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On the quality of local search for the quadratic assignment problem

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Abstract

Local search is widely used to solve approximately NP-complete combinatorial optimization problems. But, little is known about quality of obtained local minima, for a given neighborhood. We concentrate on one of the most difficult optimization problems, the Quadratic Assignment Problem, and we give an upper bound for the quality of solutions obtained with deepest local search. Moreover, other recently established results on the traveling salesman problem, the graph bipartitioning problem and the maximum independent set problem can be deduced as particular cases. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Local search; Quadratic assignment problem; Maximum independent set

1. Introduction

Local search (respectively, deepest descent local search) works in an iterative fashion by successively replacing the current solution by a better (respectively, the best) one in the neighborhood of the current solution. It terminates when a locally optimal solution is attained, i.e. when no further improvement is possible.

In [4] a new complexity class, the class polynomial local search (PLS) has been defined. It contains local search problems for which local optimality can be verified in polynomial time. There exists a notion of reduction, and a PLS problem is said to be PLS-complete if all PLS problems can be reduced to it in polynomial time. Given a PLS-complete problem, if we could find a local optimum for it in polynomial time, then we could find a local optimum for any PLS problem in polynomial time too.

If we note C_{loc}^{-} the cost of any solution which is a local minimum relative to a certain neighborhood and C_{AV} the average cost over all possible solutions, it has been proved for the traveling salesman problem that $C_{\mathrm{loc}}^{-} \leqslant C_{\mathrm{AV}}$ with the 2-exchange neighborhood [3], the 3-exchange and the 2-opt neighborhood [1]. In this paper, we extend these results to the Quadratic Assignment Problem. In the next section, we give our main result concerning quality and complexity of the deepest local search

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method with a 2-exchange neighborhood, and in Section 3 we give some applications of these results on some combinatorial problems, namely the traveling salesman, the graph bipartitioning and the maximum independent set problems.

2. The Quadratic Assignment Problem

Given two $n \times n$ symmetric matrices $F = (f_{ij})$ and $D = (d_{ij})$, with a null diagonal, the symmetric Quadratic Assignment Problem (QAP) can be stated as follows:

$$\min_{\pi \in \Pi} \sum_{i=1}^{n} \sum_{k=i+1}^{n} f_{ik} d_{\pi(i)\pi(k)},$$

where Π is the set of all permutations of $\{1, 2, \ldots, n\}$. One of the major applications of the QAP is in location theory where f_{ij} is the flow of materials from facility i to facility j, and d_{ij} represents the distance from location i to location j [5]. The objective is to find an assignment of all facilities to locations which minimizes the total cost (see [8] for a survey of this problem).

The only known methods to solve exactly the QAP are branch and bound ones. However, due to time constraints, they remain limited to problems of size around 15, and only recently, exploiting symmetries in locations distribution, problems of size up to 20 have been solved exactly [6]. In all other cases, one must use approximate methods, i.e. heuristics. Recently, a comparison of eight heuristics [2] has led to a particularly interesting result: the remarkable effectiveness of greedy local search, which sometime outperforms more elaborate heuristics like simulated annealing and tabu search.

We consider the 2-exchange neighborhood which is the standard one for the QAP. Given a permutation $\pi = (\pi(1), \ldots, \pi(i), \ldots, \pi(j), \ldots, \pi(n))$, its neighbors are the $[n(n-1)]/2$ permutations of the form $(\pi(1), \ldots, \pi(j), \ldots, \pi(i), \ldots, \pi(n))$ for $1 \leqslant i < j \leqslant n$, obtained from π by a swap.

In [9], it has been proved that the Graph Bipartitioning Problem is PLS-complete for the swap neighborhood, and since it is a particular case of the symmetric QAP with the 2-exchange neighborhood, it follows that the QAP with this neighborhood is PLS-complete.

Let $s(A)$ denote the sum of all terms of a given matrix A. Let x and y two vectors of the same dimension. The maximum (respectively, minimum) scalar product of x and y is defined by: $\langle x, y \rangle_+ = \max_{\pi \in \Pi} \langle x, \pi y \rangle$ (respectively, $\langle x, y \rangle_- = \min_{\pi \in \Pi} \langle x, \pi y \rangle$).

Let F_k and D_k denote the sum over the kth column of F and D, respectively.

Let $\langle F, D \rangle_+$ (respectively, $\langle F, D \rangle_-$) be an abbreviation for $\langle (F_1, \ldots, F_n), (D_1, \ldots, D_n) \rangle_+$ (respectively, $\langle (F_1, \ldots, F_n), (D_1, \ldots, D_n) \rangle_-$).

