Extended Hopfield Models for Combinatorial Optimization

Armelle Le Gall and Vassilis Zissimopoulos

***Abstract*— The extended Hopfield neural network proposed by Abe *et al.* for solving combinatorial optimization problems with equality and/or inequality constraints has the drawback of being frequently stabilized in states with neurons of ambiguous classification as active or inactive. We introduce in the model a competitive activation mechanism and we derive a new expression of the penalty energy allowing us to reduce significantly the number of neurons with intermediate level of activations. The new version of the model is validated experimentally on the set covering problem. Our results confirm the importance of instituting competitive activation mechanisms in Hopfield neural-network models.**

***Index Terms*—Activation mechanism, combinatorial optimization, competitive heuristic, Hopfield neural network, inequality constraints, set covering problem.**

I. Introduction

Abe *et al.* [2] have given an extension of the Hopfield model [1], [5] in order to handle equality or/and inequality constraints. The extended Hopfield model (EHM) introduces in the objective function some additional energy terms which penalize any infeasible state. For determining an appropriate expression for the penalty energy in the inequality constraints case, each inequality constraint is converted to an equality constraint. This is obtained by introducing an additional variable which is managed by a new neuron. Each new neuron is connected to the initial neurons where their corresponding variables occur in its linear combination. The disadvantage of the EHM is to produce in many cases uninterpretable stable solutions, that is, some neurons with activation levels far from the suitable ones, i.e., zero and one. The aim of this paper is to reduce such undesirable situations and to improve EHM performance by introducing a competitive activation mechanism in the model. In a relative work, to handle inequality constraints, Ohlsonn *et al.* [9] have used another expression for the additional energy term and in [10] and [11] a Potts glass theory technique using mean field theory has been developed and tested on the knapsack problem and the assignment problem.

Here, we use an analogy between handling inequality constraints strategy and an allocation problem of limited resources. For each constraint a fictional resource, the amount of which represents the degree of constraint unsatisfaction, is attributed to the corresponding new neuron. At its present form, the EHM takes as resource amounts the differences between the bounds and the current values of the linear combinations. Then, it distributes them proportionably to the receiving neuron activations.

In order to reinforce in EHM neurons repartition into two categories: active or inactive, *we institute an inequitable way to distribute the resources.* Henceforth, in the lower-bounded constraint case (positive resources) the most active neurons are being in favor whereas in the upper-bounded constraint case (negative resources) there are the most inactive neurons that are favorized. These inequitable allocation rules create inhibitory relationships between neurons which are in competition to acquire the same limited resource. For the case of positive resources Reggia has defined [12] the rules which must govern resources allocation in order to select a limited number of neurons as competition winners. This allocation method, reported as competitive activation mechanism, is well suited to resolve unexclusive in nature competitions, that is, competitions eventually requiring some collaborations between competitors for elaborating a global solution. The competitions implemented to satisfy inequality constraints are owned to this category.

Here, *we extend this competitive activation mechanism for dealing with negative resources and we deduce a new expression for the energy term which integrates the inequality constraints in the Hopfield model.* This energy generates activation rules ensuring a better neurons repartition to two sets. The set including neurons with near to zero activation and the set including neurons with near to one activation. We show also how to evaluate *the integration weights of the new penalty energy* essentially based on the idea developed by Abé [1] for dealing with the equality constraints on the traveling salesman problem. We obtain in this way for a combinatorial optimization problem that any infeasible solution located on a vertex in the proximity of a feasible solution can not be a local minimum of the network's energy.

The new method for handling inequality constraints is validated experimentally on the set covering problem (SCP) [6]. In the next section we describe briefly the EHM. In Section III we discuss different competitive mechanisms and present an extension of the EHM including competitive activation mechanisms. We derive the new activation rules and we show how to define the main involved parameters. In Section IV we discuss the set covering problem and we adapt the neural model for solving it. In Section V we present an extensive experimental study which confirms the utility of the instituted competitive activation mechanism.

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A. LeGall is with the LRI, CNRS-URA 410, Bât 490, Université de Paris Sud, 91405 Orsay Cedex, France.
V. Zissimopoulos is with the LIPN, CNRS-URA 1507, Université de Paris Nord, 93430 Villetaneuse, France.
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II. The Extended Hopfield Model

The original Hopfield model is a neural-network model adequate for solving optimization problems which can be expressed by a quadratic function of the form

$$\begin{aligned}&\text{minimize } E_1(x) = \tfrac{1}{2}\, x^t P x + q^t x\\ &\text{with } x \in \{0, 1\}^n\end{aligned} \tag{1}$$

where $x = (x_1, \cdots, x_n)^t$ is the vector of problem variables, q a nth constant vector and P a symmetric $n \times n$ matrix with $P_{ii} = 0, i = 1, \cdots, n$ [5].

The EHM proposed by Abe *et al.* [2] is able to handle the following constraints.

- The l equality constraints: $r_i^t x = s_i, i = 1, \cdots, l$, where $r_i^t = (r_{1i}, \cdots, r_{ni})$.
- The k inequality constraints: $w_i^t x \leq d_i$ or $w_i^t x \geq d_i$ where $d_i > 0$ and $w_i^t = (w_{1i}, \cdots, w_{ni})$, $w_{ji} \in \mathcal{R}$, $j = 1, \cdots, n$, $i = 1, \cdots, k$.

We describe the EHM with a slight modification allowing us to handle also inequality constraints with $d_i = 0$. This case will be useful later (Sections IV and V) and it affects the definition domain of the introduced new variables.

Handling constraints implies the weighted insertion of two energy terms into the initial objective function E_1. The first term, denoted by E_2, is related to the equality constraints and the second term, denoted by E_3, is related to the inequality constraints. The minimizing by the network function is given by $E = AE_1 + BE_2 + CE_3$ where A, B, and C are strictly positive weights.

