} x_0 \\ f_0 \end{bmatrix} = \begin{bmatrix} x_{n-1} \\ f_{n-1} \end{bmatrix} \quad \text{and} \quad w_n \begin{bmatrix} x_N \\ f_N \end{bmatrix} = \begin{bmatrix} x_n \\ f_n \end{bmatrix} \text{ for } n = 1, 2, \cdots, N.$$

To assure that the IFS $\{K: w_{1-N}\}$ constructed above is hyperbolic, we need the following

Theorem 1. *Let s_n be such that $0 \leqslant |s_n| r_n < 1$ for $n = 1, 2, \cdots, N$. Then there is a metric ρ on K, eqivalent to the Euclidean metric, such that the IFS $\{K;\ w_{1-N}\}$ is hyperbolic with respect to ρ.*

Theorem 2. *The hyperbolic IFS $\{K;\ w_{1-N}\}$ defined above has a unique attractor $G \in \mathscr{H}(K)$. Furthermore, G is the graph of a continuous function $f: I \to \mathbf{R}$ which obeys*

$$f(x_i) = f_i, \qquad i = 0, 1, \cdots, N.$$

The proofs of Theorems 1 and 2 follow closely those of Theorems 2.1 and 2.2 in [1] or Lemma 2.1 and Theorem 1 in [3], repsectively, and are therefore omitted.

Definition 1. *The function f whose graph is the attractor of an IFS as described in Theorem 2, is called a fractal interpolation function or FIF for short.*

4 Main Result

The main idea behind all the dimension calculations is to define the right covers for the graph G of the fractal function. Let us now define a class of covers that will allow to relate covers of different sizes (see also [3]).

Definition 2. *For $0<\varepsilon<1$, $\{\tau_j\}_{j=0}^m$ is called an ε-partition if*

1. $\tau_j\in(-\varepsilon/2, 1)$, *for* $j=0,1,\cdots,m$.
2. $\varepsilon/2<\tau_{j+1}-\tau_j\leqslant\varepsilon$, *for* $j=0,1,\cdots,m-1$.

A cover $C(\varepsilon)$ of G will be called an ε-column cover of G with associated ε-partition $\{\tau_j\}_{j=0}^m$ if there are positive integers $n_0,n_1,\cdots,n_m$ and real numbers $\xi_0,\xi_1,\cdots,\xi_m$ such that

$$C(\varepsilon)=\{[\tau_k,\tau_k+\varepsilon]\times[\xi_k+(j-1)\varepsilon,\xi_k+j\varepsilon]:\ j=1,2,\cdots,n_k;\ k=0,1,\cdots,m\}.$$

The class of all such covers of G is denoted by $\mathscr{C}(\varepsilon)$. Note that a cover $C(\varepsilon)\in\mathscr{C}(\varepsilon)$ consists of $\sum_{k=0}^{m}n_k$ closed $\varepsilon\times\varepsilon$ squares arranged in $m+1$ columns. Let $|C(\varepsilon)|$ denote the cardinality of $C(\varepsilon)$ and define $\mathscr{N}^*(\varepsilon)=\min\{|C(\varepsilon)|:C(\varepsilon)\in\mathbf{C}(\varepsilon)\}$.

Definition 3. *Let F be a nonempty bounded subset of $\mathbf{R}^d$ and let $\mathscr{N}(\varepsilon)$ be the smallest number of (closed) squares of side ε which can cover F. The lower and upper box-counting dimensions of F are defined respectively as*

$$\underline{\dim}_B F=\liminf_{\varepsilon>0}\frac{\log\mathscr{N}(\varepsilon)}{-\log\varepsilon},$$

$$\overline{\dim}_B F=\limsup_{\varepsilon>0}\frac{\log\mathscr{N}(\varepsilon)}{-\log\varepsilon}.$$

If these are equal we refer to the common value as the box-counting or box dimension of F

$$\dim_B F=\lim_{\varepsilon\to0^+}\frac{\log\mathscr{N}(\varepsilon)}{-\log\varepsilon}.$$

The next result shows that for the calculation of the box dimension of G it suffices to consider ε-column covers.

Lemma 1. $\mathscr{N}(\varepsilon)\leqslant\mathscr{N}^*(\varepsilon)\leqslant2\mathscr{N}(\varepsilon)$ *for all* $0<\varepsilon<1$.

Proof. See Lemma 4.1, p.1236 of [3] or Proposition 6.1, p.206 of [10].