We prove the following two theorems and corollary:

Theorem 1. *For the QAP, let C_{loc} the cost of any solution found by a deepest local search with the 2-exchange neighborhood, then the following inequality holds:*

$C_{\text{loc}}^{-} \leqslant (\langle F, D\rangle - fs(F)s(D))nC_{\text{AV}}$, *where* C_{AV} *is the average cost of all possible permutations.*

Corollary 1. *For the QAP, the following inequality holds:* $C_{\text{loc}}^{--} \leqslant (n/2)C_{\text{AV}}$. *Moreover, there is a sequence of instances for which the ratio* $C_{\max}/(n/2)C_{\text{AV}}$ *tends to infinity, where* $C_{\max}$ *is the maximum cost over all permutations.*

Theorem 2. *If the matrices F and D are positive integer ones, a deepest descent local search will reach a solution with a cost less than* $(n/2)C_{\text{AV}}$ *in at most* $O(n\log(s(F)s(D)/2))$ *iterations.*

The demonstration's technique is inspired from [3]. Let $\pi = (\pi(1), \ldots, \pi(n))$ any permutation, and π_{ij} the permutation obtained from π by swapping $\pi(i)$ and $\pi(j)$.

Let $C(\pi)$ be the cost of the permutation π:

$$C(\pi) = \frac{1}{2}\sum_{i,j=1}^{n} f_{ij}d_{\pi(i)\pi(j)}.$$

We define $\delta_{ij} = C(\pi_{ij}) - C(\pi)$.

Lemma 1. *The total difference cost relative to the 2-exchange neighborhood is given by* $\sum_{i,j=1}^{n}\delta_{ij} = 2s(F\pi(D)) - 4(n-1)C(\pi)$, *with the matrix* $\pi(D)$ *defined by* $(\pi(D))_{ij} = d_{\pi(i)\pi(j)}$.

Proof. Clearly,

$$\delta_{ij} = \sum_{k\neq i,j} f_{ik}d_{\pi(j)\pi(k)} + f_{jk}d_{\pi(i)\pi(k)} - f_{ik}d_{\pi(i)\pi(k)} - f_{jk}d_{\pi(j)\pi(k)}.$$

To simplify notations, we make the hypothesis the permutation π has been applied to the matrix D. So,

$$\begin{aligned}\delta_{ij} &= \sum_{k\neq i,j} f_{ik}d_{jk} + f_{jk}d_{ik} - f_{ik}d_{ik} - f_{jk}d_{jk}\\ &= \sum_{k=1}^{n}(f_{ik}d_{jk} + f_{jk}d_{ik} - f_{ik}d_{ik} - f_{jk}d_{jk}) + 2f_{ij}d_{ij}.\end{aligned}$$

Since

$$C(\pi) = \frac{1}{2}\sum_{i,j=1}^{n} f_{ij}d_{ij},$$

we have

$$\sum_{i,j=1}^{n}\delta_{ij} = \sum_{i,j,k=1}^{n}(f_{ik}d_{jk} + f_{jk}d_{ik} - f_{ik}d_{ik} - f_{jk}d_{jk}) + 4C(\pi).$$

By using

$$\sum_{i,j,k=1}^{n} f_{ik}d_{ik} = \sum_{i,j,k=1}^{n} f_{jk}d_{jk} = 2nC(\pi),$$

and $\sum_{i,j,k=1}^{n} f_{jk}d_{ik} = \sum_{i,j,k=1}^{n} f_{jk}d_{ki} = s(FD)$, the proof is completed. □

Lemma 2. *The average cost over all permutations is given by* $C_{AV} = s(F)s(D)/(2n(n-1))$.

Proof. By definition the average cost is $C_{AV} = \sum_{\pi} C(\pi)/n!$. Therefore, $C_{AV} = (1/2n!)\sum_{\pi}\sum_{i,j} f_{ij}d_{\pi(i)\pi(j)} = (1/2n!)\sum_{i,j} f_{ij}\sum_{\pi} d_{\pi(i)\pi(j)}$. With i and j fixed we have $\sum_{\pi} d_{\pi(i)\pi(j)} = (n-2)!s(D)$, as there are $(n-2)!$ permutations such that $\pi(i)=k$ and $\pi(j)=l$ with k and l fixed. So, $C_{AV} = (1/2n!)\sum_{i,j} f_{ij}(n-2)!\,s(D) = s(F)s(D)/(2n(n-1))$. □

We are ready now to give the proof of our main theorem.