Let us first consider the equality constraints. We are looking for a function E_2 of x whom the minimum is obtained only when every equality constraint $r_i^t x = s_i, i = 1, \cdots, l$, is satisfied. It is defined as follows:

$$E_2 = \sum_{i=1}^{l} \left[\tfrac{1}{2}\,(r_i^t x)^2 - s_i r_i^t x\right]. \tag{2}$$

By denoting $R = \sum_{i=1}^{l} r_i r_i^t$ and $s = -\sum_{i=1}^{l} s_i r_i$, E_2 can been written in a matricial form: $E_2 = 1/2 x^t R x + s^t x$. Let $T' = AP + BR$ and $b = Aq + Bs$. Then, the function $AE_1 + BE_2 = 1/2 x^t T' x + b^t x$ is a quadratic function and so, the initial Hopfield model can be used for handling equality constraints.

For handling the k inequality constraints we convert them into equality constraints by introducing new variables y_i, for $i = 1, \cdots, k$, brought together into the vector $y = (y_1, \cdots, y_k)^t$. Then, it is easy to prove the following lemma.

Lemma 1: For i varying from one to k, the inequality constraint $w_i^t x \leq d_i$ or $w_i^t x \geq d_i$ with nonnegative bound d_i is equivalent to the equality constraint $d_i' y_i - w_i^t x = 0$ with

$$d_i' = \begin{cases} d_i, & \text{if } d_i \neq 0\\ 1, & \text{otherwise}\end{cases}$$

and

$$\begin{cases} y_i \leq 0, & \text{if } w_i^t x \leq 0\\ y_i \geq 0, & \text{if } w_i^t x \geq 0\\ y_i \leq 1, & \text{if } w_i^t x \leq d_i \qquad (d_i \neq 0)\\ y_i \geq 1, & \text{if } w_i^t x \geq d_i \qquad (d_i \neq 0).\end{cases}$$

This preliminary step allows us to use the same schema as the one used for the equality constraints. So, the energy function E_3 is given by

$$E_3(x, y) = \tfrac{1}{2} \sum_{i=1}^{k} (w_i^t x - d_i' y_i)^2. \tag{3}$$

With the notations

$$\begin{aligned} V' &= -(d_1' w_1, \cdots, d_k' w_k)\text{: } n \times k \text{ matrix}\\ W' &= \sum_{i=1}^{k} w_i w_i^t\text{: } n \times n \text{ matrix}\\ D' &= \begin{pmatrix} d_1'^2 & \cdots & 0\\ 0 & & d_k'^2\end{pmatrix}\text{: } k \times k \text{ diagonal matrix.}\end{aligned}$$

E_3 can be written as follows:

$$E_3(x, y) = \tfrac{1}{2}\,(x^t, y^t)\begin{pmatrix} W' & V'\\ V'^t & D'\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}.$$

By putting $W = CW'$, $V = CV'$, $D = CD'$ and $T = T' + W$, the global energy $E = AE_1 + BE_2 + CE_3$ that the model must to minimize is written

$$E(x, y) = \tfrac{1}{2}\,(x^t, y^t)\begin{pmatrix} T & V\\ V^t & D\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} + b^t x. \tag{4}$$

We obtain for the EHM the following result handling also inequality constraints with $d_i = 0$.

Proposition 2: A two layers connectionist model whom the behavior is controlled by the dynamic system

$$\begin{aligned}\begin{pmatrix} du/dt\\ dv/dt\end{pmatrix} &= -\begin{pmatrix} \partial E/\partial x\\ \partial E/\partial y\end{pmatrix}\\ &= -\begin{pmatrix} T & V\\ V^t & D\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} - \begin{pmatrix} b\\ 0\end{pmatrix}\end{aligned} \tag{5}$$

with $u \in (-\infty, +\infty)^n$ and $x \in (0, 1)^n$ the vectors of internal and coming-out variables for the first layer

$$\forall j = 1, \cdots, n x_j = \frac{1}{2}\left(1 + \tanh \frac{u_j}{\tau_j}\right) \qquad (\text{with } \tau_j > 0)$$

and $v \in (-\infty, +\infty)^k$ and y the vectors of internal and coming-out variables for the second layer

$$\forall i = 1, \cdots, k$$

$$y_i = \begin{cases} -\exp(-v_i/\rho_i), & \text{if } y_i \leq 0\\ \exp(v_i/\rho_i), & \text{if } y_i \geq 0\\ 1 - \exp(-v_i/\rho_i), & \text{if } y_i \leq 1\\ 1 + \exp(v_i/\rho_i), & \text{if } y_i \geq 1\end{cases} \qquad (\text{with } \rho_i > 0) \tag{6}$$

converges to a local minimum of the function E.

The convergence proof consists to establish that the function E is a Lyapunov function for the dynamic system defined by (5). In [1] and [2], the conditions for the system to converge to a vertex, a point on the surface or an interior point in the n-dimensional hypercube have been studied by an eigenvalue analysis of the linearized system.

In summary, the EHM has two layers. The first layer includes n neurons Ξ_j, $j = 1, \cdots, n$, interconnected following the matrix T. Each such neuron is associated to a variable x_j. The second layer includes k neurons Υ_i, $i = 1, \cdots, k$, each one corresponding to the new variable y_i. A neuron Υ_i is linked to the Ξ_j neuron with a synaptic weight w_{ji} which is exactly the coefficient of the variable x_j in the linear combination $w_i^t x$. The model operates in a synchronous mode alternatively on the two layers until reaching a stable state. The rules which govern the evolution of neurons Ξ_j and Υ_i are, respectively,

$$\begin{aligned}\frac{dx_j}{dt} &= \frac{2}{\tau_j}\left(C \sum_{i=1,\cdots,k} w_{ji}x_j[d_i' y_i - w_i^t x] - [T_j' x + b_j]x_j\right) \\ &\quad \cdot (1 - x_j) \\ \frac{dy_i}{dt} &= -Cd_i' f(y_i)(d_i' y_i - w_i^t x) \end{aligned} \tag{7}$$

with

$$f(y_i) = \begin{cases} -y_i/\rho_i & \text{if } y_i \leq 0 \\ y_i/\rho_i & \text{if } y_i \geq 0 \\ (1 - y_i)/\rho_i & \text{if } y_i \leq 1 \\ (y_i - 1)/\rho_i & \text{if } y_i \geq 1. \end{cases} \tag{8}$$