Lemma 2. *Let $y,y_1,y_2\in\mathbf{R}$ be such that $y_1=(1-\lambda)y+\lambda y_2+\delta$ for some $\lambda\in(0,1)$ and $\delta\neq0$. Let $\varphi:\mathbf{R}\to\mathbf{R}$ be a function which satisfies $|\varphi(x_1)-\varphi(x_2)|\geqslant l|x_1-x_2|$ for $x_1,x_2\in\mathbf{R}$*

and some $l>0$. Then,

i) *if φ is increasing, concave and $\delta>0$, we have $\varphi(y_1-(1-\lambda)\varphi(y)-\lambda\varphi(y_2)\geqslant l\delta$;*

ii) *if φ is increasing, concave and $\delta>0$, we have $\varphi(y_1)-(1-\lambda)\varphi(y)-\lambda\varphi(y_2)\leqslant l\delta$;*

iii) *if φ is linear, we have $|\varphi(y_1)-(1-\lambda)\varphi(y)-\lambda\varphi(y_2)|\geqslant l|\delta|$.*

Proof. Let

$$\begin{aligned}\Gamma &= \varphi(y_1) - (1-\lambda)\varphi(y) - \lambda\varphi(y_2)\\ &= [\varphi((1-\lambda)y + \lambda y_2 + \delta) - \varphi((1-\lambda)y + \lambda y_2)]\\ &\quad + [\varphi((1-\lambda)y + \lambda y_2) - (1-\lambda)\varphi(y) - \lambda\varphi(y_2)] \equiv A + B.\end{aligned}$$

i) Since φ is increasing, concave, and $\delta>0$, we have $A,B\geqslant 0$ and $\Gamma\geqslant A=|A|\geqslant l\delta$;

ii) Since φ is increasing, convex, and $\delta<0$, we have $A,B\leqslant 0$ and $\Gamma\leqslant A=-|A|\leqslant -l|\delta|=l\delta$;

iii) Since φ is linear, we have $|\Gamma|=|\varphi(\delta)|\geqslant l|\delta|$.

To show that the term $1/\varepsilon$ becomes negligible we need the following

Lemma 3. *Let $\{(x_i,f_i):i=0,1,\cdots,N\}$ be given points and $V_k=(f_k-f_{k-1})-(f_{k+1}-f_{k-1})(x_k-x_{k-1})/(x_{k+1}-x_{k-1})\neq 0$ for some $k\in\{1,2,\cdots,N-1\}$. Choose $g_n(x)=x$ for all $n=1,2,\cdots,N$ and*

i) *if there exists a $k\in\{1,2,\cdots,N-1\}$ such that $V_k>0$, choose s_nh_n for $n=1,2,\cdots,N$ to be increasing and concave;*

ii) *if there exists a $l\in\{1,2,\cdots,N-1\}$ such that $V_l<0$, choose s_nh_n for $n=1,2,\cdots,N$ to be increasing and convex;*

iii) *if there exist $k,l\in\{1,2,\cdots,N-1\}$ such that $V_kV_l<0$, choose s_nh_n for $n=1,2,\cdots,N$ to be (all) increasing and concave or (all) increasing and convex.*

Then, if $\gamma_1=\sum_{n=1}^{N}l_n|s_n|>1$,

$$\lim_{\varepsilon\to 0^+}\varepsilon\mathcal{N}^*(\varepsilon)=\infty.$$

Proof. We shall prove only the first case; the other two can be proved using similar arguments. Let $x_k=(1-\lambda)x_{k-1}+\lambda x_{k+1}$ for some $\lambda\in(0,1)$. Then, $f_k=(1-\lambda)f_{k-1}+\lambda f_{k+1}+V_k$(see Fig. 1(a)).

Let $a=\min\{2a_n:n=1,2,\cdots,N\}$. Then $0<a\leqslant 1$, because $N\geqslant 2$. Since f is continuous and the points $(x_{k-1},f_{k-1}),(x_k,f_k),(x_{k+1},f_{k+1})\in G$ we obtain

$$\mathcal{N}^*(\varepsilon)\geqslant\frac{V_k}{\varepsilon}\text{ for all }0<\varepsilon<a.$$

From the previous lemma and since $g_n(x)=x$ for all $n=1,2,\cdots,N$ we have that $M_n(x_k,f_k)-(1-\lambda)M_n(x_{k-1},f_{k-1})-\lambda M_n(x_{k+1},f_{k+1})=s_nh_n((1-\lambda)f_{k-1}+\lambda f_{k+1}+V_k)-(1-\lambda)s_nh_n$

