Propositional Logic - An Entailment Proof

An Entailment Proof

Here is an example of a simple entailment proof that sometimes you will have to do in the homework.

Exercise. Consider the following statements in English.

1. Object A is green.
2. Object A is red or blue or not green.
3. Object A is red or blue.

Encode the above statements in propositional logic. Then prove that statement (3) follows logically from statements (1) and (2).

You need to give a **model-theoretic proof** i.e., one that is based on the definition of entailment or "logically follows" (slide 46 of file `https://cgi.di.uoa.gr/~ys02/dialekseis2020/propositional.pdf`). You are not allowed to use truth-tables or resolution.

Proof

We introduce propositional symbols $Green$, Red and $Blue$. Then, the given sentences can be encoded in propositional logic as follows:

1. $Green$
2. $Red \vee Blue \vee \neg Green$
3. $Red \vee Blue$

Now we have to prove that

$$Green \wedge (Red \vee Blue \vee \neg Green) \models (Red \vee Blue)$$

Proof (cont'd)

We will apply the definition of entailment.

Let I be an interpretation which satisfies the sentence $Green \wedge (Red \vee Blue \vee \neg Green)$. From the definition of satisfaction (slide 41), we now have that $I(Green \wedge (Red \vee Blue \vee \neg Green)) = true$.

From the definition of interpretation (slides 31-33), the above is equivalent to $I(Green) = true$ and $I(Red \vee Blue \vee \neg Green) = true$.

Similarly, from the definition of interpretation again, the second of the above statements is equivalent to $I(Red) = true$ or $I(Blue) = true$ or $I(\neg Green) = true$. The third of these statements is equivalent to $I(Green) = false$.

Proof (cont'd)

Putting all these statements about I together, we see that $I(Red) = true$ or $I(Blue) = true$ which is equivalent to $I(Red \vee Blue) = true$ i.e., I satisfies the statement $Red \vee Blue$.

The proof is now complete.