III. Competitive Activation Mechanisms

In order to estimate the capacity of the EHM to share out the neurons in active and in inactive ones, we consider an analogy between the inequality constraints and an allocation problem of limited resources $R_i = d_i' y_i - w_i^t x$ for $i = 1, \cdots, k$. Each neuron Υ_i evaluates the amount of its resource which is interpreted as a constraint unsatisfaction measure. An analysis of the EHM dynamic shows that, for a satisfied constraint, the unsatisfaction degree tends to zero because $d_i' y_i$ converges asymptotically to $w_i^t x$. For an unsatisfied constraint, $d_i' y_i$ converges asymptotically to d_i, and so, R_i converges to $d_i - w_i^t x$. By distributing its resource to the neighbors Ξ_j, a neuron Υ_i influences the evolution of the network toward to a state which should satisfy its corresponding constraint. Resources distribution is realized by sending out flows, $out_{\Upsilon_i \to \Xi_j}$, which are proportional to the amount R_i and to the activation level x_j of the receiving neuron

$$\forall i = 1, \cdots, k, \; \forall j = 1, \cdots, n, \; \text{out}_{\Upsilon_i \to \Xi_j} = w_{ji} x_j R_i. \tag{9}$$

Notice that a such distribution does not produce the desirable neurons sharing out. Indeed, when $w_i^t x \geq d_i$ (positive resource) the advantage offered to a strongly activated receiving neuron Ξ_j is almost eliminated by the inhibition term $[T_j' x + b_j]x_j$ which is also proportional to its self activation level x_j. Therefore, the activations of the less active neurons progress as far as the ones of the more active neurons. So, many neurons after networks stability will have activation values which are far from both one and zero. Also, when $w_i^t x \leq d_i$ (negative resource) the neurons which must receive large negative flows are the neurons with weak activations in order to ensure their activation decreament and let the neurons with strong activations near the bound one. However, the model in its present form tends to decrease the activation of the strongly activated neurons by sending to them large negative flows. As a consequence, a stable state will still include neurons with uninterpretable activations.

An inequitable allocation of resources achievable by instituting competitions between all Ξ_j neurons owning a common resource aims to remedy this drawback.

Definition 1: Two neurons are said to be competitors if the gain of one occurs at the expense of the other, i.e., their functional relationships are inhibitory in nature [4].

This definition of competition in neural networks, due to Grossberg, indicates inhibitory interactions but it says nothing about the underlying mechanisms involved in producing those inhibitory interactions. The *direct* or *antagonist* competition is a well-known competition mechanism. It occurs when an active entity directly interacts with rival entities by sending out negative flow in order to suppress theirs activities. Since, all neurons having lost a competition can not become active. This mechanism generates a single *winner-takes-all* behavior. But, in some applications, in order to produce a global solution some competitors must cooperate. For capturing competitive but cooperative relationships between neighbors and producing so a *multiple-winners-take-all* behavior, the following mechanism, called *indirect competition* or *competitive activation mechanism (cam)* [8], [12], [13], could be more adapted. It occurs when two rivals require and consume the same limited resource. Under the hypothesis that the connection weights and the resources are not negative, for sharing out the competitors between the losers and the winners, the resources must respect the following principles.

- The activation flows should be representative of the way by which the emitting neuron wishes allocating its resource between its neighboring neurons. For this, the activation flows are positive and their amount is equal to the resource.
- An inhibition term in the activation rule should decrease the activations of the neurons not acquiring enough resources. This term prevents network's saturation. For a neuron Ξ_j, the activation rule looks as: $dx_j/dt = [In_j - Ih_j x_j](1 - x_j)$ with In_j the sum of input activation flows and $Ih_j x_j$ its inhibition term. The factor $(1 - x_j)$ and the inhibition term, which is proportional to its self activation x_j, are used to prevent the activations from overstepping their upper or lower bound.
- Large activation flows should be sent to the stronger activated neurons in order to they acquire the active level in spite of their inhibition terms increament.

We return now in the combinatorial optimization problem and the EHM. In general, many variables equal to one are needed to satisfy a lower-bounded inequality constraint. By using the EHM this type of constraint is solved by generating positive resources. So, the competitive mechanism is well suited to use. But, for upper-bounded inequality constraints the resources are negative and their satisfaction, in general, needs several variables equal to zero. So, it is worthful to extend the *cam* for handling also negative resources. This can be done by sending inhibitory flows tending to decrease the activation of the receiving neurons preferably those having weak activations. In summary, ensuring *cam* in the EHM the following conditions are required. These conditions reflect the previously described principles of *cam* but with positive and negative resources.

1)

$$\forall\,\Xi_j,\ \mathrm{out}_{\Upsilon_i\to\Xi_j}\begin{cases}\geq 0, & \text{if the constraint type is } w_i^t x \geq d_i \quad (R_i \geq 0)\\ \leq 0, & \text{if the constraint type is } w_i^t x \leq d_i \quad (R_i \leq 0).\end{cases}$$

2) $\mathrm{out}_{\Upsilon_i} = \sum_{j=1}^{n} \mathrm{out}_{\Upsilon_i\to\Xi_j} = R_i$.

3) If $w_{ji} = w_{pi} \neq 0$ and $x_j > x_p$ then

$$\frac{\mathrm{out}_{\Upsilon_i\to\Xi_j}}{\mathrm{out}_{(\Upsilon_i\to\Xi_p}}\begin{cases}> x_j/x_p, & \text{if } R_i > 0\\ < x_j/x_p, & \text{if } R_i < 0.\end{cases}$$

It can be seen that in the EHM the flows $out_{\Upsilon_i\to\Xi_j} = w_{ji}x_jR_i$ which distribute the resource $R_i = d_i'y_i - w_i^t x$ do not respect the second and the third properties of *cam*. In fact, the sign of the sent flows is identical to the one of the resource R_i but the global amount emitting by the neuron Υ_i, i.e., $\mathrm{out}_{\Upsilon_i} = w_i^t xR_i$, is not equal to the resource value. For the third condition, let consider two competitors Ξ_j and Ξ_p requiring the same resource R_i and having $w_{ji} = w_{pi}$. Clearly, they receive flows with a ratio equal to the ratio of their activation levels

$$\forall\,\Xi_j,\,\Xi_p,\ w_{ji} = w_{pi},\ \frac{\mathrm{out}_{\Upsilon_i\to\Xi_j}}{\mathrm{out}_{\Upsilon_i\to\Xi_p}} = \frac{x_j}{x_p}.$$

Under only the influence of the neuron Υ_i, if the two activation levels x_j and x_p are almost identical, the two neurons are going to evolute identically, since we have $dx_j/dt \simeq dx_p/dt$. In the next section, we show how to ensure in EHM the two not respected conditions of *cam*.