$(f_{k-1})-\lambda s_n h_n(f_{k+1})\geq l_n|s_n|V_k$ for $n=1,2,\cdots,N$. Since $w_n(G)\subset G$ for $n=1,2,\cdots,N$, the points $w_n(x_{k-1},f_{k-1})=(L_n(x_{k-1}),M_n(x_{k-1},f_{k-1}))$, $w_n(x_k,f_k)=(L_n(x_k),M_n(x_k,f_k))$, $w_n(x_{k+1},f_{k+1})=(L_n(x_{k+1}),M_n(x_{k+1},f_{k+1}))\in G$, so $M_n(x_k,f_k)=(1-\lambda)M_n(x_{k-1},f_{k-1})+\lambda M_n(x_{k+1},f_{k+1})+\mu_n|s_n|l_nV_k\in f(L_n([0,1]))$ for some $\mu_n\geq 1$ and $L_n(x_k)=(1-\lambda)L_n(x_{k-1})+\lambda L_n(x_{k+1})$. Hence, to cover $\bigcup_{n=1}^{N}L_n([0,1])\times f(L_n([0,1]))$ we need

$$\mathcal{N}^*(\varepsilon)\geq\sum_{n=1}^{N}\mu_n l_n|s_n|\frac{V_k}{\varepsilon}\geq\sum_{n=1}^{N}l_n|s_n|\frac{V_k}{\varepsilon}\quad\text{for all}\quad 0<\varepsilon<a^2$$

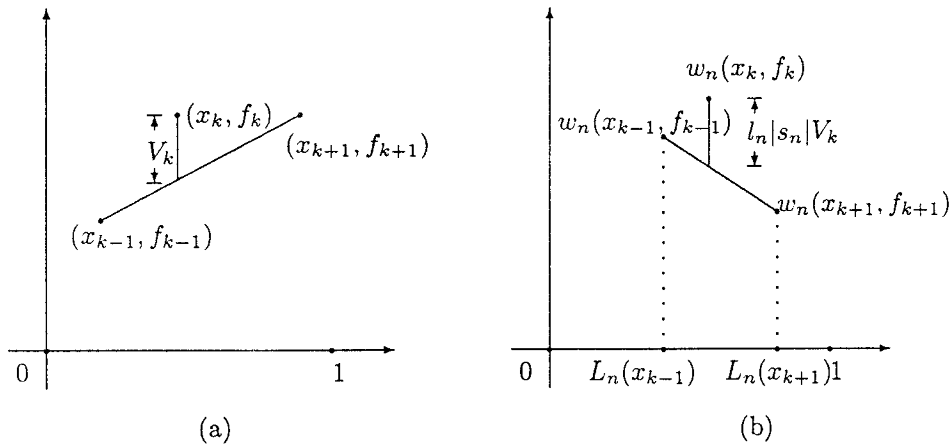


Figure 1

(see Fig. 1(b)). By induction

$$\mathcal{N}^*(\varepsilon)\geq\Big(\sum_{n=1}^{N}l_n|s_n|\Big)^m\frac{V_k}{\varepsilon}\quad\text{for}\quad 0<\varepsilon<a^{m+1}\text{ and }m\in\mathbf{N}.$$

Since $\sum_{n=1}^{N}l_n|s_n|>1$ and $V_k>0$ the lemma is proved.

The next theorem serves as a generalization of Theorem 4, p. 1236 of [3] to the case of nonaffine fractal functions, which are more flexible, since they can deal with a wider (than in the affine case) range of applications.

Theorem 3. *Let f be the function with the graph G generated by the hyperbolic IFS $\{K;w_{1-N}\}$.*

(a) *If $\sum_{n=1}^{N}r_n|s_n|>1$ and $(x_i,f_i)\in K$ for $i=0,1,\cdots,N$ are not all collinear, then $\overline{\dim}_B G\leq D$, where $D\in(1,2)$ is the unique real solution of $\sum_{n=1}^{N}r_n|s_n|a_n^{D-1}=1$.*

(b) *If the hyperbolic IFS satisfies in addition the conditions of Lemma 3, then $d\leq\underline{\dim}_B G$, where $d\in(1,2)$ is the unique real solution of $\sum_{n=1}^{N}l_n|s_n|a_n^{d-1}=1$.*