Proof of the Theorem 1. We note, by analogy with the space Laplacian operator, $\nabla^2 C(\pi)$ the average cost difference between the permutation π and its neighbors.

$$\begin{aligned}\nabla^2 C(\pi) &= \frac{\sum_{i,j=1}^{n} \delta_{ij}}{n(n-1)} \\ &= \frac{4C(\pi) - 4nC(\pi) + 2s(F\pi(D))}{n(n-1)} \quad \text{(by Lemma 1)} \\ &= \frac{-4C(\pi)}{n} + \frac{2s(F\pi(D))}{n(n-1)}.\end{aligned}$$

The key idea is to bound $\nabla^2 C(\pi)$ and to use the fact that at each step there always exists an exchange i,j such that $\delta_{i,j} \leqslant \nabla^2 C(\pi)$.

We put $\bar{C}(\pi) = C(\pi) - C_{AV}$. Thus,

$$\nabla^2 C(\pi) = \frac{-4\bar{C}(\pi)}{n} - \frac{4C_{AV}}{n} + \frac{2s(F\pi(D))}{n(n-1)}.$$

By using Lemma 2 we obtain

$$\nabla^2 C(\pi) = \frac{-4\bar{C}(\pi)}{n} - \frac{2s(F)s(D)}{n^2(n-1)} + \frac{2s(F\pi(D))}{n(n-1)}.$$

The last equation is rewritten

$$\nabla^2 C(\pi) = \frac{-4\bar{C}(\pi)}{n} + \alpha(\pi),$$

where

$$\alpha(\pi) = \frac{2s(F\pi(D))}{n(n-1)} - \frac{2s(F)s(D)}{n^2(n-1)}.$$

Since the inequality

$$s(F\pi(D)) \leqslant \frac{s(F)s(D)}{n}$$

does not always hold, $\alpha(\pi)$ is not always negative, see Remark 1.

Let $\alpha = \max_\pi \alpha(\pi) = \max_\pi (2/n(n-1))(s(F\pi(D)) - s(F)s(D)/n)$.

Then, $\nabla^2 C$ is bounded by

$$\nabla^2 C(\pi) \leqslant \frac{-4}{n}(C(\pi) - C_{\mathrm{AV}}) + \alpha.$$

So, since there always exists an exchange i,j such that $\delta_{i,j} \leqslant \nabla^2 C(\pi)$, the deepest local search algorithm picks a movement at least as good as this one, and the cost of the solution is at the next step less than $C(\pi) - (4/n)(C(\pi) - C_{\mathrm{AV}}) + \alpha$.

Iterating this process gives a limit cost l which verifies $l = l - (4/n)(l - C_{\mathrm{AV}}) + \alpha$. It follows that $l = \alpha n/4 + C_{\mathrm{AV}}$, and thus,

$$\frac{l}{C_{\mathrm{AV}}} = 1 + \frac{\alpha n}{4C_{\mathrm{AV}}}.$$

By using Lemma 2 and the expression of α we obtain

$$\frac{l}{C_{\mathrm{AV}}} = \max_\pi \frac{s(F\pi(D))}{s(F)s(D)} n.$$

By the fact that

$$(F\pi(D))_{ij} = \sum_{k=1}^{n} f_{ik} d_{\pi(k)\pi(j)},$$

we have

$$s(F\pi(D)) = \sum_{i,j,k=1}^{n} f_{ik} d_{\pi(k)\pi(j)} = \sum_{i,k=1}^{n} f_{ik} \sum_{j=1}^{n} d_{\pi(j)\pi(k)} = \sum_{k=1}^{n} F_k D_{\pi(k)},$$

and so, $\max_\pi s(F\pi(D)) = \langle F, D\rangle_+$.

Therefore,

$$\frac{C_{\mathrm{loc}}^-}{C_{\mathrm{AV}}} \leqslant \frac{l}{C_{\mathrm{AV}}} \leqslant \frac{\langle F, D\rangle_+}{s(F)s(D)} n. \qquad \square$$

Remark 1. The average cost of $\alpha(\pi)$ is null, i.e. $aver_\pi\, \alpha(\pi) = 0$.