A. The Competitive Activation Extended Hopfield Model

The second condition, requiring that neuron Υ_i sends out a global quantity of flows equal to its resource amount R_i, is simply obtained by normalization of the flows $\mathrm{out}_{\Upsilon_i\to\Xi_j}$, i.e., by sending flows proportional to the rate $w_{ji}x_j/w_i^t x$, $i = 1,\cdots,k$, $j = 1,\cdots,n$. However, a such expression is partially correct; it still remains to favor the best competitors, that is, the most active (respectively, the less active) in the lower bounded inequality constraints case (respectively, the upper bounded inequality constraints). For this, before the normalization phase, we raise into the linear combination $w_i^t x$ the variables x_j to a power α_i. This modification generates flows proportional to $w_{ji}x_j^{\alpha_i}/w_i^t x^{\alpha_i}$ where x^{α_i} denotes the vector $(x_1^{\alpha_i},\cdots,x_n^{\alpha_i})^t$. The power α_i is allowing us to accentuate the existing gap between the activation levels of neurons Ξ_j. Indeed

if $w_{ji} = w_{pi} \neq 0$ and $x_j > x_p$ then

$$\frac{\mathrm{out}_{\Upsilon_i\to\Xi_j}}{\mathrm{out}_{\Upsilon_i\to\Xi_p}} = \left[\frac{x_j}{x_p}\right]^{\alpha_i}\begin{cases}> x_j/x_p, & \text{if } \alpha_i > 1\\ < x_j/x_p, & \text{if } 0 < \alpha_i < 1.\end{cases}$$

In this expression, if the type of ith constraint is $w_i^t x \geq d_i$ ($w_i^t x \leq d_i$) the condition $\alpha_i > 1$ (respectively, $0 < \alpha_i < 1$) ensures the third condition of *cam*.

Notice that the contribution of the variable x_j is always bounded by the weight w_{ji}. So, for the lower bounded constraints ($w_i^t x \geq d_i$) the linear combination is lower estimated, i.e., $w_i^t x^{\alpha_i} < w_i^t x$ since $\alpha_i > 1$. For the upper bounded constraints ($w_i^t x \leq d_i$) the linear combination is upper estimated, i.e., $w_i^t x^{\alpha_i} > w_i^t x$ since $0 < \alpha_i < 1$. Thereby, the inequality constraints satisfaction requires a large number of variables to tend to an activation level sufficiently near to one or zero. Intuitively, these observations show that the instituting *cam* contributes significantly to direct network's convergence to a vertex of the hypercube.

Let now study the dynamics of the new extended Hopfield model (EHM)*cam* with the instituted *cam*. The normalization of the activation flows is obtained by applying the logarithm function on each term of the inequality constraints before their transformations into equality constraints.

Lemma 3: The k inequality constraints $w_i^t x \leq d_i$ or $w_i^t x \geq d_i$ with bounds $d_i \neq 0$ are equivalent to the equality constraints $d_i'y_i - \ln(w_i^t x^{\alpha_i}) = 0$ for $i = 1,\cdots,k$ provided that

$$d_i' = \begin{cases}1, & \text{if } d_i = 1\\ \ln(d_i), & \text{otherwise}\end{cases}$$

and

$$\begin{cases}y_i \leq 0, & \text{if } w_i^t x \leq 1\\ y_i \geq 0, & \text{if } w_i^t x \geq 1\\ y_i \leq 1, & \text{if } w_i^t x \leq d_i \qquad (d_i \neq 1)\\ y_i \geq 1, & \text{if } w_i^t x \geq d_i \qquad (d_i \neq 1).\end{cases}$$

The corresponding to the inequality constraints new energy in the (EHM)*cam* is now written

$$Ecam_3(x,\,y) = \frac{1}{2}\sum_{i=1}^{k}\frac{C_i}{\alpha_i}\left[\ln(w_i^t x^{\alpha_i}) - d_i'y_i\right]^2. \tag{10}$$

The Proposition 2 can be stated as follows.

Proposition 4: A two layers connectionist model whom the behavior is controlled by the dynamic system

$$\begin{pmatrix}du/dt\\ dv/dt\end{pmatrix} = -\begin{pmatrix}\partial Ecam/\partial x\\ \partial Ecam/\partial y\end{pmatrix} = -\begin{pmatrix}[T'x + b] + \sum_{i=1}^{k} C_i\,\dfrac{w_{ji}x_j^{\alpha_j - 1}}{w_i^t x_{\alpha_i}}\left[\ln(w_i^t x^{\alpha_i}) - d_i'y_i\right]\\ \cdot\dfrac{C_i}{\alpha_i}\,d_i'[d_i'y_i - \ln(w_i^t x_{\alpha_i})]\end{pmatrix} \tag{11}$$

with $u \in (-\infty,+\infty)^n$ and $x \in (0,1)^n$ the vectors of internal and coming-out variables for the first layer: $\forall\,j = 1,\cdots,n$ $x_j = \frac{1}{2}[1 + \tanh(u_j/\tau_j)]$ (with $\tau_j > 0$) and $v \in (-\infty,+\infty)^k$ and y the vectors of internal and coming-out variables for the second layer: $\forall\,i = 1,\cdots,k$

$$y_i = \begin{cases}-\exp(-v_i/\rho_i), & \text{if } y_i \leq 0\\ \exp(v_i/\rho_i), & \text{if } y_i \geq 0 \qquad (\text{with}\\ 1 - \exp(-v_i/\rho_i), & \text{if } y_i \leq 1 \qquad \rho_i > 0)\\ 1 + \exp(v_i/\rho_i), & \text{if } y_i \geq 1\end{cases} \tag{12}$$

converges to a local minimum of the function

$$\begin{aligned}Ecam(x,\,y) &= AE_1(x) + BE_2(x) + Ecam_3(x,\,y)\\ &= \frac{1}{2}x^tT'x + b^tx + \frac{1}{2}\sum_{i=1}^{k}\frac{C_i}{\alpha_i}\\ &\quad\cdot\left[\ln(w_i^t x^{\alpha_i}) - d_i'y_i\right]^2.\end{aligned} \tag{13}$$