Proof. Let $0<\varepsilon<a$ be given and $C(\varepsilon)$ a minimal ε-column cover of G with associated ε-partition $\{\tau_j\}_{j=0}^{m}$. For $n\in\{1,2,\cdots,N\}$, let $[\tau_k,\tau_l+\varepsilon]$ be the smallest interval that covers $I_n=[x_{n-1},x_n]$. Denote by

$$C_n(\varepsilon)=\{[\tau_i,\tau_i+\varepsilon]\times[\xi_i+(j-1)\varepsilon,\xi_j+j\varepsilon]:j=1,2,\cdots,n_i,\ [\tau_i,\tau_i+\varepsilon]\subset[\tau_k,\tau_l+\varepsilon]\}$$

the "restriction" of $C(\varepsilon)$ to I_n and by $\mathcal{N}_n(\varepsilon)=|C_n(\varepsilon)|$ its cardinality. Note that $\mathcal{N}_n(\varepsilon)$ denotes the number of squares of side length ε that intersect $w_n(G)$ for $n=1,2,\cdots,N$. Since there are at most two columns in $C_n(\varepsilon)\cap C_{n+1}(\varepsilon)$ and that f is uniformly bounded on I, there is some constant $A_1>0$ such that

$$\sum_{n=1}^{N}\mathcal{N}_n(\varepsilon)\leqslant\mathcal{N}^*(\varepsilon)+A_1\varepsilon^{-1}.$$

Now suppose that $n\in\{1,2,\cdots,N\}$ is such that $s_n\neq0$. Then w_n is invertible. Consider a typical column $\mathscr{R}$ in $C_n(\varepsilon)$ that consists of $n_0\varepsilon\times\varepsilon$ squares. Obviosuly $\mathscr{R}$ has the width ε and height $n_0\varepsilon$. Since $w_n(G)\subset\bigcup_{\mathscr{R}\subset C_n(\varepsilon)}\mathscr{R}$, we have $G\subset\bigcup_{\mathscr{R}\subset C_n(\varepsilon)}w_n^{-1}(\mathscr{R})$. If $(x,y),(x',y')\in\mathscr{R}$, then, since $|x-x'|\leqslant\varepsilon$, $|y-y'|\leqslant n_0\varepsilon$ and

$$w_n^{-1}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\dfrac{x-d_n}{a_n}\\ h_n^{-1}\left(\dfrac{y-c_ng_n(\frac{x-d_n}{a_n})-e_n}{s_n}\right)\end{bmatrix},$$

$w_n^{-1}(\mathscr{R})$ has the maximum height

$$\left|h_n^{-1}\left(\frac{y-c_ng_n(\frac{x-d_n}{a_n})-e_n}{s_n}\right)-h_n^{-1}\left(\frac{y'-c_ng_n(\frac{x'-d_n}{a_n})-e_n}{s_n}\right)\right|$$

$$\leqslant\frac{1}{l_n|s_n|}\left|(y-y')-c_n\left[g_n(\frac{x-d_n}{a_n})-g_n(\frac{x'-d_n}{a_n})\right]\right|$$

$$\leqslant\frac{1}{l_n|s_n|}n_0\varepsilon+\frac{|c_n|}{l_n|s_n|a_n}m_n\varepsilon.$$

Thus, the inverse image, $w_n^{-1}(\mathscr{R})$, of $\mathscr{R}$ is a set that is inside a rectangle of the width ε/a_n and height $\frac{1}{l_n|s_n|}n_0\varepsilon+\frac{|c_n|}{l_n|s_n|a_n}m_n\varepsilon$, which can be covered by

$$\left[n_0\frac{a_n}{l_n|s_n|}+\frac{|c_n|}{l_n|s_n|}m_n\right]+1$$

squares of the side ε/a_n as in Fig. 2, where $[\cdot]$ denotes the integer part of a number. Since there are at most $2a_n/\varepsilon+2$ columns in $C_n(\varepsilon)$, $\bigcup_{\mathscr{R}\subset C_n(\varepsilon)}w_n^{-1}(\mathscr{R})$ can be covered by

$$\frac{a_n}{l_n|s_n|}\mathcal{N}_n(\varepsilon)+\frac{c_n}{l_n|s_n|}m_n(\frac{2a_n}{\varepsilon}+2)=\frac{a_n}{l_n|s_n|}\mathcal{N}_n(\varepsilon)+\frac{A_n}{\varepsilon}$$

$\varepsilon/a_n \times \varepsilon/a_n$ squares for some constant $A_n > 0$. Therefore

$$\mathcal{N}^*\left(\frac{\varepsilon}{a_n}\right) \leqslant \frac{a_n}{l_n|s_n|}\mathcal{N}_n(\varepsilon) + \frac{A_n}{\varepsilon},$$

or equivalently,

$$\mathcal{N}(\varepsilon) \geqslant \frac{l_n|s_n|}{a_n}\mathcal{N}^*\left(\frac{\varepsilon}{a_n}\right) - \frac{A_n}{\varepsilon}.$$