By symmetry,

$$\sum_\pi \overline{C}(\pi) = 0 \quad \text{and} \quad \sum_\pi \nabla^2 C(\pi) = \sum_\pi \nabla^2 \overline{C}(\pi) = 0.$$

Therefore,

$$\sum_{\pi\in\Pi} s(F\pi(D)) = \frac{\sum_{\pi\in\Pi} s(F)s(\pi(D))}{n} = \frac{n!s(F)s(D)}{n}$$
$$= (n-1)!s(F)s(D),$$

and

$$s(F\pi(D))_{\text{aver}} = \frac{s(F)s(D)}{n}.$$

Which implies $\nabla^2 C(\pi) \simeq -(4/n)(C(\pi) - C_{\text{AV}})$. □

Remark 2. Let C^-_{loc} the cost of a local maximum solution obtained by a deepest local search. Changing matrix D with $-D$ and using $\langle F,-D\rangle_+ = -\langle F,D\rangle_-$, we deduce from Theorem 1 that

$$C^+_{\text{loc}} \geqslant \frac{\langle F,D\rangle_-}{s(F)s(D)} nC_{\text{AV}}.$$

In order to prove Corollary 1 we need the following result.

Lemma 3. *We have for the maximum scalar product* $1/n \leqslant \langle F,D\rangle_+/s(F)s(D) \leqslant 1/2$.

Proof. Without loss of generality, we make the hypothesis F_i and D_i are arranged in increasing order.

Let π_k, for $1 \leqslant k \leqslant n$, be the circular permutation of $\{1,\ldots,n\}$ such that $\pi_k(1) = k$.

We put $S_k = \{(i,\pi_k(i)) \mid i = 1,\ldots,n\}$, and $S = \{1,\ldots,n\}^2$.

The S_k form a partition of S, i.e. $\bigcup_{k=1}^n S_k = S$, and $S_i \cap S_j = \emptyset$ for $i \neq j$.

Notice that $\langle F,D\rangle_+ = \sum_{(i,j)\in S_1} F_iD_j$, and

$$s(F)s(D) = \sum_{(i,j)\in S} F_iD_j = \sum_{k=1}^n \sum_{(i,j)\in S_k} F_iD_j.$$

For the left-hand part of the inequality, we have

$$n\langle F,D\rangle_+ = n\sum_{(i,j)\in S_1} F_iD_j \geqslant \sum_{k=1}^n \sum_{(i,j)\in S_k} F_iD_j = s(F)s(D).$$

For the right-hand part of the inequality, we have

$$\frac{\langle F,D\rangle_+}{s(F)s(D)} \leqslant \frac{1}{2} \Leftrightarrow 2\sum_{i=1}^n F_iD_i \leqslant \left(\sum_{i=1}^n F_i\right)\left(\sum_{i=1}^n D_i\right)$$
$$\Leftrightarrow \sum_{i=1}^n F_iD_i \leqslant \sum_{i=1}^n \left(F_i \sum_{\substack{j=1 \\ j\neq i}}^n D_j\right),$$

which is always true, since the D_i are arranged in increasing order, for all $i<n$, F_iD_i is bounded by F_iD_{i+1}, and therefore, by $F_i\sum_{j\neq i}D_j$. Also, F_nD_n is majored by $F_n\sum_{j\neq n}D_j$, because $D_n\leqslant D_1+D_2+\cdots+D_{n-1}$ as the matrix D is symmetric with a null diagonal. □

Notice that the upper bound is tight. In fact, consider the following input data:

$$F=\begin{pmatrix}0 & 0 & 0\\ 0 & \ddots & \alpha\\ 0 & \alpha & 0\end{pmatrix} \quad \text{and} \quad D=\begin{pmatrix}0 & 0 & 0\\ 0 & \ddots & \beta\\ 0 & \beta & 0\end{pmatrix} \qquad \alpha,\beta\in\mathbb{Z}^*$$

Obviously, $\langle F,D\rangle_+=\langle(0,\ldots,0,\alpha,\alpha),(0,\ldots,0,\beta,\beta)\rangle_+=2\alpha\beta$ and $s(F)s(D)=4\alpha\beta$. It follows that $\langle F,D\rangle_+/s(F)s(D)=1/2$.

Proof of Corollary 1. The inequality $C^-_{\mathrm{loc}}\leqslant(n/2)C_{\mathrm{AV}}$ follows by applying Lemma 3 to Theorem 1.

As concerns the second part of the corollary which characterizes the quality of the given upper bound, let us consider the following instances:

Take $F=D$ with

$f_{ii}=0\ \forall i\in\{1,\ldots,n\}$,

$f_{n(n-1)}=f_{(n-1)n}=n^2$,

$f_{ij}=1$, otherwise.