The convergence proof is similar to this one in [1] and [2] and the activation rules for the neurons Ξ_j and Υ_i have as follows:

$$\frac{dx_j}{dt} = \frac{2}{\tau_j}\left(\sum_{i=1,\cdots,k} C_i\,\frac{w_{ji}x_j^{\alpha_i}}{w_i^t x^{\alpha_i}}\,[d_i' y_i - \ln(w_i^t x^{\alpha_i})] - [T_j' x + b_j]x_j\right)(1 - x_j) \tag{14}$$

$$\frac{dy_i}{dt} = -\frac{C_i}{\alpha_i}\,d_i' f(y_i)(d_i' y_i - \ln(w_i^t x^{\alpha_i})). \tag{15}$$

Then, the new expressions of the resources $R_i^{\alpha_i}$ attributed to the neurons Υ_i and the activation flows $out_{\Upsilon_i \to \Xi_j}$ spread from Υ_i to neurons Ξ_j are equal to

$$R_i^{\alpha_i} = d_i' y_i - \ln(w_i^t x^{\alpha_i}), \tag{16}$$

$$\begin{aligned}\mathrm{out}_{\Upsilon_i \to \Xi_j} &= \frac{w_{ji}x_j^{\alpha_i}}{w_i^t x^{\alpha_i}}\,[d_i' y_i - \ln(w_i^t x^{\alpha_i})]\\ &= \frac{w_{ji}x_j^{\alpha_i}}{w_i^t x^{\alpha_i}}\,R_i^{\alpha_i}.\end{aligned} \tag{17}$$

By checking the three conditions it can easily be seen that the (EHM)*cam* incorporates the *cam.*

B. Weights Determination

Following the idea of Abè [1] for defining the value of the weight B in relation to A for the equality constraints on the traveling salesman problem, we prove the following result for the weights A, B, and C_i, $i = 1, \cdots, k$, which is valid for a combinatorial optimization problem with equality and inequality constraints. Let us consider two adjacent vertices c and $c(j)$ in the hypercube $[0, 1]^n$ which differ only on their jth component.

Theorem 5: If the coefficients w_{ji} and d_i are nonnegative integers and the weights A, B, and C_i are defined with respect to

$$B > 2A\,\frac{\max\limits_{c\in\Gamma}\max\limits_{j=1,\cdots,n}(0,\, E_1(c) - E_1(c(j)))}{\min\limits_{\substack{i=1,\cdots,l\\ j=1,\cdots,n}} r_{ji}^2} \quad \forall i = 1, \cdots, k \tag{18}$$

$$C_i > 2A\,\frac{\max\limits_{c\in\Gamma}\max\limits_{j=1,\cdots,n}[E_1(c) - E_1(c(j))]}{\frac{1}{\alpha_i}\,[\ln(d_i op_i 1) - \ln(d_i)]^2} \tag{19}$$

where Γ is the set of feasible solutions and op_i an addition if the constraint is of the type $w_i^t x \le d_i$ or subtraction if the constraint is of the type $w_i^t x \ge d_i$, then any vertex of the hypercube violating at least one constraint and including in its neighborhood a feasible solution can not be a local minimum for $Ecam$.

Proof: Let $c \in \Gamma$ and $c(j) \notin \Gamma$ violating at least an equality constraint, say p, or an inequality constraint, say q. Obviously, the vertex $c(j)$ is not a local minimum if the weights A, B, and C_i take values such that $Ecam(c(j)) > Ecam(c)$. That is

$$\begin{aligned}&A[E_1(c(j)) - E_1(c)] + B[E_2(c(j)) - E_2(c)]\\ &\quad + [Ecam_3(c(j)) - Ecam_3(c)] > 0.\end{aligned} \tag{20}$$

On the vertex $c(j)$ violating the pth equality constraint the term $E_2(c(j))$ is increased with regard to $E_2(c)$ by at least a cost P_{e_p} equal to

$$\begin{aligned}P_{e_p} &= \tfrac{1}{2}\,[r_p^t c(j)]^2 - s_p \times r_p^t c(j) - \tfrac{1}{2}\,[r_p^t c]^2 + s_p \times r_p^t c\\ &= [r_p^t c + r_{jp}(c(j)_j - c_j)]\\ &\quad \times \left[\tfrac{1}{2}\,(r_p^t c + r_{jp}[c(j)_j - c_j]) - s_p\right] + \tfrac{1}{2}\,s_p^2\\ &\qquad (\text{since, } r_p^t c = s_p)\\ &= [s_p + r_{jp}[1 - 2c_j]] \times \left[\tfrac{1}{2}\,(s_p + r_{jp}[1 - 2c_j]) - s_p\right]\\ &\quad + \tfrac{1}{2}\,s_p^2 \qquad (\text{since, } c(j)_j = 1 - c_j)\\ &= [s_p + r_{jp}[1 - 2c_j]] \times \left[\tfrac{1}{2}\,r_{jp}[1 - 2c_j] - \tfrac{1}{2}\,s_p\right] + \tfrac{1}{2}\,s_p^2\\ &= \tfrac{1}{2}\,r_{jp}^2 \qquad (\text{since, } (1 - 2c_j)^2 = 1).\end{aligned}$$

So, we have $E_2(c(j)) - E_2(c) \ge \frac{1}{2}\,r_{jp}^2$ and $Ecam_3(c(j)) \ge Ecam_3(c) = 0$. If the weight B is such that $B > 2A[E_1(c) - E_1(c(j))]/r_{jp}^2$, then (20) is always verified.