Summing over n yields

$$\sum_{n=1}^{N}\frac{l_n|s_n|}{a_n}\mathcal{N}^*\left(\frac{\varepsilon}{a_n}\right) - \frac{\beta_1}{\varepsilon} \leqslant \sum_{n=1}^{N}\mathcal{N}_n(\varepsilon) - \frac{A_1}{\varepsilon} \leqslant \mathcal{N}^*(\varepsilon) \tag{1}$$

for some positive constant β_1.

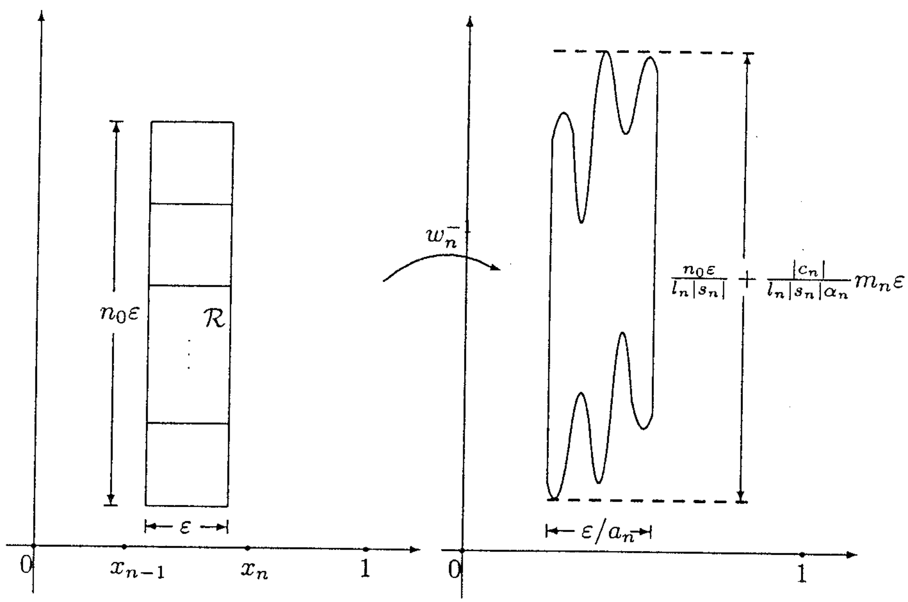


Figure 2 The rectangle $\mathscr{R}$ and its image under the map w_n^{-1}

Next an upper bound for $\mathcal{N}^*(\varepsilon)$ is obtained. Fix an $n\in\{1,\cdots,N\}$. Let D_n be a minimal ε/a_n-column cover of G and $\mathscr{R}$ a typical column of D_n that consists of $n_0\varepsilon/a_n\times\varepsilon/a_n$ squares. Note that $w_n(\mathscr{R})$ is a set that is inside a parallelogram of width ε and height $|s_n| r_n n_0 \frac{\varepsilon}{a_n} + |c_n|\frac{\varepsilon}{a_n}m_n$, which can be covered by

$$\left[n_0\frac{r_n|s_n|}{a_n} + \frac{|c_n|}{a_n}m_n\right] + 1$$

$\varepsilon\times\varepsilon$ squares as in Fig. 3. This way a cover $C_n(\varepsilon)$ of $w_2(G)$ consisting of $\varepsilon\times\varepsilon$ squares is generated. Since there are at most $2a_n/\varepsilon+2$ columns of D_n, $\bigcup_{\mathscr{R}\subset D_n} w_n(\mathscr{R})$ can be covered by

$$\frac{r_n|s_n|}{a_n}|D_n| + \frac{|c_n|}{a_n}m_n\left(\frac{2a_n}{\varepsilon}+2\right) = \frac{r_n|s_n|}{a_n}|D_n| + \frac{B_n}{\varepsilon}$$

squares ofthe side ε for some constant $B_n>0$. Therefore

$$|C_n(\varepsilon)| \leqslant \frac{r_n|s_n|}{a_n}\mathcal{N}^*(\frac{\varepsilon}{a_n}) + \frac{B_n}{\varepsilon}.$$