By Lemma 2.

$$\frac{n}{2}C_{\mathrm{AV}}=\frac{s(F)(D)}{4(n-1)}=\frac{(n-1)(3n+2)^2}{4},$$

and obviously, $C_{\max}=n(n-1)/2-1+n^4$. Thus, the ratio $C_{\max}/(n/2)C_{\mathrm{AV}}$ tends to infinity. □

Proof of theorem 2. We have seen that at each step the cost $C(\pi)$ of the current solution π becomes less than $C(\pi)-(4/n)(C(\pi)-C_{\mathrm{AV}})+\alpha$. If we note $\tilde{C}(\pi)=C(\pi)-l$ (l is the limit cost defined above) it is equivalent to say that a each step, $\tilde{C}(\pi)$ becomes less than $(1-4/n)\tilde{C}(\pi)$. Indeed, at each step $C(\pi)-l$ becomes less than $C(\pi)-(4/n)(C(\pi)-C_{\mathrm{AV}})+\alpha-(\alpha n/4+C_{\mathrm{AV}})=(1-4/n)(C(\pi)-\alpha n/4-C_{\mathrm{AV}})$. Since F and D are integer matrices, the cost at each step must decrease at least one. So, if we have L such that $C_{\max}-l\leqslant 2^L$ the number of steps t until $\tilde{C}(\pi)$ is less than zero satisfies $(1-4/n)^t<2^{-L}$, that is $t>-L/\log_2(1-4/n)\sim O(nL)$ when $n\to\infty$. We have $C_{\max}-l\leqslant C_{\max}\leqslant s(F)s(D)/2$, and the result follows. □

Notice that we do not claim that the entire local search will use at most this polynomial number of iterations. Since the QAP is PLS-complete, this would be equivalent to say that $P=NP$! Certainly, for some instances the search will go on.

3. Applications

In this section, we apply results of the previous section to some particular cases of the QAP, namely the Symmetric Traveling Salesman Problem and the Graph Bipartitioning Problem. We also introduce two new optimization problems and we study the Maximum Independent Set Problem.

Notice that, when one of the matrices, say F, has constant row sums, i.e. $Fe = \lambda e$, for e the vector of all ones, then $\langle F, D\rangle_+ / s(F)s(D) = 1/n$, and it follows that $C_{\text{loc}}^- \leqslant C_{\text{AV}}$ from Theorem 1 (one can also notice that $\alpha(\pi) = 0$).

3.1. The Symmetric Traveling Salesman Problem

Obviously, the Symmetric Traveling Salesman Problem with n cities can be seen as a particular case of QAP by considering D to be the distance matrix and F to be defined by $f_{i,i+1} = f_{i+1,i} = 1$ with $1 \leqslant i \leqslant n-1$, $f_{n,1} = f_{1,n} = 1$ and $f_{ij} = 0$ otherwise.

Using the above remark, we obtain for the 2-exchange neighborhood, $C_{\text{loc}}^- \leqslant C_{\text{AV}}$, a result also previously found by Grover [3].

In the sequel, given a graph, we must select a fixed number of vertices in order to maximize or minimize an associated cost. The neighborhood of a solution consists of all solutions that can be obtained from the current one by selecting an unselected vertex and unselecting a selected vertex. We call this neighborhood the swap neighborhood.

3.2. The Graph Bipartitioning Problem

Recall that given a graph with an even number n of vertices the Graph Bipartitioning Problem consists of finding a partition of the vertices into two equal-sized subsets A and B such that the number of edges having one extremity in A and the other in B is minimized.

For this problem D is the adjacency matrix of the graph, and

$$F = \begin{pmatrix} 0 & U \\ U & 0 \end{pmatrix},$$

where U is the $n/2 \times n/2$ matrix, with $u_{ij} = 1$, $i, j = 1, \ldots, n/2$.

Using the above remark, we obtain for the swap neighborhood, $C_{\text{loc}}^- \leqslant C_{\text{AV}}$, a result previously found by Grover [3].

3.3. Generalized maximum independent set and maximum clique problems

We define two new NP-complete optimization problems which are obvious generalizations the first of the maximum independent set and the second one of the maximum clique problem.

Given a graph $G=(V,E)$ and a number m $(m\leqslant|V|)$, find m vertices of G such that the number of edges in the subgraph induced by these vertices is minimum (respectively maximum).

These problems are NP-hard. In fact, if we could solve them in polynomial time, then by varying m between 1 and $|V|$ we could find a maximum independent set and a maximum clique in polynomial time too.