On $c(j)$ violating the qth inequality constraint the energy $Ecam_3(c(j))$ is at least equal to

$$P_{i_q} = \frac{1}{2}\,\frac{C_q}{\alpha_q}[\ln(w_q^t c(j)^{\alpha_q}) - d_q' y(j)_q]^2. \tag{21}$$

A study of the (EHM)*cam* dynamics (15) gives that $d_q' y(j)_q$ converges asymptotically to $\ln(d_q)$. So, when the network reaches convergence, P_{i_q} can be approximated by: $P_{i_q} \approx \frac{1}{2}\,(C_q/\alpha_q)[\ln(w_q^t c(j)^{\alpha_q}) - \ln(d_q)]^2$. Since $w_q^t c(j)^{\alpha_q}$ is bounded below by $d_q + 1$ if $w_q^t x \le d_q$ and upper bounded by $d_q - 1$ if $w_q^t x \ge d_q$ (we recall that w_{jq} and d_q are integers), we obtain that

$$P_{i_q} \ge \begin{cases}\dfrac{1}{2}\,\dfrac{C_q}{\alpha_q}\,[\ln(d_q + 1) - \ln(d_q)]^2, \\ \text{if the type of the constraint is } w_q^t x \le d_q\\ \dfrac{1}{2}\,\dfrac{C_q}{\alpha_q}\,[\ln(d_q - 1) - \ln(d_q)]^2, \\ \text{if the type of the constraint is } w_q^t x \ge d_q.\end{cases}$$

Since $c \in \Gamma$, we have $Ecam_3(c) = 0$ and $E_2(c(j)) \ge E_2(c)$. So, by taking the weights A and C_q as indicated in (19), (20) is verified. ■

Notice that the weight C_q must be chosen at least equal to the inverse of the minimal cost

$$P_{i_q} = \frac{1}{2}\,\frac{1}{\alpha_q}\,[\ln(d_q op_q 1) - \ln(d_q)]^2, \qquad q = 1, \cdots, k.$$

So, its value depends on the bounds d_q and the type of the constraints which fix the operators op_q.

Let us see the case $d_q = 1$. If the constraint is lower bounded, i.e., $w_q^t x \ge 1$, then the operator op_q is a subtraction. Since the cost P_{i_q} tends to infinity, the weight C_q can take any positive value. If the constraint is upper bounded, i.e., $w_q^t x \le 1$, the operator op_q is an addition and the cost P_{i_q} becomes equal to $P_{i_q} = \frac{1}{2}\,1/\alpha_q[\ln(2)]^2$. It is clearly possible to find a value for the weight C_q that satisfies condition (19). For the case $d_q > 1$, for satisfying the condition (19), the weight C_q must take large values since whatever the constraint type the quantities $\ln(d_q+1)-\ln(d_q)$ or $\ln(d_q-1)-\ln(d_q)$ are small. Obviously, the network dynamic will be principally controlled by the constraints and less by the objective function. Consequently, fixing a value for the weight C_q is possible only when the global amount $\sum_j \text{out}_{\Upsilon_i \to \Xi_j}$ of flows emitting by the neuron Υ_q is sufficiently large as long as its associated constraint remains unsatisfied. The neperien logarithm function is well adapted when the (EHM)*cam* must handle inequality constraint of type $w_i^t x \geq 1$. For the constraints $w_i^t x \leq 1$, it would be necessary in practice to use a logarithm with a smaller base in order to reduce the weight C_q. On the other hand, for the inequality constraints with large bound d_i the alone introduction of the logarithm function is rather unadvised. We can use it, for example, with a hyperbolic tangent. This function preserves around the bound d_i a sufficient emitting of flows. Nevertheless, the choice of any other function different to the logarithm will imply the lost of the equality between the global quantity of flows emitting by a neuron and the amount of its resource. However, a such choice must preserve the two other principles of *cam.*

IV. The Set Covering Problem

We consider a set Q with k elements and a collection R of n weighted subsets of Q. For $j = 1, \cdots, n$ the jth set is labeled S_j, and its weight c_j. Each element q_i of Q is supposed being included to at least one set of the collection R. The exact composition of each set is given by the $k \times n$ matrix W: w_{ji} is equal to one if the ith element belongs to the jth set, and zero otherwise.

A *cover* is a subcollection S of sets from R not necessarily disjoint which covers all elements of Q, i.e., such that every element of Q belongs to, at least, one member of S. The weighted minimum set covering problem (WSCP) consists in finding a cover minimizing the total weight $\sum_{S_j \in S} c_j$ for a given instance $I = (R, Q)$. This problem is known to be NP-hard [7] and it can be formulated as an integer linear program as follows:

$$\begin{aligned} &\text{minimize } c^t x \\ &\text{subject to: } w_i^t x \geq 1 \qquad \forall i = 1, \cdots, k \\ &\text{with } x \in \{0, 1\}^n. \end{aligned} \tag{22}$$

In order to evaluate the quality of neural solutions we use the greedy algorithm due to Johnson [6] as a comparison measure. This algorithm chooses at each step the set S_j with the larger ratio $|S_j|/c_j$ to put in the solution as long as all elements of Q are not covered. Next, the introduced set and their elements are removed from R and Q and the cardinalities of the other sets are updated. With such an algorithm solutions minimality and sets irredundance are not always guaranteed. A worst case tight bound has been established by Johnson [6] and Chvátal [3] equal to $\sum_{j=1}^{d}(1/j)$ times the weight of an optimal cover, where d is the size of the largest set S_j. This approximation ratio is reachable for a series of particular unweighted instances. We use a such instance for evaluating the performance of EHM and (EHM)*cam* neural networks.

In the EHM, the covering constraints are introduced by the penalty function

$$E_3 = \tfrac{1}{2} \sum_{i=1}^{k} [w_i^t x - y_i]^2 \qquad \text{with } y_i \geq 1,\, i = 1, \cdots, k.$$

The derived activation rules [(7) and (8)] are

$\diamondsuit$ for the sets S_j

$$dx_j/dt = \left(C \sum_{i=1}^{k} w_{ji} x_j (y_i - w_i^t x) - A c_j x_j \right)(1 - x_j)$$

$\diamondsuit$ for the elements q_i: $dy_i/dt = -C(y_i - 1)(y_i - w_i^t x)$, with $C > A \max_{j=1,\cdots,n} c_j$.

In the (EHM)*cam*, the covering constraints are introduced by the function

$$Ecam_3 = \frac{1}{2} \sum_{i=1}^{k} \frac{C_i}{\alpha_i} \left[\ln(w_i^t x^{\alpha}) - y_i\right]^2$$

with $\alpha_i > 1$ and $y_i \geq 0$, $i = 1, \cdots, k$.