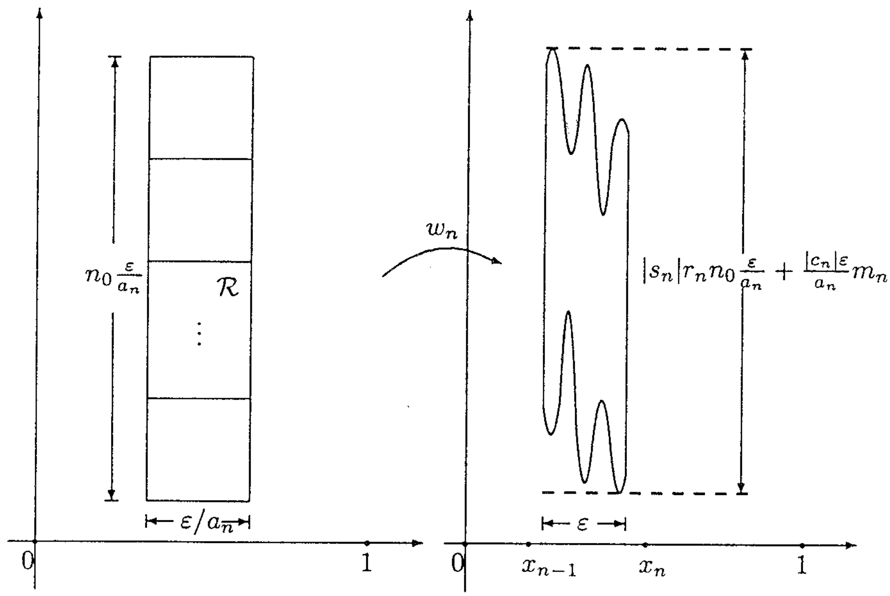


Figure 3 The rectangle $\mathcal{R}$ and its image under the map w_n

The union $\bigcup_{n=1}^{N} C_n(\varepsilon)$ is a cover of G, but in general may not be an ε-column cover of G because the columns of $C_n(\varepsilon)$ may not join up properly with those of $C_{n+1}(\varepsilon)$; however, an ε-column cover $C(\varepsilon)$ can be constructed from $\bigcup_{n=1}^{N} C_n(\varepsilon)$ by replacing at most two columns from $C_n(\varepsilon)\bigcup C_{n+1}(\varepsilon)$ with at most two properly spaced columns. Thus, there exists a positive constant β_2 such that

$$\mathcal{N}^*(\varepsilon) \leqslant \sum_{n=1}^{N} \frac{r_n|s_n|}{a_n}\mathcal{N}^*(\frac{\varepsilon}{a_n}) + \frac{\beta_2}{\varepsilon}. \tag{2}$$

From (1) and (2) we have established that $\mathcal{N}^*(\varepsilon)$ satisfies the functional inequality

$$\sum_{n=1}^{N} \frac{l_n|s_n|}{a_n}\mathcal{N}^*(\frac{\varepsilon}{a_n}) - \frac{\beta_1}{\varepsilon} \leqslant \mathcal{N}^*(\varepsilon) \leqslant \sum_{n=1}^{N} \frac{r_n|s_n|}{a_n}\mathcal{N}^*(\frac{\varepsilon}{a_n}) + \frac{\beta_2}{\varepsilon}$$

for all $0<\varepsilon<a$ and some $\beta_1, \beta_2>0$.

(a) Note that if $\gamma_2 = \sum_{n=1}^{N} r_n|s_n|$ then $\gamma_2 \geqslant \gamma_1 > 1$. Select $\varepsilon_0>0, k_2>0$ so that

$$\mathcal{N}^*(\varepsilon) \leqslant \frac{\beta_2}{1-\gamma_2}\varepsilon^{-1} + k_2\varepsilon^{-D} \tag{3}$$

for $\varepsilon_0 \leqslant \varepsilon \leqslant \varepsilon_0/a$, where $a=\min\{a_n : n=1,\cdots,N\}$. If $A\varepsilon_0 \leqslant \varepsilon \leqslant \varepsilon_0$, where $A=\max\{a_n : n=1,$

$\cdots,N\}$, then $\varepsilon_0\leqslant\varepsilon/a_n\leqslant\varepsilon_0/a$ and thus

$$\mathscr{N}^*(\frac{\varepsilon}{a_n})\leqslant\frac{\beta_2}{1-\gamma_2}\frac{a_n}{\varepsilon}+k_2(\frac{a_n}{\varepsilon})^D.$$