In the sequel, we consider that $d_1, d_2, \ldots, d_n$ are the degrees of vertices of G arranged in decreasing order.

Proposition 1. *The local optimal solution found by a deepest local search with the swap neighborhood satisfies $C^-_{\mathrm{loc}}\leqslant((m-1)/2(n-1))(d_1+d_2-\cdots+d_m)$ (respectively, $C^+_{\mathrm{loc}}\geqslant((m-1)/2(n-1))(d_n+d_{n-1}+\cdots+d_{n-m+1})$) for the minimization (respectively, maximization) problem.*

Proof. One can modelize them by a QAP with D the adjacency matrix of graph G and $F=(f_{ij})$, where $f_{ij}=1$ if $i\neq j$, $1\leqslant i,\ j\leqslant m$, and $f_{ij}=0$ otherwise.

Then with a permutation π are associated m vertices $v_{i_1},\ldots,v_{i_m}$ of G such that $\forall k,\ \pi(i_k)\leqslant m$ and the cost of π is equal to the number of edges in the subgraph induced by these vertices. Conversely, with each set of m vertices $v_{i_1},\ldots,v_{i_m}$ of G corresponds a class of permutations: those such that $\forall k,\ \pi(i_k)\leqslant m$.

We have $s(F)=m(m-1)$ and $s(D)=2|E|$. By using Lemma 2 we obtain $C_{\mathrm{AV}}=s(F)s(D)/(2n(n-1))=[(m-1)m/((n-1)n)]|E|$. Moreover, we have $\langle F,D\rangle_1=(m-1)(d_1+d_2+\cdots+d_m)$, and the result follows by Theorem 1. □

3.4. The Maximum Independent Set Problem

Proposition 2. *If $d_1+d_2+\cdots+d_k\leqslant\lfloor 2c(n-1)/(k-1)\rfloor_*$, with $2\leqslant k\leqslant n$ and c any integer, the deepest local search with the swap neighborhood finds an independent set with at least $k-c+1$ vertices. By definition, $\lfloor x\rfloor_*$ is equal to $x-1$ if x is an integer, and $\lfloor x\rfloor$ otherwise.*

Proof. Let k vertices chosen randomly in a graph G, they are called "marked vertices". The cost of a configuration is equal to the number of edges in the subgraph induced by the marked vertices. A swap consists of simultaneously mark an unmarked vertex, and unmark a marked vertex. Consider now, the deepest local search algorithm which at each step examines all the swaps (a polynomial number) and make the best one. The condition $d_1+d_2+\cdots+d_k\leqslant\lfloor 2c(n-1)/(k-1)\rfloor_*$ implies that $[(k-1)/(2(n-1))](d_1+d_2+\cdots+d_k)\leqslant c-1$. Therefore, by using Proposition 1 we deduce that the attained local optimum includes at most $c-1$ edges. It is sufficient now to remove at most $c-1$ marked vertices in order to obtain an independent set. □

This result can be compared with the following theorem.

Theorem 3 (Meyer [7]). *If* $d_1+d_2+\cdots+d_{k'} \leqslant n-k'$, *with* $2 \leqslant k' \leqslant n$, *then any maximal independent set has at least* k' *vertices.*

Indeed, notice that the maxima independent sets are precisely the local minima for the above local search algorithm.

Notice that our result is more flexible than Meyer's theorem. By varying the parameter c we can choose the strength of the hypothesis. In return, our bound is sometimes less good than the Meyer's one. Notice, however that this is a deductive result, while our result has the advantage to be constructive.

4. Conclusions

We have given an upper bound for the solutions obtained by local search algorithms, for the Quadratic Assignment Problem with respect to the 2-exchange neighborhood. However, the question remains open to show whether or not this bound is attained. It would be interesting to have an upper bound for C_{loc}^{-} where the size n of the problem does not occur. But notice that if we conserve the shape of the bound of Theorem 1, that is the term $\langle F, D\rangle_+ / s(F)s(D)$, the bound cannot be improved. Indeed, recall that when restricted to the Symmetric Traveling Salesman Problem, this bound becomes C_{AV}, a result which is optimal (think at the special case when all entries of the distance matrix are equal, then all the solutions have the same cost equal to C_{AV}). Our general result, when it is applied to other optimization problems seen as particular cases of the Quadratic Assignment Problem provides some known results. Continuing in this way we could provide a hierarchy of combinatorial optimization problems relative to the quality of solutions obtained with local search algorithms.

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