As the k constraints are the same, we take $C_i = C$ and $\alpha_i = \alpha\ \forall i = 1, \cdots, k$ (with $C > 0$ and $\alpha > 1$). The derived activation rules [(14) and (15)] are

$\diamondsuit$ for the sets S_j

$$dx_j/dt = \left[C \sum_{i=1}^{k} \frac{w_{ji} x_j^{\alpha}}{w_i^t x^{\alpha}} \left(y_i - \ln(w_i^t x^{\alpha})\right) - A c_j x_j \right] \times (1 - x_j)$$

$\diamondsuit$ for the elements q_i

$$dy_i/dt = -C/\alpha y_i (y_i - \ln(w_i^t x^{\alpha})).$$

By (19), the weight C can take any positive value and A can be fixed to one.

V. Experimental Results

In order to extract the cover from the state of the network at convergence we consider two experimentally established thresholds L^{inf} and L^{sup} fixed, respectively, to the values 0.3 and 0.7. The first one is the threshold below which the neuron is considered as inactive. The second is the threshold up which the neuron is active. Any activation level located between these two thresholds is considered as intermediate and so, it is ambiguous to interpret.

To break the symmetry and thus prevent the system from settling down into an unstable equilibrium state, the initial activation levels are randomly attributed in a fixed interval. A weak width for this interval gives equivalent chances for any set to belong to the cover. The interval chosen has a width equal to 0.1 centered around 0.5. In our experiments we have considered that the model reaches convergence when two consecutive states are almost identical. Their differences are measured by the evolution rate [9]: $\Lambda = (1/n) \sum_{j=1}^{n} [x_j(t + \Delta t) - x_j(t)]^2$. We have fixed the limit for Λ equal to 10^{-10}. The time step Δt was equal to $\Delta t = 0.01$. This large time step has sometimes as effect to violate the activation bounds. In such cases, the activations are reset nearer the bounds zero or one. The weights were fixed to $A = 1$ and $C = 2$ which also are valid for the EHM (see Section IV).

The first objective of our experiments was to compare the two versions of the extended Hopfield model on their capacity to share out the neurons corresponding to the sets of the collection R. We have considered a randomly generated instance containing 60 sets and 30 elements. The Fig. 1 reports, for the EHM and for the (EHM)*cam*, the evolution of the activation levels x_j, $j = 1, \cdots, n$. For the (EHM)*cam* different values of the power α were studied.

For the EHM, case a, we observe the existence of a large number of neurons with ambiguous interpretation. So, a cover cannot be deduced from the final state. For the (EHM)*cam*, cases b, c, and d, the repartition of the sets into two active/inactive categories is realized as soon as $\alpha > 1$. For $\alpha = 2$ (case c), about a thousand iterations are needed. For $\alpha = 3$ (case d), a hundred of iterations are sufficient to yield a cover. But, the rapidity of convergence is achieved in prejudice of the solution quality. Indeed, the cardinality of the solution returned by the model for $\alpha = 3$ is more expensive than the one obtained for $\alpha = 2$ (11 sets instead of nine). It is clear that when the convergence is too fast, the favor conceded by an element is so great for the most active neurons that any weak difference generated during the initialization phase affects the solution. Our study with different values of α has shown that the best experimental behavior is achieved for the (EHM)*cam* when the parameter $\alpha = 2$. The sets are clearly shared out and the final solution is relatively not conditioned by the initialization interval.

The second part of our experimental study consists in investigating the behavior of EHM and (EHM)*cam* for $\alpha = 2$ on some particular instances of Johnson. We have observed a clear repartition of the neurons into active and inactive by both models. However, the (EHM)*cam* converges faster than EHM. The final state for both models is composed exclusively by neurons giving the optimal solution. We see that such instances, which are difficult for the sequential heuristic, become easy for the studied neural networks.

In the third part, we evaluate the (EHM)*cam* on its capacity of settling down into states without intermediate activations and its solution quality in comparison with the greedy algorithm. We do not report results on the EHM since, for many treated instances, many neurons had involved with ambiguous classification. By considering the same hard limits L^{inf} (0.3) and L^{sup} (0.7) the produced solutions were not enough interesting.

We have considered different groups of 50 randomly generated unweighted graphs. They are specified by giving (see Table I).

- The size of the instance $I = (R, Q)$ defined by the number $|R|$ of sets composing the collection R and the number $|Q|$ of elements included in the basic set Q.
- The number $|q_i|$ of sets containing an element of Q. An interval of authorized values is given for this parameter.
- The maximum number of elements included into a set S_j of the collection R. In order to make the competitions