Therefore, using also (2) and (3),

$$\mathscr{N}^*(\varepsilon)\leqslant\sum_{n=1}^{N}\frac{r_n|s_n|}{a_n}(\frac{\beta_2}{1-\gamma_2})\frac{a_n}{\varepsilon}+\frac{k_2}{\varepsilon^D}\sum_{n=1}^{N}\frac{r_n|s_n|}{a_n}a_n^D+\beta_2\varepsilon^{-1}$$

$$=\frac{\beta_2}{1-\gamma_2}\varepsilon^{-1}\sum_{n=1}^{N}r_n|s_n|+\frac{k_2}{\varepsilon^D}\sum_{n=1}^{N}r_n|s_n|a_n^{D-1}+\beta_2\varepsilon^{-1}$$

$$=\frac{\beta_2\gamma_2}{1-\gamma_2}\varepsilon^{-1}+k_2\varepsilon^{-D}+\beta_2\varepsilon^{-1}=\beta_2(\frac{\gamma_2}{1-\gamma_2}+1)\varepsilon^{-1}+k_2\varepsilon^{-D}$$

$$=\frac{\beta_2}{1-\gamma_2}\varepsilon^{-1}+k_2\varepsilon^{-D}$$

for $A\varepsilon_0\leqslant\varepsilon\leqslant\varepsilon_0$. The preceding argument serves as an induction step: Suppose that for $A^k\varepsilon_0\leqslant\varepsilon\leqslant\varepsilon_0,k\in\mathbf{N}$

$$\mathscr{N}^*(\varepsilon)\leqslant\frac{\beta_2}{1-\gamma_2}\varepsilon^{-1}+k_2\varepsilon^{-D},$$

then it must also hold for $A^{k+1}\varepsilon_0\leqslant\varepsilon\leqslant\varepsilon_0$. Since $A^k\to0$ as $k\to\infty$,

$$\mathscr{N}^*(\varepsilon)\leqslant\frac{\beta_2}{1-\gamma_2}\varepsilon^{-1}+k_2\varepsilon^{-D}<k_2\varepsilon^{-D}$$

for all $\varepsilon\in(0,\varepsilon_0]$. Therefore,

$$\frac{\log\mathscr{N}^*(\varepsilon)}{-\log\varepsilon}\leqslant D+\frac{\log k_2}{-\log\varepsilon},$$

which implies that $\overline{\dim}_B G\leqslant D$.

(b) By Lemma 3 there exists $\delta_0>0$ so that

$$\varepsilon\mathscr{N}^*(\varepsilon)\geqslant2\frac{\beta_1}{\gamma_1-1}\quad\text{for}\quad0<\varepsilon\leqslant\delta_0/a.$$

Then

$$\varepsilon\mathscr{N}^*(\varepsilon)\geqslant\frac{\beta_1}{\gamma_1-1}+\frac{\beta_1}{\gamma_1-1}\geqslant\frac{\beta_1}{\gamma_1-1}+\frac{k_1}{\varepsilon^{d-1}},$$

for $\delta_0\leqslant\varepsilon\leqslant\frac{\delta_0}{a}$ and $0<k_1<\frac{\beta_1}{\gamma_1-1}\delta_0^{d-1}$. Working in an analogous way, we finally have

$$k_1\varepsilon^{-d}\leqslant\frac{\beta_1}{\gamma_1-1}\varepsilon^{-1}+k_1\varepsilon^{-d}\leqslant\mathscr{N}^*(\varepsilon)$$

for all $\varepsilon\in(0,\delta_0]$. Therefore, $\underline{\dim}_B G\geqslant d$.

The proof of the theorem is complete.

The following corollary is, in turn, a generalisation of Theorem 5, p. 1240 of [3], because it does not confine the functions g_n only to the case $g_n(x)=x$, but permits freedom of selection according, of course, to the restrictions set.

Corollary 1. *With the same notation as above, but with $h_n(y)=y$,*

(a) *if* $\sum_{n=1}^{N}|s_n|>1$ *and the interpolation points do not all lie on a single straight line, the upper box-counting diemnsion of G is the unique real solution D of*

$$\sum_{n=1}^{N}|s_n|\,|a_n|^{D-1}=1;$$

(b) *if in addition the conditions of Lemma* 3 *are satisfied, the box-counting dimension of G is the unique real solution D of*

$$\sum_{n=1}^{N}|s_n|\,|a_n|^{D-1}=1.$$

Example 1. Let $I=[0,1]$, $Y=\mathbf{R}$ and let $\{(0,0),(1/2,1),(1,0)\}$ be a given set of data. Define the functions $L_n:I\to I$ by