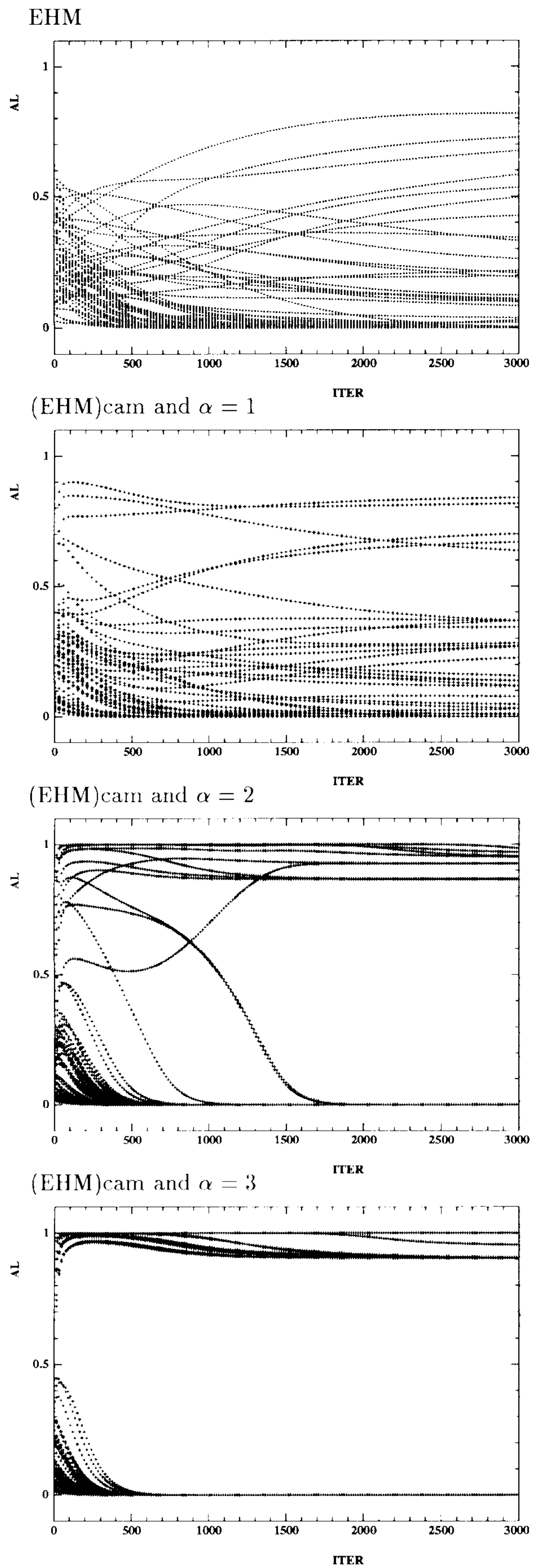

Fig. 1. Activation level (AL) evolution in function of the number of iterations (ITER) for the sets of a randomly generated instance.

TABLE I
NEURAL NETWORK (NN) PERFORMANCE (SOLUTION QUALITY AND AVERAGE CPU TIME IN SECONDS) IN COMPARISON TO A HEURISTIC (H) FOR DIFFERENT INSTANCE SIZES AND DENSITIES

$\|q_i\|$	$\|R\| \times \|Q\|$	H	=	NN	T_H	T_{NN}
[1..5]	80 40	6	28	66	0.1	15.3
	100 30	6	42	52	0.09	12.5
	100 50	0	36	64	0.17	25.5
	150 50	6	24	70	0.37	34.8
	average	4.5	32.5	63.0		
[3..5]	80 40	4	28	68	0.09	21.3
	100 30	4	26	70	0.09	15
	100 50	14	38	48	0.16	32.4
	150 50	2	18	80	0.25	41.7
	average	6.0	27.5	66.5		
[3..7]	80 40	12	34	54	0.09	25.1
	100 30	12	34	54	0.08	16.4
	100 50	16	34	50	0.15	37.5
	150 50	4	12	84	0.25	44.3
	average	11.0	28.5	60.5		
[3..10]	80 40	18	34	48	0.09	23.3
	100 30	12	36	52	0.08	17.3
	100 50	12	18	70	0.14	42.3
	150 50	16	22	62	0.22	57.1
	average	14.5	27.5	58.0		
[5..10]	80 40	24	38	38	0.08	29.9
	100 30	14	42	44	0.08	18.6
	100 50	22	24	54	0.14	50.8
	150 50	12	24	64	0.22	60.8
	average	18.0	32.0	50.0		

more uncertain this number was limited to seven for each instance.

Table I gives the performance of the two methods from solution quality point of view and CPU required time. For each group of instances, the percentages of the cases where the heuristic has given a better solution than the neural network (column "H"), an equivalent solution (column "="), and finally a worst solution (column "NN") are reported. Also the average required CPU times (on a SPARC 10) for the heuristic T_H and the neural network T_{NN} are presented.

Even with the instituted *cam* in some cases the network settles into a state with some neurons having intermediate activations. The percentage of such solutions increases with the average degree of the basic elements, but at average never it exceeds 20%. More often the number of concerned neurons is about three or five but never exceeds the eight. These neurons with intermediate activations are considered as active and are introduced into the solution. This has always given feasible solutions but had introduced some redundant sets.

For a percentage of cases ranging between 50–66.5%, the NN presents a strictly better performance than H. The most favorable cases correspond to the groups of instances for which few number of sets contain each element of Q, i.e., $|q_i| \in [1 \cdots 5]$ or $|q_i| \in [3 \cdots 5]$. Two reasons explain the excellent results of the NN for those groups of instances. Firstly, because very few solutions (about 8% instead of 14–20% for other groups) contain neurons with intermediate activations. The second reason is the low efficiency of H when the instance is made up of some basic elements included only in one set. Obviously, the insertion of these sets into the cover is necessary and if they have a weak cardinality the greedy algorithm introduces them only during the last steps. It can be seen, in Table I, that for high densities, i.e., $|q_i| \in [3 \cdots 7]$, $|q_i| \in [3 \cdots 10]$, or $|q_i| \in [5 \cdots 10]$, the NN gives a strictly better solution than H in a percentage of cases ranging between 50–60%, and an equivalent solution in about 30% of cases. A degradation of the NN performance is observed when the density increases. However, the NN performs still better than H. In the worst case ($|q_i| \in [5 \cdots 10]$), a better cover is given by the NN in 50% of cases, comparatively to only 18% for the H.

Notice that the NN requires larger CPU time than H but it remains at average lower than one minute. This time would sometimes be acceptable if solutions quality is the main objective while a massively parallel implementation let us expecting significant improvements.

VI. CONCLUSION

We have proposed a new expression for the penalty energy handling inequality constraints in Hopfield models. The derived rules introduce competitions between the variables involved into the same constraint and solve them with the competitive activation mechanism. The treatment of the upper bounded inequality constraints generates competitions for which the objective is negative resources acquisition. In consequence, we have extended the competitive activation mechanism for dealing also with negative resources. The great interest for this mechanism is its capacity to distribute the neurons into two active/inactive categories. This point remedies a drawback of the extended Hopfield models. Validation was given through an extensive experimental study on the set cover problem.

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Armelle Le Gall received the Ph.D. degree in computer science from the University of Paris-Sud in 1997.

She is a Research Assistant at the University of Paris-Nord. Her research interests are incremental and adaptive algorithms for solving combinatorial optimization problems and neural networks.

Vassilis Zissimopoulos received the bachelor’s degree from Athens University, Greece, and the Ph.D. degree in computer science at the University of Paris-Sud in 1984.

He is currently Professor of computer science at the University of Paris-Nord. His research interests include approximation algorithms, local search, neural networks in combinatorial optimization, incremental algorithms, task allocation, static and dynamic scheduling, load balancing, and adaptive algorithms.