$$L_n(x)=\frac{1}{2}(-1)^{n-1}x+(n-1),\qquad n=1,2.$$

Let $g_1,g_2\in C(I)$. Define mappings $M_n:I\times Y\to Y$ by $M_n(x,y)=g_n(x)+s_ny, n=1,2$. Fig. 4 shows the graph of a nonaffine fractal interpolation function for $g_1(x)=x$, $g_2(x)=x^2$ and $s_1=s_2=3/4$, which has an upper box dimension bounded by

$$\overline{\dim}_B(G)\leqslant 1+\frac{\log(|s_1|+|s_2|)}{\log 2}=\frac{\log 3}{\log 2}\cong 1.585.$$

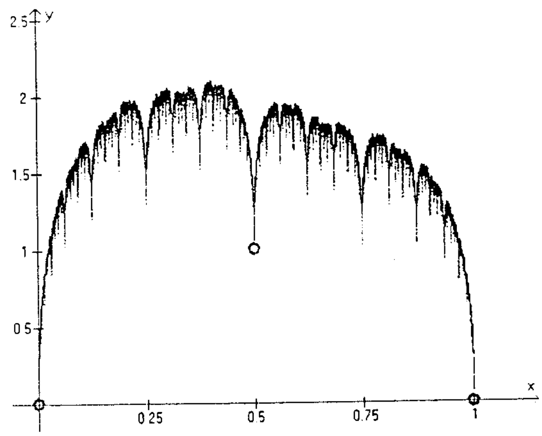


Figure 4 The graph of a nonaffine FIF

Example 2. *Let* $I=[0,1], Y=[1,6]$ *and let* $\{(0,2),(1/3,4),(2/3,5/2),(1,2)\}$ *be a given set of data. Define the functions* $L_n:I\to I$ *by*

$$L_n(x)=\frac{1}{3}[x+(n-1)],\qquad n=1,2,3.$$

Let $g_1,g_2,g_3\in C(I)$. *Define mappings* $M_n:I\times Y\to Y$ *by* $M_n(x,y)=c_ng_n(x)+s_nh_n(y)+e_n, n$

$=1,2,3$. Fig. 5(a) shows the graph of such a fractal interpolation function for

$$g_1(x) = g_2(x) = g_3(x) = x,\ h_1(y) = y^2,\ h_2(y) = h_3(y) = y$$

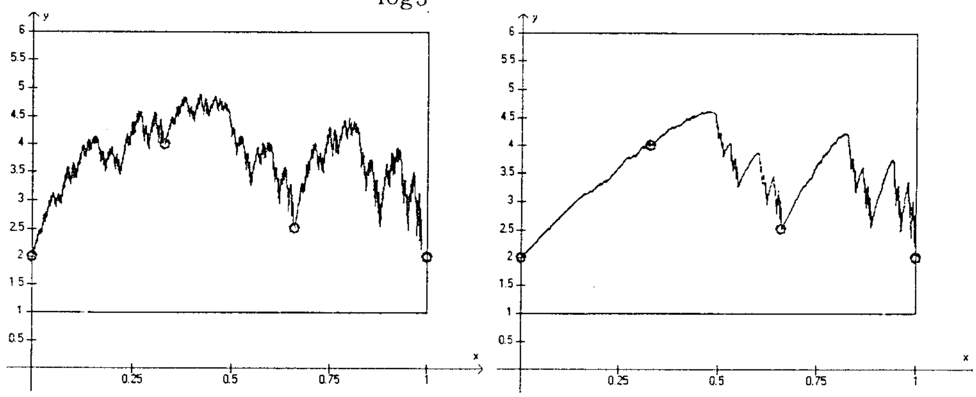
and

$$s_1 = 1/16,\ s_2 = 1/2 \text{ and } s_3 = 3/4,$$

the upper box-counting dimension of which satisfy

$$1.29 \cong 1 + \frac{\log(11/8)}{\log 3} \leqslant \underline{\dim}_B(G) \leqslant \overline{\dim}_B(G)$$

$$\leqslant 1 + \frac{\log 2}{\log 3} \cong 1.63.$$



(a) (b)

Figure 5 The graphs of (a) a nonaffine FIF and (b) an affine FIF

Figure 5(b) shows the graph of an affine fractal interpolation function using almost the same parameters as before except that $h_1(y)=y$, which has a box-counting dimension of

$$\dim_B(G) = 1 + \frac{\log(|s_1| + |s_2| + |s_3|)}{\log 3} \cong 1.248.$$

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